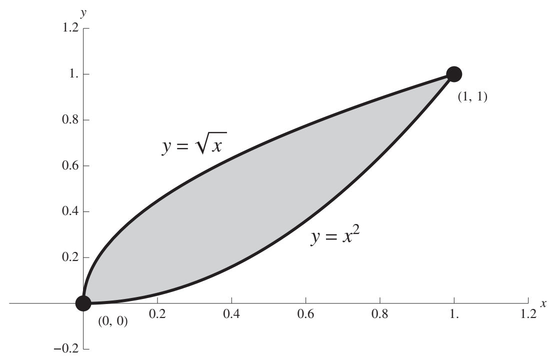
# Chapter 6 Applications of the Integral

## 6.1 Area Between Graphs

### Concepts and Vocabulary

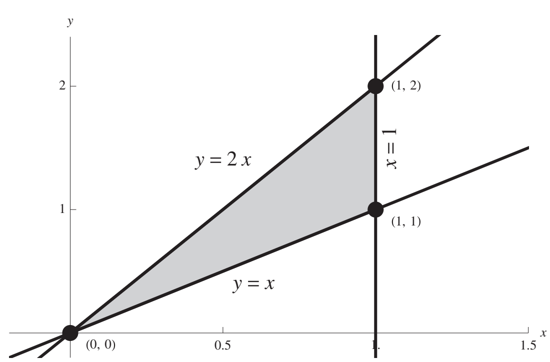
1. The two graphs meet at $x = 0$ and at $x = 1$, and in the interval $[0, 1]$, we have $\sqrt{x} \geq x^2$, so the integral is $\int_0^1 \left(\sqrt{x} - x^2\right)\, dx$. See the figure below:




2. Partitioning the $y$-axis, the two graphs meet at $y = -1$ and again at $y = 1$; when $-1 \leq y \leq 1$, we have $y^2 \leq 1$, so the integral is $\int_{-1}^1 (1 - y^2)\, dy$.

### Skill Building

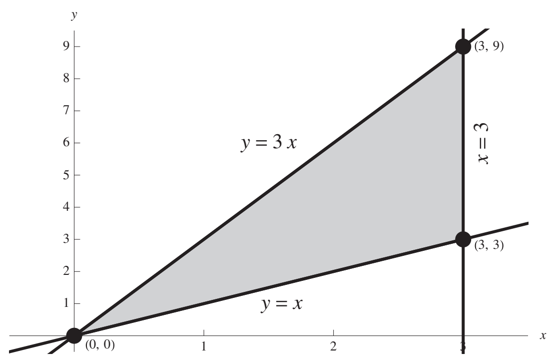
3. The region is shown below:




Partitioning along the $x$-axis, we have $x \leq 2x$, so the area is

$$\int_0^1 (2x - x)\, dx = \int_0^1 x\, dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2} - 0 = \boxed{\frac{1}{2}}.$$

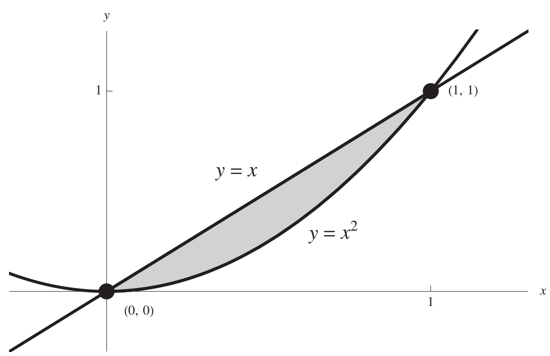
4. The region is shown below:




Partitioning along the $x$-axis, we have $x \leq 3x$, so the area is

$$\int_0^3 (3x - x)\,dx = \int_0^3 2x\,dx = \left[x^2\right]_0^3 = 9 - 0 = \boxed{9}.$$

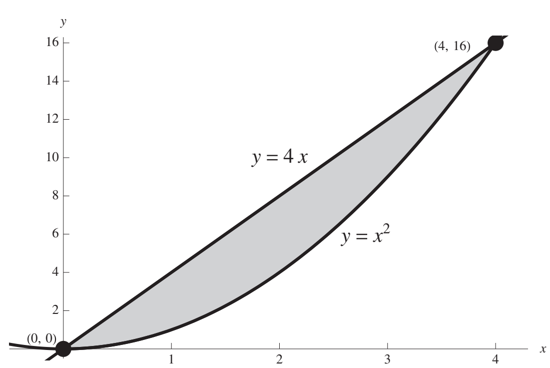
5. The region is shown below:




The two curves intersect when $x^2 = x$, so when $x = 0$ and $x = 1$. Partitioning along the $x$-axis, we have $x^2 \leq x$, so the area is

$$\int_0^1 (x - x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \left(\frac{1}{2} - \frac{1}{3}\right) - (0 - 0) = \boxed{\frac{1}{6}}.$$

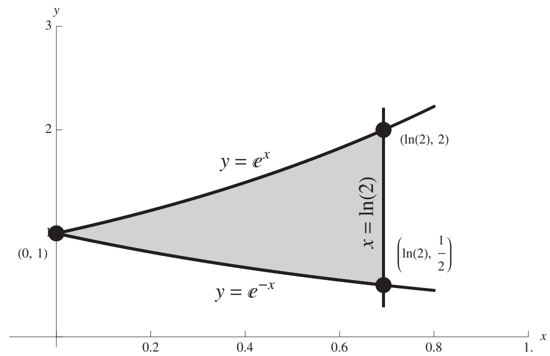
6. The region is shown below:



The two curves intersect when $x^2 = 4x$, so when $x = 0$ and $x = 4$. Partitioning along the $x$-axis, we have $x^2 \le 4x$, so the area is

$$\int_0^4 (4x - x^2)\,dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \left(32 - \frac{64}{3}\right) - (0 - 0) = \boxed{\frac{32}{3}}.$$

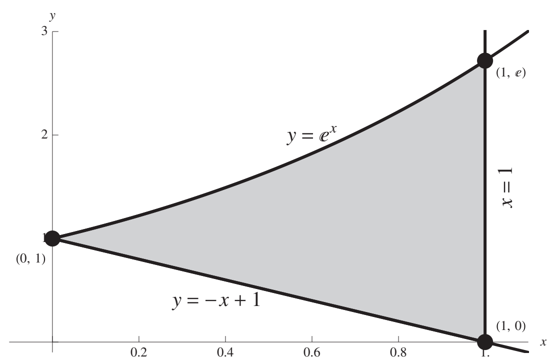
7. The region is shown below:




Partitioning along the $x$-axis, we have $e^{-x} \le e^x$, so the area is

$$\int_0^{\ln 2} \left(e^x - e^{-x}\right)\,dx = \left[e^x + e^{-x}\right]_0^{\ln 2} = \left(e^{\ln 2} + e^{-\ln 2}\right) - (e^0 + e^0) = \left(2 + \frac{1}{2}\right) - (1+1) = \boxed{\frac{1}{2}}.$$

8. The region is shown below:

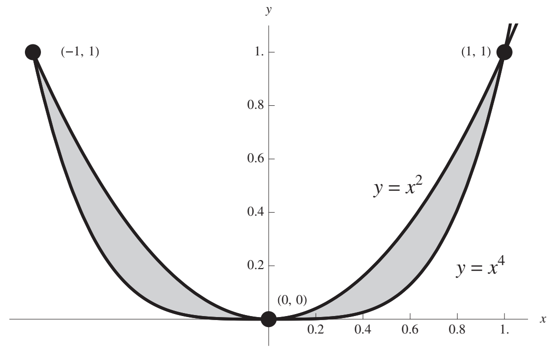



Partitioning along the $x$-axis, we have $-x + 1 \le e^x$, so the area is

$$\int_0^1 (e^x - (-x+1))\,dx = \int_0^1 (e^x + x - 1)\,dx = \left[e^x + \frac{1}{2}x^2 - x\right]_0^1$$

$$= \left(e^1 + \frac{1}{2} - 1\right) - (e^0 + 0 - 0) = \boxed{e - \frac{3}{2}}.$$

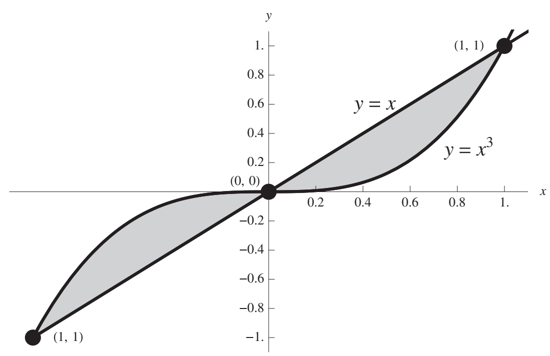
9. The region is shown below:




The two curves intersect when $x^2 = x^4$, so when $x = -1$, $x = 0$, and $x = 1$. Partitioning along the $x$-axis, we have $x^4 \le x^2$, so the area is

$$\int_{-1}^{1} \left(x^2 - x^4\right)\, dx = \left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_{-1}^{1} = \left(\frac{1}{3} - \frac{1}{5}\right) - \left(-\frac{1}{3} + \frac{1}{5}\right) = \boxed{\frac{4}{15}}.$$

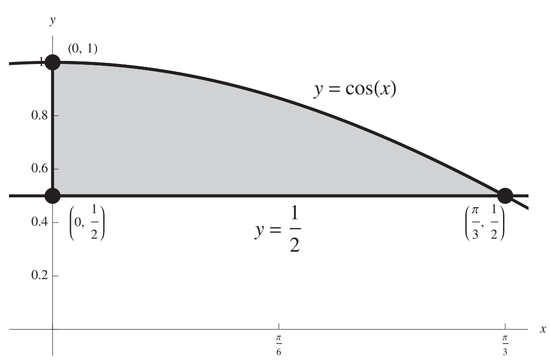
10. The region is shown below:




The two curves intersect when $x = x^3$, so when $x = -1$, $x = 0$, and $x = 1$. Since the region between $x = -1$ and $x = 0$ is below the $x$-axis, we compute the area from $x = 0$ to $x = 1$ and double it so that we count that area as positive. Partitioning along the $x$-axis, we have $x^3 \le x$, so the area is

$$2\int_{0}^{1} \left(x - x^3\right)\, dx = 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_{0}^{1} = 2\left(\left(\frac{1}{2} - \frac{1}{4}\right) - (0 - 0)\right) = \boxed{\frac{1}{2}}.$$

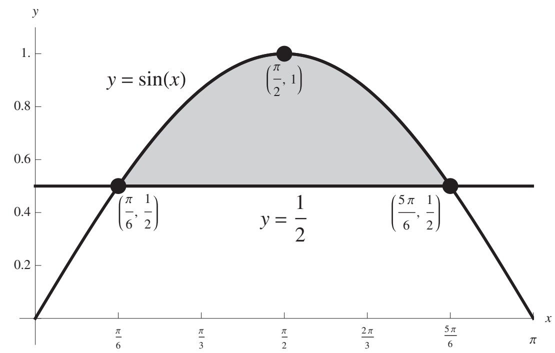
11. The region is shown below:



Partitioning along the $x$-axis, we have $\frac{1}{2} \le \cos x$, so the area is

$$\int_0^{\pi/3} \left(\cos x - \frac{1}{2}\right)\, dx = \left[\sin x - \frac{1}{2}x\right]_0^{\pi/3} = \left(\sin\frac{\pi}{3} - \frac{1}{2}\cdot\frac{\pi}{3}\right) - (\sin 0 - 0) = \boxed{\frac{\sqrt{3}}{2} - \frac{\pi}{6}}.$$

12. The region is shown below:

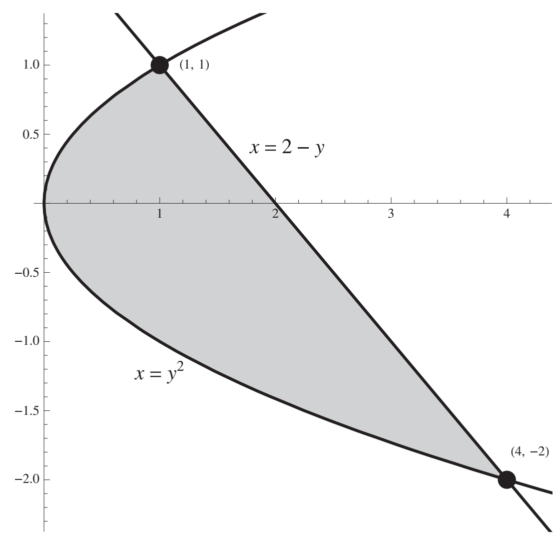



The two curves intersect when $\sin x = \frac{1}{2}$, so at $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ and $\left(\frac{5\pi}{6}, \frac{1}{2}\right)$. Partitioning along the $x$-axis, we have $\frac{1}{2} \le \sin x$, so the area is

$$\int_{\pi/6}^{5\pi/6} \left(\sin x - \frac{1}{2}\right)\, dx = \left[-\cos x - \frac{1}{2}x\right]_{\pi/6}^{5\pi/6} = \left(-\cos\frac{5\pi}{6} - \frac{1}{2}\cdot\frac{5\pi}{6}\right) - \left(-\cos\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\pi}{6}\right)$$

$$= \left(\frac{\sqrt{3}}{2} - \frac{5\pi}{12}\right) - \left(-\frac{\sqrt{3}}{2} - \frac{\pi}{12}\right) = \boxed{\sqrt{3} - \frac{\pi}{3}}.$$

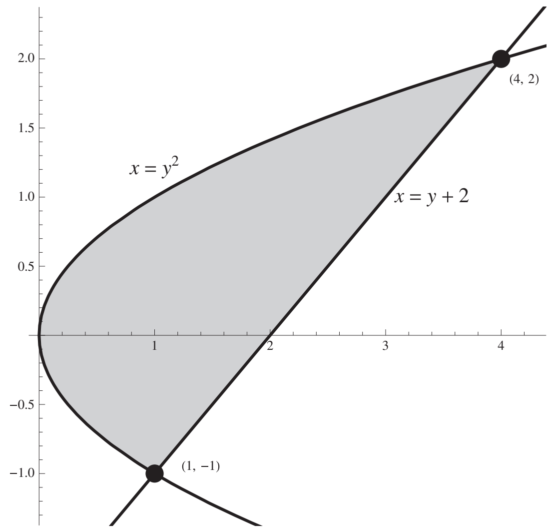
13. The region is shown below:




The two curves intersect when $2 - y = y^2$, so when $y^2 + y - 2 = (y+2)(y-1) = 0$; this happens at the points $(1, 1)$ and $(4, -2)$. Partitioning along the $y$-axis, we have $y^2 \le 2 - y$, so the area is

$$\int_{-2}^{1} \left(2 - y - y^2\right)\, dy = \left[2y - \frac{1}{2}y^2 - \frac{1}{3}y^3\right]_{-2}^{1} = \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) = \boxed{\frac{9}{2}}.$$

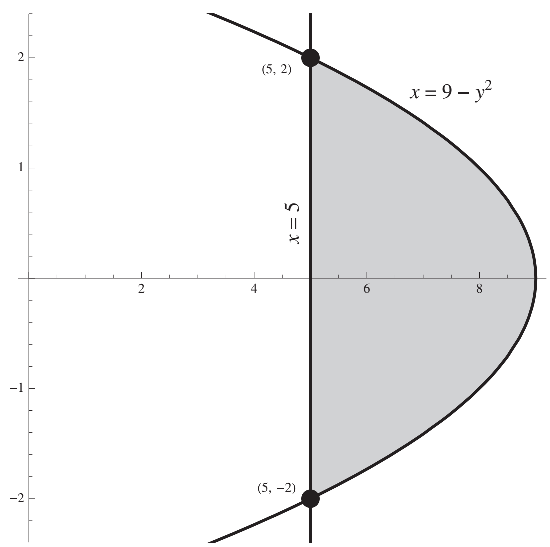
14. The region is shown below:




The two curves intersect when $y+2=y^2$, so when $y^2-y-2=(y-2)(y+1)=0$; this happens at the points $(4,2)$ and $(1,-1)$. Partitioning along the $y$-axis, we have $y^2 \le y+2$, so the area is

$$\int_{-1}^{2}\left(y+2-y^2\right)\,dy=\left[\frac{1}{2}y^2+2y-\frac{1}{3}y^3\right]_{-1}^{2}=\left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)=\boxed{\frac{9}{2}}.$$

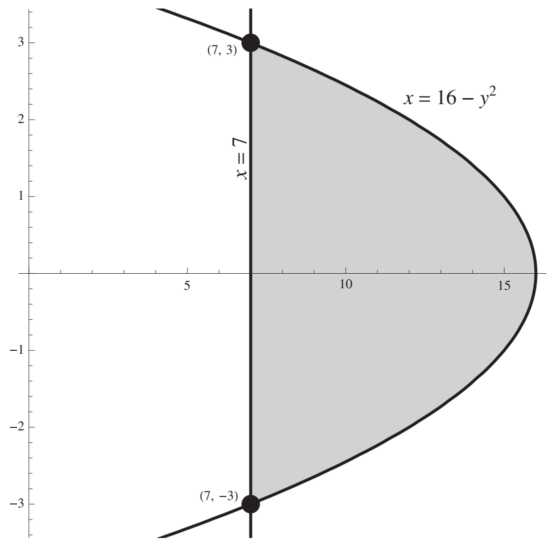
15. The region is shown below:




The two curves intersect where $9-y^2=5$, so when $y=-2$ and $y=2$. Partitioning along the $y$-axis, we have $5 \le 9-y^2$, so the area is

$$\int_{-2}^{2}(9-y^2-5)\,dy=\int_{-2}^{2}(4-y^2)\,dy=\left[4y-\frac{1}{3}y^3\right]_{-2}^{2}=\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)=\boxed{\frac{32}{3}}.$$

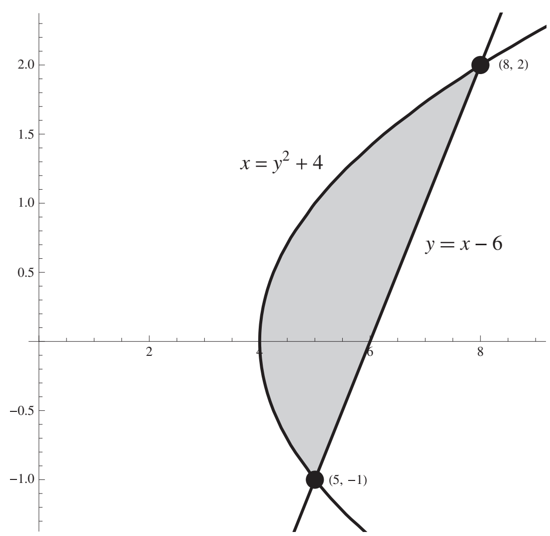
16. The region is shown below:




The two curves intersect where $16 - y^2 = 7$, so when $y = -3$ and $y = 3$. Partitioning along the $y$-axis, we have $7 \le 16 - y^2$, so the area is

$$\int_{-3}^{3} (16 - y^2 - 7)\, dy = \int_{-3}^{3} (9 - y^2)\, dy = \left[9y - \frac{1}{3}y^3\right]_{-3}^{3} = (27 - 9) - (-27 + 9) = \boxed{36}.$$

17. The region is shown below:

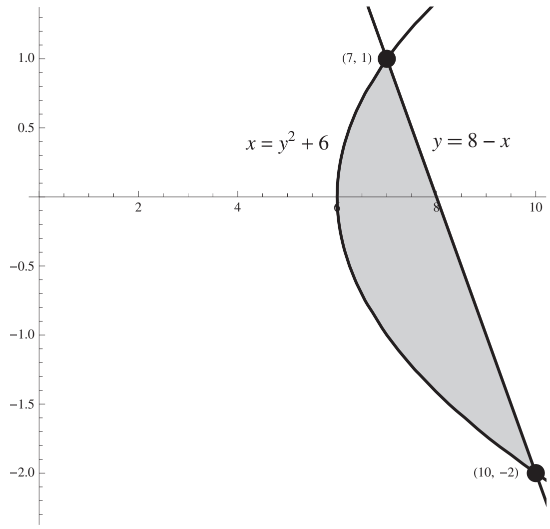



Solve the second equation for $x$ to get $x = y + 6$. The two curves intersect where $y^2 + 4 = y + 6$, so when $y^2 - y - 2 = (y - 2)(y + 1) = 0$. Therefore the intersection points are $(8, 2)$ and $(5, -1)$. Partitioning along the $y$-axis, we see that $y^2 + 4 \le y + 6$, so the area is

$$\int_{-1}^{2} \left(y + 6 - (y^2 + 4)\right) dy = \int_{-1}^{2} \left(-y^2 + y + 2\right) dy = \left[-\frac{1}{3}y^3 + \frac{1}{2}y^2 + 2y\right]_{-1}^{2}$$

$$= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) = \boxed{\frac{9}{2}}.$$

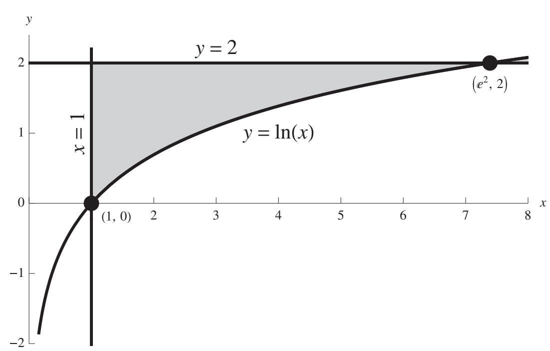
18. The region is shown below:




Solve the second equation for $x$ to get $x = 8 - y$. The two curves intersect where $y^2 + 6 = 8 - y$, so when $y^2 + y - 2 = (y+2)(y-1) = 0$. Therefore the intersection points are $(7,1)$ and $(10,-2)$. Partitioning along the $y$-axis, we see that $y^2 + 6 \le 8 - y$, so the area is

$$\int_{-2}^{1} \left(8 - y - (y^2+6)\right)\, dy = \int_{-2}^{1} (-y^2 - y + 2)\, dy = \left[-\frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y\right]_{-2}^{1}$$
$$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \boxed{\frac{9}{2}}.$$

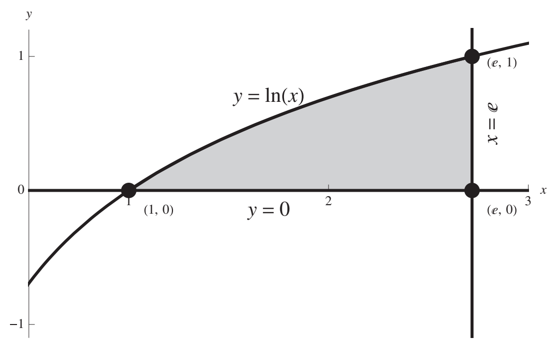
19. The region is shown below:




Solving $y = \ln x$ for $x$ gives $x = e^y$. Then partitioning along the $y$-axis, we see that $1 \le e^y$, so the area is

$$\int_0^2 (e^y - 1)\, dy = [e^y - y]_0^2 = (e^2 - 2) - (e^0 - 0) = \boxed{e^2 - 3}.$$

20. The region is shown below:




Solving $y = \ln x$ for $x$ gives $x = e^y$. Then partitioning along the $y$-axis, we see that $e^y \le e$, so the area is

$$\int_0^1 (e - e^y)\, dy = [ey - e^y]_0^1 = (e - e) - (0 - 1) = \boxed{1}.$$

21. This region is bounded above by $y = \cos x$ and below by $y = -\sin x$, and it extends from $x = -\frac{\pi}{4}$ to $x = \frac{3\pi}{4}$. So we integrate by subdividing along the $x$ axis, and we get

$$\int_{-\pi/4}^{3\pi/4} (\cos x - (-\sin x))\, dx = \int_{-\pi/4}^{3\pi/4} (\cos x + \sin x)\, dx = [\sin x - \cos x]_{-\pi/4}^{3\pi/4}$$
$$= \left(\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = \boxed{2\sqrt{2}}.$$

22. This region consists of two areas, one below the $x$ axis and one above. For $x < 0$, we have $\sqrt[3]{x} < x^3$, while for $x > 0$ we have $\sqrt[3]{x} > x^3$. So the total area is

$$\int_{-1}^{0} \left(x^3 - \sqrt[3]{x}\right)\, dx + \int_0^1 \left(\sqrt[3]{x} - x^3\right)\, dx = \left[\frac{1}{4}x^4 - \frac{3}{4}x^{4/3}\right]_{-1}^{0} + \left[\frac{3}{4}x^{4/3} - \frac{1}{4}x^4\right]_0^1$$
$$= \left((0-0) - \left(\frac{1}{4} - \frac{3}{4}\right)\right) + \left(\left(\frac{3}{4} - \frac{1}{4}\right) - (0-0)\right)$$
$$= \boxed{1}.$$

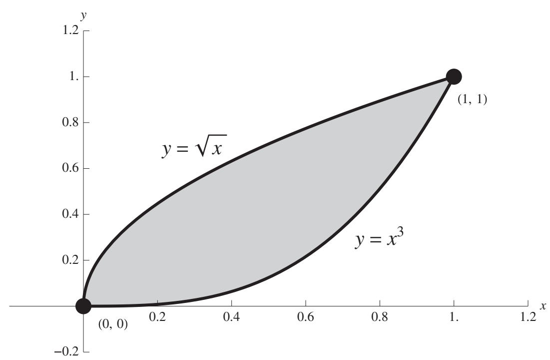
23. If we try to partition along the $x$-axis, we will require two integrals, one for $-5 \le x \le 3$ and the other for $3 \le x \le 4$, since the upper boundof the area changes equations at $x = 3$. So we partition along the $y$-axis. The equation of the parabola is $x = -y^2 + 4$; solving the linear equation for $x$ gives $x = 2y + 1$. Since $-y^2 + 4 \ge 2y + 1$ throughout the region of integration, the area is

$$\int_{-3}^{1} (-y^2 + 4 - (2y+1))\, dy = \int_{-3}^{1} (-y^2 - 2y + 3)\, dy = \left[-\frac{1}{3}y^3 - y^2 + 3y\right]_{-3}^{1}$$
$$= \left(-\frac{1}{3} - 1 + 3\right) - (9 - 9 - 9) = \boxed{\frac{32}{3}}.$$

24. Integrate by subdividing along the $x$-axis. The upper bound is $-x+3$, while the lower bound is $\frac{4}{x+2}$. So the area is

$$\int_{-1}^{2}\left(-x+3-\frac{4}{x+2}\right)dx = \left[-\frac{1}{2}x^2+3x-4\ln|x+2|\right]_{-1}^{2}$$
$$= (-2+6-4\ln 4)-\left(-\frac{1}{2}-3-4\ln 1\right) = \boxed{\frac{15}{2}-4\ln 4}.$$

25. The region is shown below:




The two curves intersect when $\sqrt{x}=x^3$, which is at the points $(0,0)$ and $(1,1)$. So:

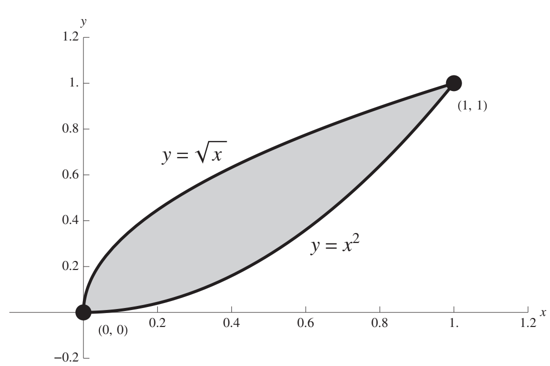
**(a)** Partitioning the $x$-axis, the area is

$$\int_0^1 (\sqrt{x}-x^3)\,dx = \left[\frac{2}{3}x^{3/2}-\frac{1}{4}x^4\right]_0^1 = \left(\frac{2}{3}-\frac{1}{4}\right)-(0-0) = \boxed{\frac{5}{12}}.$$

**(b)** Partitioning the $y$-axis, we must first solve the two equations for $x$. This gives $x=y^2$ and $x=\sqrt[3]{y}$. Then the area between the curves is

$$\int_0^1 (\sqrt[3]{y}-y^2)\,dy = \left[\frac{3}{4}y^{4/3}-\frac{1}{3}y^3\right]_0^1 = \left(\frac{3}{4}-\frac{1}{3}\right)-(0-0) = \boxed{\frac{5}{12}}.$$

26. The region is shown below:




The two curves intersect when $\sqrt{x}=x^2$, which is at the points $(0,0)$ and $(1,1)$. So:

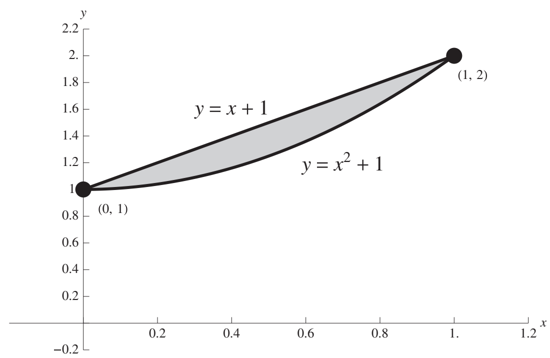
**(a)** Partitioning the $x$-axis, the area is

$$\int_0^1 (\sqrt{x}-x^2)\,dx = \left[\frac{2}{3}x^{3/2}-\frac{1}{3}x^3\right]_0^1 = \left(\frac{2}{3}-\frac{1}{3}\right)-(0-0) = \boxed{\frac{1}{3}}.$$

**(b)** Partitioning the $y$-axis, we must first solve the two equations for $x$. This gives $x = y^2$ and $x = \sqrt{y}$. Then the area between the curves is

$$\int_0^1 (\sqrt{y} - y^2)\,dy = \left[\frac{2}{3}y^{3/2} - \frac{1}{3}y^3\right]_0^1 = \left(\frac{2}{3} - \frac{1}{3}\right) - (0-0) = \boxed{\frac{1}{3}}.$$

27. The region is shown below:




The curves $y = x^2 + 1$ and $y = x + 1$ intersect when $x^2 + 1 = x + 1$, so when $x^2 = x$. As a result, the intersection points are $(0,1)$ and $(1,2)$. So:

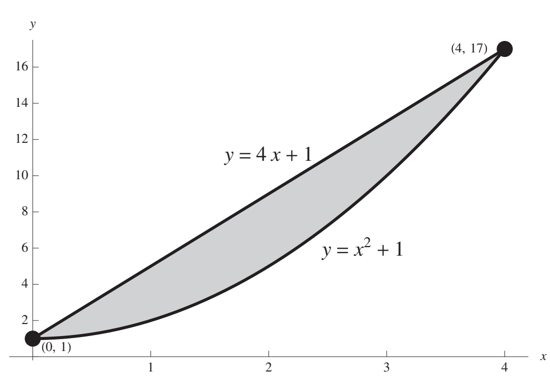
**(a)** Partitioning the $x$-axis, the area is

$$\int_0^1 ((x+1)-(x^2+1))\,dx = \int_0^1 (x-x^2)\,dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \left(\frac{1}{2} - \frac{1}{3}\right) - (0-0) = \boxed{\frac{1}{6}}.$$

**(b)** Partitioning the $y$-axis, we must first solve the two equations for $x$. This gives $x = \sqrt{y-1}$ and $x = y - 1$. Then the area between the curves is

$$\int_1^2 (\sqrt{y-1} - (y-1))\,dy = \int_1^2 (1 - y + \sqrt{y-1})\,dy = \left[y - \frac{1}{2}y^2 + \frac{2}{3}(y-1)^{3/2}\right]_1^2$$
$$= \left(2 - 2 + \frac{2}{3}\right) - \left(1 - \frac{1}{2} + 0\right) = \boxed{\frac{1}{6}}.$$

28. The region is shown below:




The curves $y = x^2 + 1$ and $y = 4x + 1$ intersect when $x^2 + 1 = 4x + 1$, so when $x^2 = 4x$, or at the points $(0,1)$ and $(4,17)$. So:

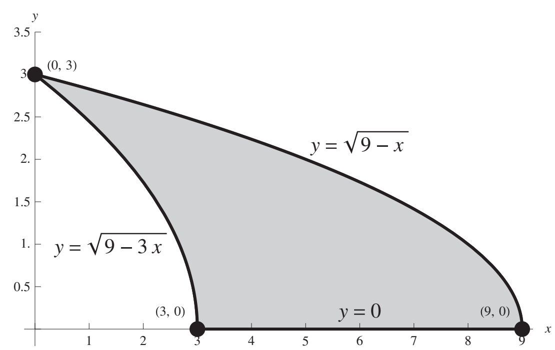
**(a)** Partitioning the $x$-axis, the area is

$$\int_0^4 ((4x+1)-(x^2+1))\,dx = \int_0^4 (4x-x^2)\,dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \left(32 - \frac{64}{3}\right) - (0-0) = \boxed{\frac{32}{3}}.$$

**(b)** Partitioning the $y$-axis, we must first solve the two equations for $x$. This gives $x = \sqrt{y-1}$ and $x = \frac{y-1}{4}$. Then the area between the curves is

$$\int_1^{17} \left(\sqrt{y-1} - \frac{y-1}{4}\right) dy = \int_1^{17} \left(\sqrt{y-1} - \frac{1}{4}y + \frac{1}{4}\right) dy$$
$$= \left[\frac{2}{3}(y-1)^{3/2} - \frac{1}{8}y^2 + \frac{1}{4}y\right]_1^{17} = \left(\frac{128}{3} - \frac{289}{8} + \frac{17}{4}\right) - \left(0 - \frac{1}{8} + \frac{1}{4}\right) = \boxed{\frac{32}{3}}.$$

29. The region is shown below:




These curves intersect where $9 - x = 9 - 3x$, so at $(0, 3)$. So:

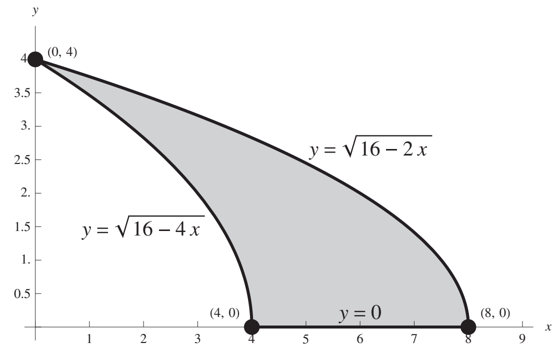
**(a)** From $x = 0$ to $x = 3$, the upper bound is $\sqrt{9-x}$ and the lower bound is $\sqrt{9-3x}$, but from $x = 3$ to $x = 9$ the lower bound is 0. Therefore the area is

$$\int_0^3 (\sqrt{9-x} - \sqrt{9-3x})\,dx + \int_3^9 \sqrt{9-x}\,dx$$
$$= \left[-\frac{2}{3}(9-x)^{3/2} + \frac{2}{9}(9-3x)^{3/2}\right]_0^3 + \left[-\frac{2}{3}(9-x)^{3/2}\right]_3^9$$
$$= \left((-4\sqrt{6}+0) - (-18+6)\right) + \left(0 - (-4\sqrt{6})\right) = \boxed{12}.$$

**(b)** Solving both equations for $x$ gives $x = 9 - y^2$ and $x = 3 - \frac{1}{3}y^2$, and the first curve forms the upper bound for integration when subdividing along the $y$-axis. Then the area is

$$\int_0^3 \left((9-y^2) - \left(3 - \frac{1}{3}y^2\right)\right) dy = \int_0^3 \left(6 - \frac{2}{3}y^2\right) dy$$
$$= \left[6y - \frac{2}{9}y^3\right]_0^3$$
$$= (18-6) - (0-0) = \boxed{12}.$$

30. The region is shown below:




These curves intersect where $16-2x=16-4x$, so at $(0,4)$.

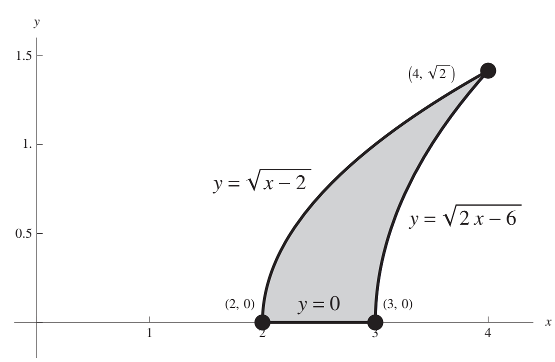
**(a)** From $x=0$ to $x=4$, the upper bound is $\sqrt{16-2x}$ and the lower bound is $\sqrt{16-4x}$, but from $x=4$ to $x=8$ the lower bound is 0. Therefore the area is

$$\begin{aligned}\int_0^4 \left(\sqrt{16-2x}-\sqrt{16-4x}\right)dx+\int_4^8 \sqrt{16-2x}\,dx &= \left[-\frac{1}{3}(16-2x)^{3/2}+\frac{1}{6}(16-4x)^{3/2}\right]_0^4+\left[-\frac{1}{3}(16-2x)^{3/2}\right]_4^8\\ &= \left(\left(-\frac{16}{3}\sqrt{2}+0\right)-\left(-\frac{64}{3}+\frac{32}{3}\right)\right)+0-\left(-\frac{16}{3}\sqrt{2}\right)\\ &= \boxed{\frac{32}{3}}.\end{aligned}$$

**(b)** Solving both equations for $x$ gives $x=8-\frac{1}{2}y^2$ and $x=4-\frac{1}{4}y^2$, and the first curve forms the upper bound for integration when subdividing along the $y$-axis. Then the area is

$$\begin{aligned}\int_0^4 \left(\left(8-\frac{1}{2}y^2\right)-\left(4-\frac{1}{4}y^2\right)\right)dy &= \int_0^4\left(4-\frac{1}{4}y^2\right)dy\\ &= \left[4y-\frac{1}{12}y^3\right]_0^4\\ &= 16-\frac{16}{3}+(0-0)=\boxed{\frac{32}{3}}.\end{aligned}$$

31. The region is shown below:




These curves intersect when $\sqrt{x-2}=\sqrt{2x-6}$, so at $x=4$, which is the point $(4,\sqrt{2})$. Therefore:

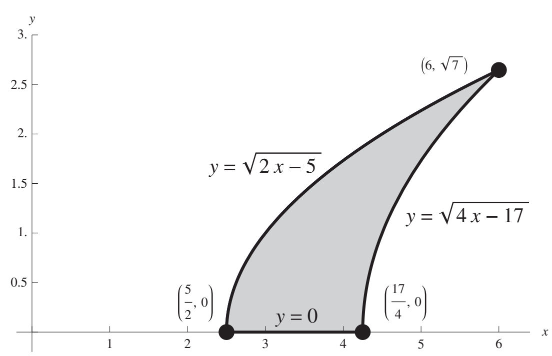
**(a)** From $x = 2$ to $x = 3$, the lower bound is zero and the upper bound is $\sqrt{x-2}$; from $x = 3$ to $x = 4$ the lower bound is $\sqrt{2x-6}$. Therefore the area is

$$\begin{aligned}\int_2^3 \sqrt{x-2}\,dx + \int_3^4 \left(\sqrt{x-2} - \sqrt{2x-6}\right)\,dy &= \left[\frac{2}{3}(x-2)^{3/2}\right]_2^3 + \left[\frac{2}{3}(x-2)^{3/2} - \frac{1}{3}(2x-6)^{3/2}\right]_3^4 \\ &= \left(\frac{2}{3} - 0\right) + \left(\left(\frac{4}{3}\sqrt{2} - \frac{2}{3}\sqrt{2}\right) - \left(\frac{2}{3} - 0\right)\right) \\ &= \boxed{\frac{2}{3}\sqrt{2}}.\end{aligned}$$

**(b)** Solving both equations for $x$ gives $x = 3 + \frac{1}{2}y^2$ and $x = 2 + y^2$; the first curve is the upper bound for integration when subdividing along the $y$-axis. Then the area is

$$\begin{aligned}\int_0^{\sqrt{2}} \left(\left(3 + \frac{1}{2}y^2\right) - (2 + y^2)\right)\,dy &= \int_0^{\sqrt{2}} \left(1 - \frac{1}{2}y^2\right)\,dy \\ &= \left[y - \frac{1}{6}y^3\right]_0^{\sqrt{2}} \\ &= \left(\sqrt{2} - \frac{\sqrt{2}}{3}\right) - (0 - 0) \\ &= \boxed{\frac{2}{3}\sqrt{2}}.\end{aligned}$$

32. The region is shown below:




These curves intersect when $2x - 5 = 4x - 17$, so when $x = 6$, at the point $(6, \sqrt{7})$. Therefore:

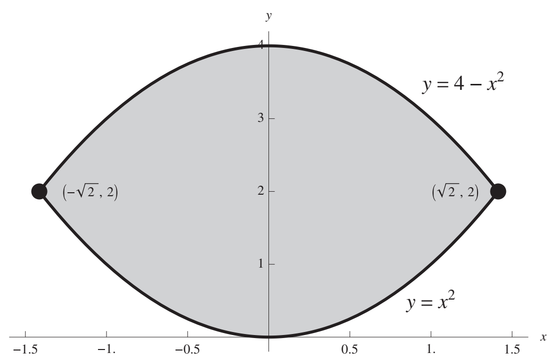
**(a)** From $x = \frac{5}{2}$ to $x = \frac{17}{4}$, the upper bound is $\sqrt{2x-5}$ and the lower bound is 0. From $x = \frac{17}{4}$ to $x = 6$ the lower bound is $\sqrt{4x-17}$. So the area is

$$\begin{aligned}&\int_{5/2}^{17/4} \sqrt{2x-5}\,dx + \int_{17/4}^{6} \left(\sqrt{2x-5} - \sqrt{4x-17}\right)\,dx \\ &\quad= \left[\frac{1}{3}(2x-5)^{3/2}\right]_{5/2}^{17/4} + \left[\frac{1}{3}(2x-5)^{3/2} - \frac{1}{6}(4x-17)^{3/2}\right]_{17/4}^{6} \\ &\quad= \frac{7}{12}\sqrt{14} + \left(\frac{7}{6}\sqrt{7} - \frac{7}{12}\sqrt{14}\right) \\ &\quad= \boxed{\frac{7}{6}\sqrt{7}}.\end{aligned}$$

**(b)** Solving both equations for $x$ gives $x = \frac{y^2+5}{2}$ and $x = \frac{y^2+17}{4}$; the second equation forms the upper bound for the integrand. So the area is

$$\int_0^{\sqrt{7}} \left(\frac{y^2+17}{4} - \frac{y^2+5}{2}\right) dy = \int_0^{\sqrt{7}} \left(\frac{7}{4} - \frac{y^2}{4}\right) dy$$
$$= \left[\frac{7}{4}y - \frac{y^3}{12}\right]_0^{\sqrt{7}}$$
$$= \left(\frac{7}{4}\sqrt{7} - \frac{7}{12}\sqrt{7}\right) - (0-0)$$
$$= \boxed{\frac{7}{6}\sqrt{7}}.$$

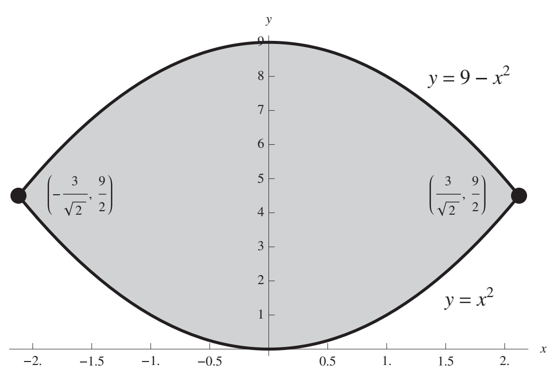
33. The region is shown below:




The two graphs intersect when $4 - x^2 = x^2$, so when $x = \pm\sqrt{2}$. Therefore the area between the curves is

$$\int_{-\sqrt{2}}^{\sqrt{2}} (4 - x^2 - x^2)\, dx = \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 2x^2)\, dx = \left[4x - \frac{2}{3}x^3\right]_{-\sqrt{2}}^{\sqrt{2}} = \boxed{\frac{16}{3}\sqrt{2}}.$$

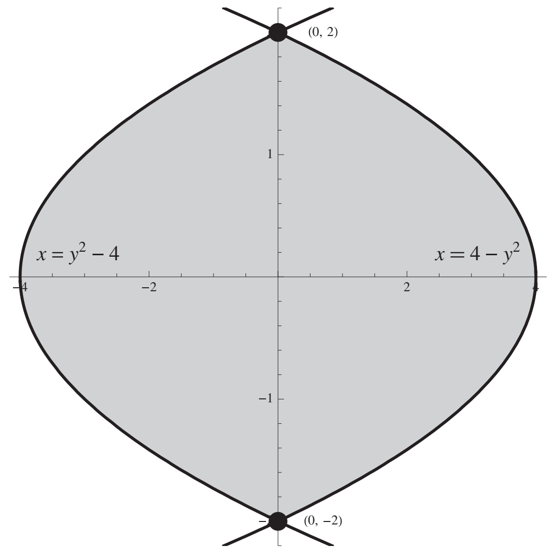
34. The region is shown below:




The graphs intersect when $9 - x^2 = x^2$, so when $x = \pm\frac{3\sqrt{2}}{2}$. Therefore the area between the curves is

$$\int_{-3\sqrt{2}/2}^{3\sqrt{2}/2} (9 - x^2 - x^2)\, dx = \int_{-3\sqrt{2}/2}^{3\sqrt{2}/2} (9 - 2x^2)\, dx = \left[9x - \frac{2}{3}x^3\right]_{-3\sqrt{2}/2}^{3\sqrt{2}/2} = \boxed{18\sqrt{2}}.$$

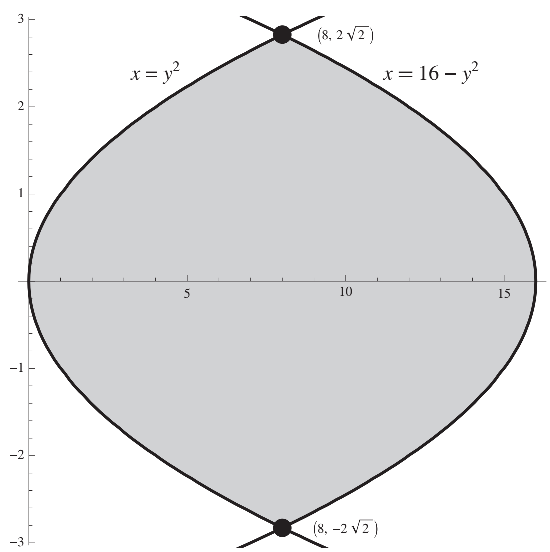
35. The region is shown below:




For these curves, we partition along the $y$-axis. The graphs intersect when $y^2 - 4 = 4 - y^2$, which is when $y = -2$ and $y = 2$. So the area between the curves is

$$\int_{-2}^{2} ((4 - y^2) - (y^2 - 4))\, dy = \int_{-2}^{2} (8 - 2y^2)\, dy = \left[8y - \frac{2}{3}y^3\right]_{-2}^{2} = \boxed{\frac{64}{3}}.$$

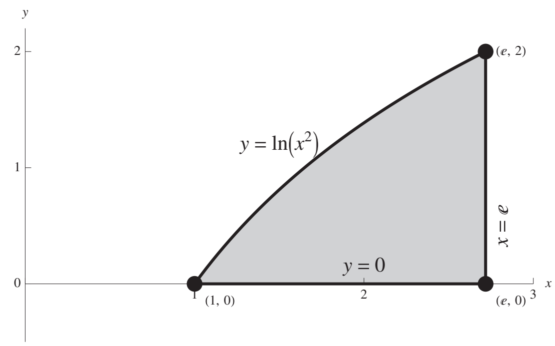
36. The region is shown below:




For these curves, we partition along the $y$-axis. The graphs intersect when $y^2 = 16 - y^2$, so for $y = \pm\sqrt{8} = \pm 2\sqrt{2}$. Therefore the area between the curves is

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} (16 - y^2 - y^2)\, dy = \int_{-2\sqrt{2}}^{2\sqrt{2}} (16 - 2y^2)\, dy = \left[16y - \frac{2}{3}y^3\right]_{-2\sqrt{2}}^{2\sqrt{2}} = \boxed{\frac{128}{3}\sqrt{2}}.$$

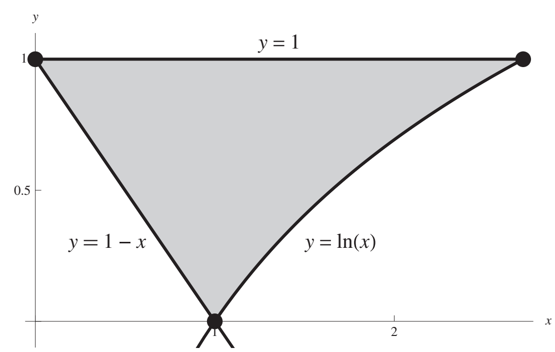
37. The region is shown below:




Note that $\ln x^2 = 2\ln x$. This area can be computed more easily by subdividing along the $y$-axis. Solving $y = 2\ln x$ for $x$ gives $x = e^{y/2}$. The line $x = e$ meets this curve at $y = 2$, so the region extends from $y = 0$ to $y = 2$ and is bounded on the left by $e^{y/2}$ and on the right by $e$. Therefore the area of the region is

$$\int_0^2 \left(e - e^{y/2}\right)\,dy = \left[ey - 2e^{y/2}\right]_0^2 = (2e - 2e) - (0 - 2) = \boxed{2}.$$

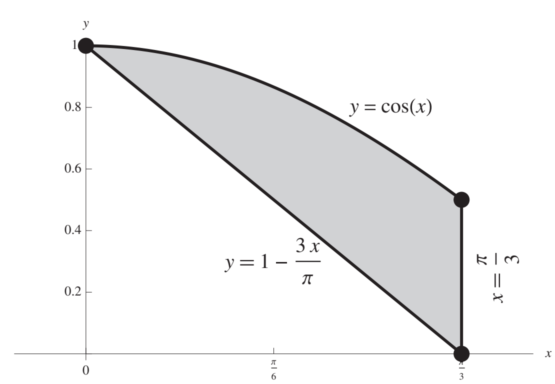
38. The region is shown below:




We partition along the $y$-axis so that we do not have to split the computation into two integrals. $y = 1 - x$ becomes $x = 1 - y$, and $y = \ln x$ becomes $x = e^y$. The region extends from $y = 0$ to $y = 1$, so the area of the region is

$$\int_0^1 (e^y - (1 - y))\,dy = \int_0^1 \left(y - 1 + e^y\right)\,dy = \left[\frac{1}{2}y^2 - y + e^y\right]_0^1 = \boxed{e - \frac{3}{2}}.$$

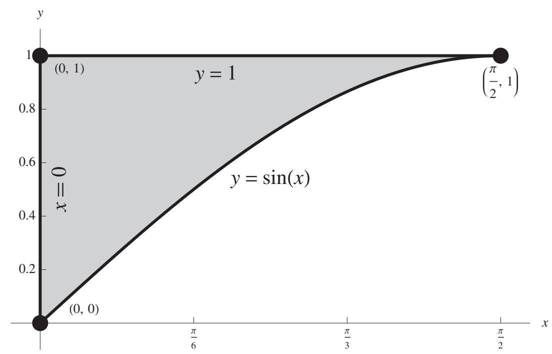
39. The region is shown below:



The area of the region is

$$\int_0^{\pi/3}\left(\cos x-\left(1-\frac{3}{\pi}x\right)\right)\,dx=\int_0^{\pi/3}\left(\frac{3}{\pi}x-1+\cos x\right)\,dx=\left[\frac{3}{2\pi}x^2-x+\sin x\right]_0^{\pi/3}=\boxed{\frac{\sqrt{3}}{2}-\frac{\pi}{6}}.$$

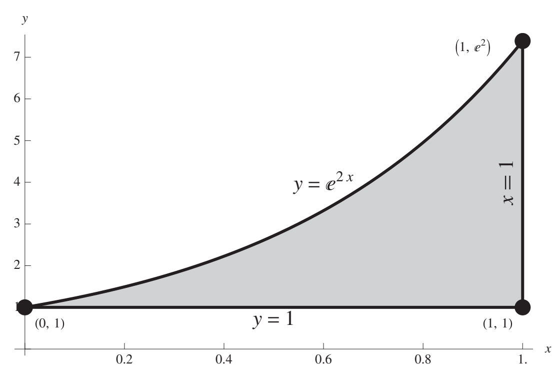
40. The region is shown below:




The region extends from $x=0$ to $x=\frac{\pi}{2}$, so the area is

$$\int_0^{\pi/2}(1-\sin x)\,dx=[x+\cos x]_0^{\pi/2}=\boxed{\frac{\pi}{2}-1}.$$

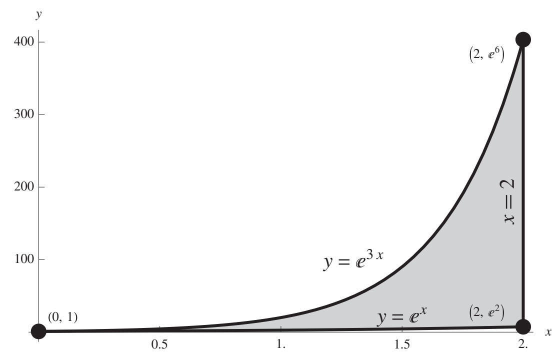
41. The region is shown below:




The area of the region is

$$\int_0^1(e^{2x}-1)\,dx=\left[\frac{1}{2}e^{2x}-x\right]_0^1=\boxed{\frac{1}{2}e^2-\frac{3}{2}}.$$

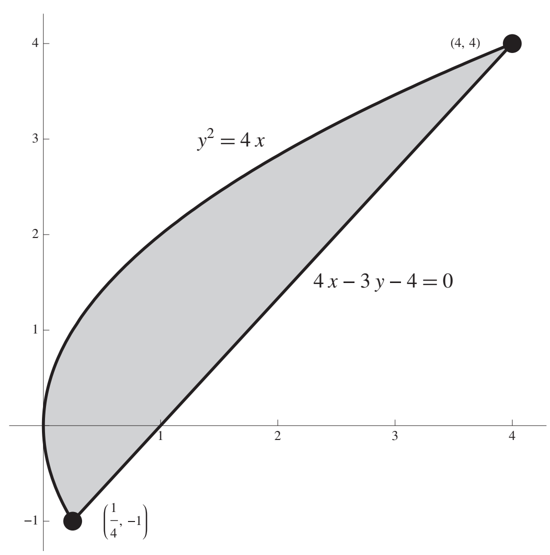
42. The region is shown below:




The area of the region is

$$\int_0^2 (e^{3x} - e^x)\,dx = \left[\frac{1}{3}e^{3x} - e^x\right]_0^2 = \boxed{\frac{1}{3}e^6 - e^2 + \frac{2}{3}}.$$

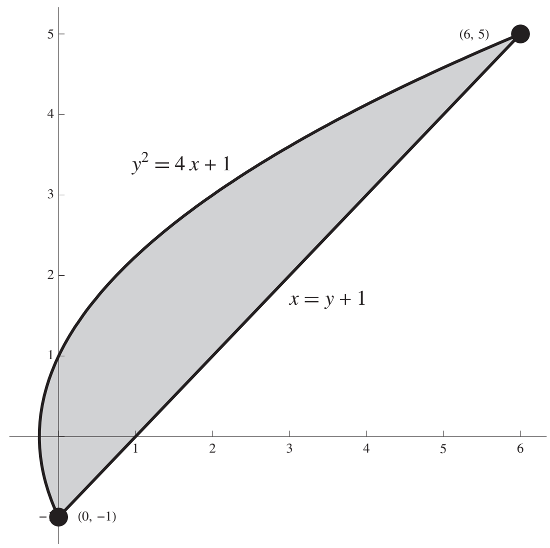
43. The region is shown below:




We partition along the $y$-axis. Solving these equations for $x$ gives $x = \frac{y^2}{4}$ and $x = 1 + \frac{3}{4}y$. The curves intersect where $y^2 - 3y - 4 = 0$ (since $y^2 = 4x$); this factors as $(y-4)(y+1)$, so the $y$-coordinates of the intersection points are $-1$ and 4. Therefore the area of the region is

$$\int_{-1}^4 \left(1 + \frac{3}{4}y - \frac{y^2}{4}\right) dy = \left[y + \frac{3}{8}y^2 - \frac{1}{12}y^3\right]_{-1}^4 = \boxed{\frac{125}{24}}.$$

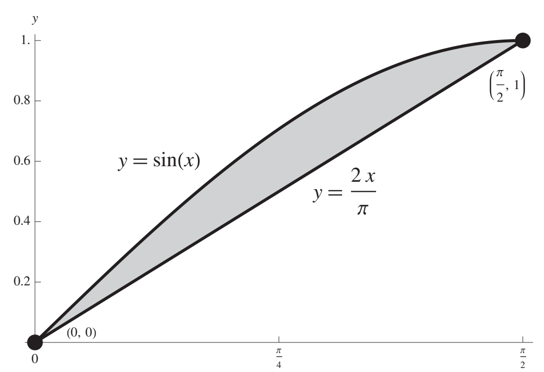
44. The region is shown below:




Partition along the $y$-axis. The two curves have equations $x = \frac{1}{4}(y^2 - 1)$ and $x = y + 1$; these intersect where $y + 1 = \frac{1}{4}(y^2 - 1)$, so when $y^2 - 4y - 5 = 0$. This happens for $y = 5$ and $y = -1$. Therefore the area of the region is

$$\int_{-1}^{5} \left( y + 1 - \frac{1}{4}(y^2 - 1) \right) dy = \int_{-1}^{5} \left( -\frac{1}{4}y^2 + y + \frac{5}{4} \right) dy = \left[ -\frac{1}{12}y^3 + \frac{1}{2}y^2 + \frac{5}{4}y \right]_{-1}^{5} = \boxed{9}.$$

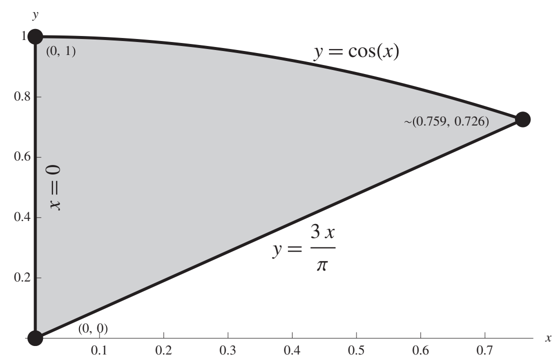
45. The region is shown below:




The graphs intersect at $x = 0$ and again at $x = \frac{\pi}{2}$, where $\sin \frac{\pi}{2} = 1$ and $\frac{2}{\pi}x = \frac{2}{\pi} \cdot \frac{\pi}{2} = 1$. So the area of the region is

$$\int_{0}^{\pi/2} \left( \sin x - \frac{2}{\pi}x \right) dx = \left[ -\cos x - \frac{1}{\pi}x^2 \right]_{0}^{\pi/2} = \boxed{-\frac{\pi}{4} + 1}.$$

46. The region is shown below:




The intersection point of these graphs has no simple exact form, but is approximately equal to 0.759. So the area of the region is approximately

$$\int_0^{0.759} \left(\cos x - \frac{3}{\pi}x\right)\,dx = \left[\sin x - \frac{3}{2\pi}x^2\right]_0^{0.759} \boxed{\approx 0.413}.$$

**Applications and Extensions**

47. The slope of $BC$ is $\frac{4-1}{-2-1} = -1$, so the point of tangency of $AD$ with the parabola is a point where the slope of the tangent to the parabola is $-1$. With $y = x^2$, we have $y' = 2x$, so the slope is $-1$ at $x = -\frac{1}{2}$. Therefore the point of tangency is $\left(-\frac{1}{2}, \frac{1}{4}\right)$, so the line $AD$ has equation

$$y - \frac{1}{4} = -1\left(x - \left(-\frac{1}{2}\right)\right), \text{ or } y = -x - \frac{1}{4}.$$

Line $BC$ has equation

$$y - 1 = -1(x - 1), \text{ or } y = -x + 2.$$

So the area of the parallelogram is

$$\int_{-2}^{1} \left((-x+2) - \left(-x - \frac{1}{4}\right)\right)\,dx = \int_{-2}^{1} \left(\frac{9}{4}\right)\,dx = \frac{27}{4}.$$

The shaded area, on the other hand, has area

$$\int_{-2}^{1} \left((-x+2) - x^2\right)\,dx = \int_{-2}^{1} (-x^2 - x + 2)\,dx = \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^{1} = \frac{9}{2}.$$

Finally, $\frac{27}{4} \cdot \frac{2}{3} = \frac{9}{2}$, and Archimedes' result follows.

48. Subdividing along the $x$-axis, we get for the area of the region

$$\int_0^{1/2} (8x - x)\,dx + \int_{1/2}^{1} \left(\frac{1}{x^2} - x\right)\,dx = \left[\frac{7}{2}x^2\right]_0^{1/2} + \left[-\frac{1}{x} - \frac{1}{2}x^2\right]_{1/2}^{1} = \frac{7}{8} + \frac{5}{8} = \frac{3}{2}.$$

A triangle with base 1 and height $h$ has area $\frac{1}{2} \cdot 1 \cdot h = \frac{h}{2}$; since we want $\frac{h}{2} = \frac{3}{2}$, we have $\boxed{h = 3}$.

49. **(a)** The points of intersection occur where the values of the two functions are equal:

$$\begin{aligned}\cos^2 x &= \sin^2 x\\ 1-\sin^2 x &= \sin^2 x\\ 2\sin^2 x &= 1\\ \sin^2 x &= \frac{1}{2}\\ \sin x &= \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}\\ x &= \pm\frac{\pi}{4}+2n\pi \text{ or } \pm\frac{3\pi}{4}+2n\pi, n \text{ any integer}\end{aligned}$$

For the region shown, this is $x=-\frac{\pi}{4}$ and $x=\frac{\pi}{4}$.

The points of intersection are $\left(-\frac{\pi}{4},\, \sin^2\left(-\frac{\pi}{4}\right)\right) = \left(-\frac{\pi}{4}, \left(-\frac{1}{\sqrt{2}}\right)^2\right) = \boxed{\left(-\frac{\pi}{4}, \frac{1}{2}\right)}$ and $\left(\frac{\pi}{4},\, \sin^2\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{4}, \left(-\frac{1}{\sqrt{2}}\right)^2\right) = \boxed{\left(\frac{\pi}{4}, \frac{1}{2}\right)}$

**(b)** The lower limit of integration is $x=0$, the $y$-axis, and the upper limit is $x=\frac{\pi}{4}$, from Part (a).
The area is the integral of the difference between the two functions over this interval:

$$\begin{aligned}A &= \int_0^{\pi/4}\left(\cos^2 x-\sin^2 x\right)\, dx\\ &= \int_0^{\pi/4}\cos 2x\, dx\end{aligned}$$

Let

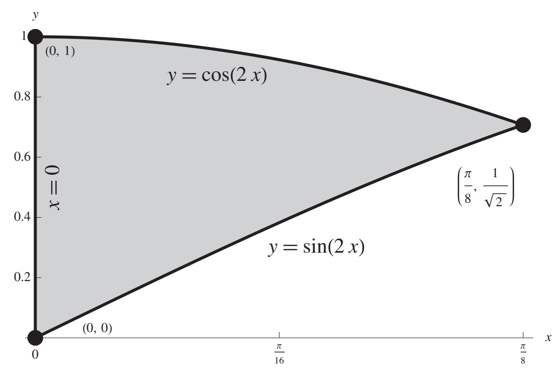
$$u = 2x$$

Then

$$\begin{aligned}du &= 2\, dx\\ dx &= \frac{1}{2}\, du\\ x = 0 &\to u = 2(0) = 0\\ x = \frac{\pi}{4} &\to u = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_0^{\pi/4}\cos 2x\, dx &= \frac{1}{2}\int_0^{\pi/2}\cos u\, du\\ &= \frac{1}{2}\left[\sin u\right]_0^{\pi/2}\\ &= \frac{1}{2}\left[\sin\frac{\pi}{2}-(-\sin 0)\right]\\ &= \frac{1}{2}(1-0)\\ &= \boxed{\frac{1}{2}}\end{aligned}$$

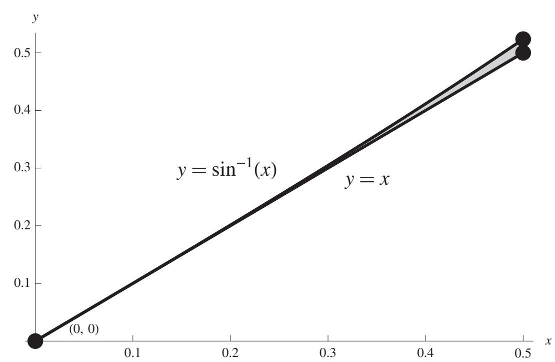
50. The region is shown below:




The region extends from $x = 0$ to $x = \frac{\pi}{8}$, so the area is

$$\int_0^{\pi/8} (\cos 2x - \sin 2x)\, dx = \left[\frac{1}{2}\sin 2x + \frac{1}{2}\cos 2x\right]_0^{\pi/8} = \boxed{\frac{\sqrt{2}}{2} - \frac{1}{2}}.$$

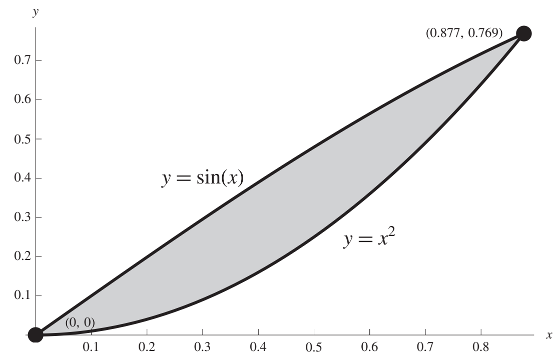
51. The region is shown below:




Since it is easier to integrate $\sin y$ than it is to integrate $\sin^{-1} x$, we choose to partition along the $y$-axis. The two equations are then $x = \sin y$ and $x = y$. The graph of $x = \sin y$ intersects the line $x = \frac{1}{2}$ at $y = \frac{\pi}{6}$, while the graph of $x = y$ intersects that line at $y = \frac{1}{2}$. So for $0 \le y \le \frac{1}{2}$, the region is bounded to the right by $x = y$ and to the left by $x = \sin y$, while for $\frac{1}{2} \le y \le \frac{\pi}{6}$, it is bounded to the right by $x = \frac{1}{2}$ and to the left by $x = \sin y$. As a result, the total area is

$$\begin{aligned}\int_0^{1/2} (y - \sin y)\, dy + \int_{1/2}^{\pi/6} \left(\frac{1}{2} - \sin y\right) dy &= \left[\frac{1}{2}y^2 + \cos y\right]_0^{1/2} + \left[\frac{1}{2}y + \cos y\right]_{1/2}^{\pi/6} \\ &= \left(-\frac{7}{8} + \cos\frac{1}{2}\right) + \left(\frac{\pi}{12} + \frac{\sqrt{3}}{2} - \frac{1}{4} - \cos\frac{1}{2}\right) \\ &= \boxed{-\frac{9}{8} + \frac{\sqrt{3}}{2} + \frac{\pi}{12}}.\end{aligned}$$

52. **(a)** The two curves are shown below:




**(b)** The curves intersect when $x^2 = \sin x$. Now, $x = 0$ is such a value of $x$; using technology we find the other value to be $x \approx 0.877$. The corresponding $y$ values are $y = 0$ and $y = 0.877^2 \approx 0.769$, so the intersection points are $\boxed{(0,0) \text{ and } \approx (0.877, 0.769)}$.

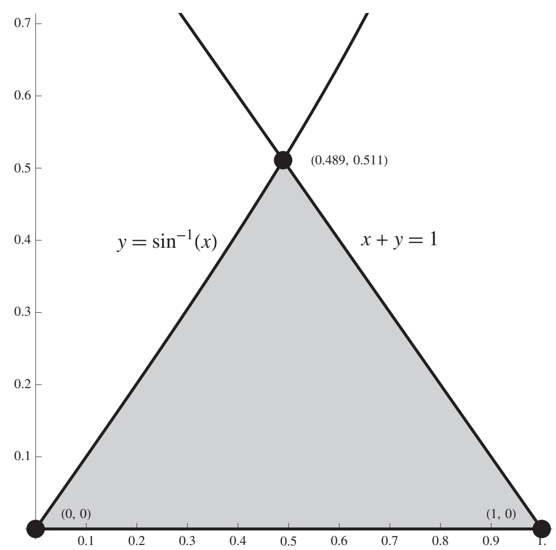
**(c)** The area is

$$\int_0^{0.877} (\sin x - x^2)\, dx = \left[-\cos x - \frac{1}{3}x^3\right]_0^{0.877} \boxed{\approx 0.136}.$$

**(d)** Solving the two equations for $x$ gives $x = \sqrt{y}$ and $x = \sin^{-1} y$. So the area between the curves is

$$\int_0^{0.769} (\sqrt{y} - \sin^{-1} y)\, dy \boxed{\approx 0.136}.$$

53. **(a)** The three curves are shown below:

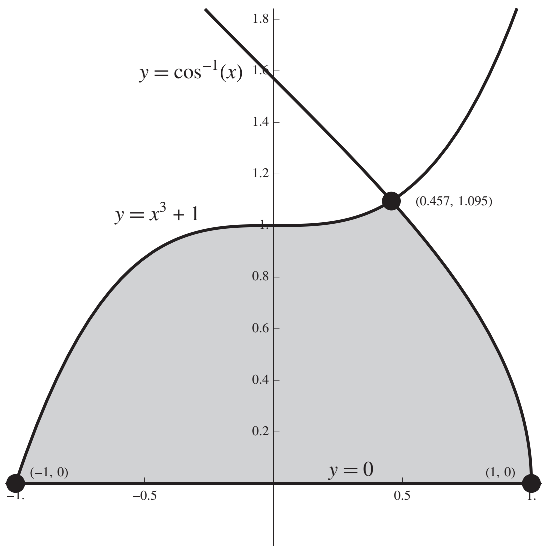



**(b)** The graphs of $\sin^{-1} x$ and $y = 0$ intersect at the point $\boxed{(0,0)}$, and $x + y = 1$ and $y = 0$ intersect at the point $\boxed{(1,0)}$. The third point of intersection is where $y = \sin^{-1} x = 1 - x$; solving $\sin^{-1} x = 1 - x$ using technology gives $x \approx 0.489$, so that $y = 1 - 0.489 = 0.511$, and the intersection point is $\boxed{\approx (0.489, 0.511)}$.

**(c)** Partitioning along the $y$-axis allows us to use just one integral, bounded above by $x = 1 - y$ and below by $x = \sin y$, so its area is

$$\int_0^{0.511} (1 - y - \sin y)\, dy = \left[ y - \frac{1}{2}y^2 + \cos y \right]_0^{0.511} \boxed{\approx 0.253}.$$

54. **(a)** The three curves are shown below:

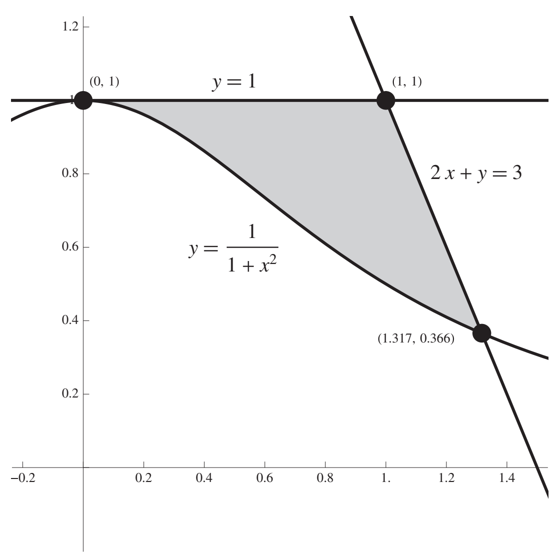



**(b)** The graphs of $\cos^{-1} x$ and $y = 0$ intersect on the $x$-axis, at $\boxed{(1, 0)}$, and $y = x^3 + 1$ and $y = 0$ intersect on the $x$-axis, at $\boxed{(-1, 0)}$. The third point of intersection is determined by solving $\cos^{-1} x = x^3 + 1$; using technology we get $x \approx 0.457$, so that $y \approx 0.457^3 + 1 \approx 1.095$, and the third point of intersection is $\boxed{\approx (0.457, 1.095)}$.

**(c)** Partitioning along the $y$-axis allows us to use just one integral, bounded above by $x = \cos y$ and below by $x = \sqrt[3]{y - 1}$, so its area is

$$\int_0^{1.095} \left( \cos y - \sqrt[3]{y - 1} \right) dy = \left[ \sin y - \frac{3}{4}(y - 1)^{4/3} \right]_0^{1.095} \boxed{\approx 1.606}.$$

55. **(a)** The three curves are shown below:

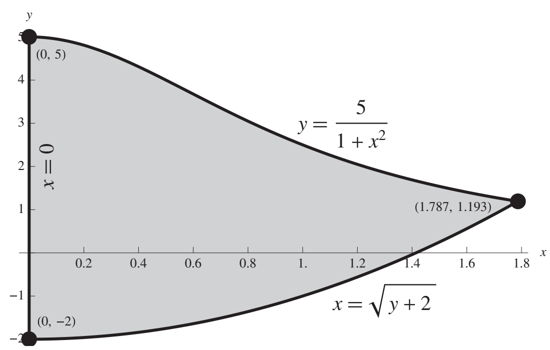


**(b)** $2x + y = 3$ and $y = 1$ intersect at $x = 1$, which is the point $\boxed{(1,1)}$. $y = 1$ intersects $y = \frac{1}{x^2+1}$ when $\frac{1}{x^2+1} = 1$, so when $x = 0$; this point is $\boxed{(0,1)}$. Finally, $2x + y = 3$ intersects $y = \frac{1}{x^2+1}$ when $3 - 2x = \frac{1}{x^2+1}$. Clearing fractions and simplifying gives $2x^3 - 3x^2 + 2x - 2 = 0$. The only real root of this cubic is, using technology, $x \approx 1.317$. At that value, $y = 3 - 2 \cdot 1.317 \approx 0.366$, so the third point of intersection is $\boxed{\approx (1.317, 0.366)}$.

**(c)** Partitioning along the $y$-axis allows us to use just one integral, bounded above by $x = \frac{1}{2}(3 - y)$ and below by $x = \sqrt{\frac{1}{y} - 1}$, so its area is

$$\int_{0.366}^{1} \left(\frac{1}{2}(3-y) - \sqrt{\frac{1}{y} - 1}\right) dy \boxed{\approx 0.295}.$$

56. **(a)** The three curves are shown below:

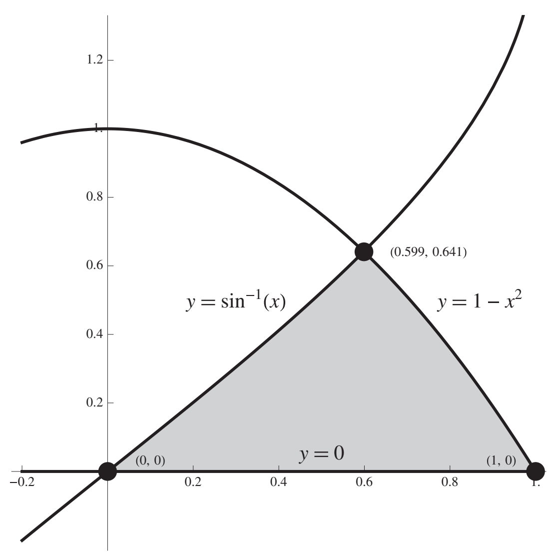



**(b)** $x = 0$ and $y = \frac{5}{1+x^2}$ intersect at $\boxed{(0,5)}$, while $x = 0$ and $x = \sqrt{y+2}$ intersect at $\boxed{(0,-2)}$. To find the third intersection point, we must solve the system $y = \frac{5}{1+x^2}$, $x = \sqrt{y+2}$; using technology, we get $(x, y)\ \boxed{\approx (1.787, 1.193)}$.

**(c)** Partition along the $x$-axis; then the second equation becomes $y = x^2 - 2$ and the area of the region is

$$\int_{0}^{1.787} \left(\frac{5}{1+x^2} - (x^2 - 2)\right) dx \boxed{\approx 6.975}.$$

57. **(a)** The three curves are shown below:



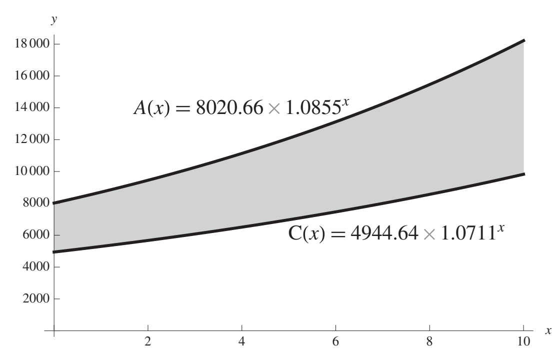
**(b)** $y = 1 - x^2$ and $y = 0$ intersect at $(-1, 0)$ and at $\boxed{(1,0)}$; the point we are concerned with here is $(1, 0)$. The graph shows that $y = \sin^{-1} x$ and $y = 0$ intersect at $\boxed{(0,0)}$. The third point of intersection is found by solving $1 - x^2 = \sin^{-1} x$; using technology we get $x \approx 0.599$, so that $y = 1 - 0.599^2 \approx 0.641$ and the third intersection point is $\boxed{\approx (0.599, 0.641)}$.

**(c)** Partitioning along the $y$-axis allows us to use a single integral, bounded above by $x = \sqrt{1 - y}$ and below by $x = \sin y$; the area of the region is then

$$\int_0^{0.641} \left(\sqrt{1-y} - \sin y\right)\, dy \boxed{\approx 0.325}.$$

## Challenge Problems

58. The two cost curves, with $A(x)$ the upper curve, are shown below:



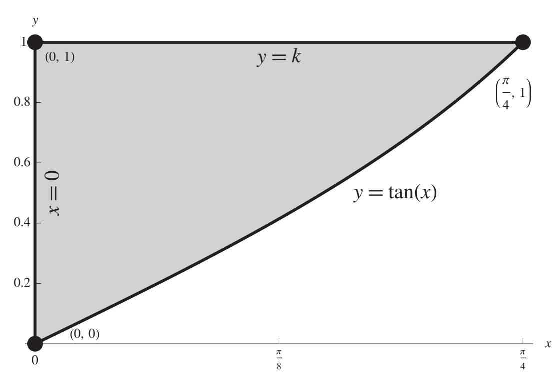

**(a)** The area between the curves is

$$\int_0^{10} (8020.6596 \cdot 1.0855^x - 4944.6424 \cdot 1.0711^x)\, dx$$

$$= \left[\frac{8020.6596}{\ln 1.0855} 1.0855^x - \frac{4944.6424}{\ln 1.0711} 1.0711^x\right]_0^{10} \boxed{\approx \$53{,}213}.$$

**(b)** The total average additional amount of money spent by a family of four in the U.S. for health care exceeded by about \$53,213 the amount spent by a family of four in Canada. Answers may vary.

59. The region (for $k = 1$) is shown below:



**(a)** Partitioning along the $x$-axis, the upper edge of the region is $y = k$ and the lower edge is $y = \tan x$. The bounds of integration range from $x = 0$ to $x = \tan^{-1} k$, so the area is

$$A = \int_0^{\tan^{-1} k} (k - \tan x)\, dx = [kx + \ln \cos x]_0^{\tan^{-1} k} = k \tan^{-1} k + \ln \cos \tan^{-1} k.$$

Now, since $x$ is positive, $\tan^{-1} k$ is in the first quadrant, so $\cos \tan^{-1} k = \frac{1}{\sqrt{1+k^2}}$, as can be seen by drawing a right triangle in the first quadrant whose tangent function is equal to $k$. Therefore we get for the area

$$A = k \tan^{-1} k + \ln \frac{1}{\sqrt{1+k^2}} = \boxed{k \tan^{-1} k - \frac{1}{2} \ln(1+k^2)}.$$

**(b)** When $k = 1$,

$$A = 1 \cdot \tan^{-1} 1 - \frac{1}{2} \ln(1 + 1^2) = \boxed{\frac{\pi}{4} - \frac{1}{2} \ln 2}.$$

**(c)** Using the Chain Rule,

$$\begin{aligned}
\frac{dA}{dt} &= \frac{dA}{dk} \cdot \frac{dk}{dt} \\
&= \left( \tan^{-1} k + \frac{k}{1+k^2} - \frac{1}{2(1+k^2)} \cdot 2k \right) \cdot \frac{dk}{dt} \\
&= \left( \tan^{-1} k + \frac{k}{1+k^2} - \frac{k}{1+k^2} \right) \cdot \frac{dk}{dt} \\
&= \tan^{-1} k \cdot \frac{dk}{dt}.
\end{aligned}$$

When $\frac{dk}{dt} = \frac{1}{10}$ and $k = 1$, we get

$$\frac{dA}{dt} = \tan^{-1} 1 \cdot \frac{1}{10} = \boxed{\frac{\pi}{40} \text{ square units per second.}}$$

60. **(a)** If you reflect the first-quadrant shaded region across the $x$-axis, you get another region that is the same size as the first; the union of these two regions is a region whose area is $A^*$ in the problem statement. The point where the line $y = -x$ meets the hyperbola in the fourth quadrant is the corner of this region; its coordinates are $(\cosh t, -\sinh t)$. So we get a triangle with vertices $(0, 0)$, $(\cosh t, \sinh t)$, and $(\cosh t, -\sinh t)$. This triangle has base (the vertical) $2 \sinh t$ and height $\cosh t$, so that its area is $\frac{1}{2} \cdot 2 \sinh t \cdot \cosh t = \sinh t \cosh t$. Since $\cosh^2 t - \sinh^2 t = 1$, we can rewrite this as $\cosh t \sqrt{\cosh^2 t - 1}$; since $x = \cosh t$, this becomes $x\sqrt{x^2 - 1}$. To compute the area of $A^*$, we must subtract from the area of the triangle the area under the hyperbola between $x = 1$ and $x = \cosh t$. We can compute this area by computing the area under the hyperbola in the first quadrant and doubling it, so (using $u$ as a dummy variable), we get

$$2 \int_1^x \sinh u\, du = 2 \int_1^x \sqrt{\cosh^2 u - 1}\, du = 2 \int_1^x \sqrt{u^2 - 1}\, du.$$

Therefore

$$A^* = x\sqrt{x^2 - 1} - 2 \int_1^x \sqrt{u^2 - 1}\, du.$$

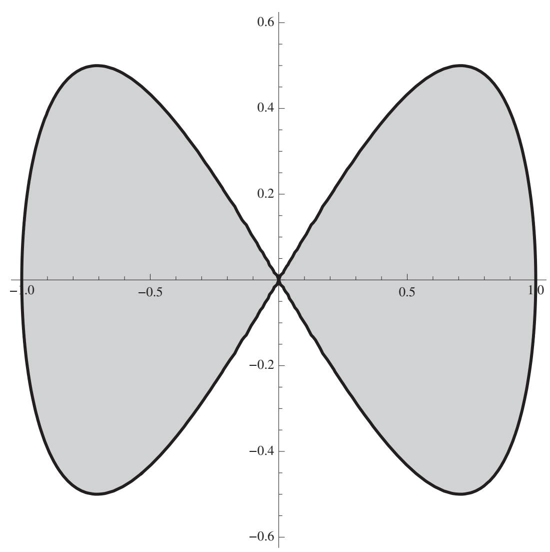
**(b)** Using the Fundamental Theorem of Calculus, we have

$$\begin{aligned}\frac{dA^*}{dx} &= \sqrt{x^2-1} + x \cdot \frac{1}{2}(x^2-1)^{-1/2} \cdot 2x - 2\sqrt{x^2-1} \\ &= -\sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}} \\ &= \frac{1}{\sqrt{x^2-1}}.\end{aligned}$$

**(c)** Since $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$, we see from part (b) that $A^* = \cosh^{-1} x + C$ for some constant $C$. When $x = 1$, the area is empty, so that $A^* = 0$. Therefore we get $0 = \cosh^{-1} 1 + C = 0 + C$, so that $C = 0$. Substituting gives $A^* = \cosh^{-1} x$.

**(d)** Finally, since $x = \cosh t$ and $t \geq 0$, we get $t = \cosh^{-1} x = A^*$.

61. The region is shown below:




Since $(-y)^2 = y^2$ and $(-x)^2 - (-x)^4 = x^2 - x^4$, the graph is symmetric about both the $x$ and $y$-axes. So we can compute its area by computing the portion of the area in the first quadrant and multiplying by 4. In the first quadrant, the equation of the curve is $y = \sqrt{x^2 - x^4} = x\sqrt{1-x^2}$. Note also that if $x > 1$, then $x^2 - x^4 < 0$, so there are no points of the curve to the right of $x = 1$; therefore, the integral to compute the area will go from $x = 0$ to $x = 1$. Using the substitution $u = 1 - x^2$, $du = -2x\,dx$, then $x = 0$ corresponds to $u = 1$ and $x = 1$ to $u = 0$, so the total area is

$$4\int_0^1 x\sqrt{1-x^2}\,dx = -4 \cdot \frac{1}{2}\int_1^0 u^{1/2}\,du = -2\left[\frac{2}{3}u^{3/2}\right]_1^0 = \boxed{\frac{4}{3}}.$$

## AP® Practice Problems

1. Determine the point(s) of intersection of $f(x) = x^2$ and $g(x) = 3x$ as upper and lower limits of integration.

$$
\begin{aligned}
3x &= x^2 \\
x^2 - 3x &= 0 \\
x(x-3) &= 0 \\
x = 0 \quad & x = 3 \\
A &= \int_0^3 \left(3x - x^2\right) dx \\
&= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 \\
&= \frac{27}{2} - \frac{27}{3} = \boxed{\frac{9}{2}}
\end{aligned}
$$

CHOICE C

2. Determine the point of intersection of $y = e^{x/3}$ and $y = 1$ as the lower limit of integration, $x = 3$ being the upper limit of integration.

$$
\begin{aligned}
e^{x/3} &= 1 \\
x &= 0 \\
A &= \int_0^3 \left(e^{x/3} - 1\right) dx \\
&= \left[3e^{x/3} - x\right]_0^3 \\
&= \left(3e^1 - 3\right) - \left(3e^0 - 0\right) \\
&= 3e - 3 - 3 \\
&= \boxed{3e - 6}
\end{aligned}
$$

CHOICE D

3. $$
\begin{aligned}
\int_{-2}^{5} [f(x) + e^x]\, dx &= \int_{-2}^{5} f(x)\, dx + \int_{-2}^{5} e^x\, dx \\
&= 4 - 4 + 4 + [e^x]_{-2}^{5} \\
&= 4 + e^5 - e^{-2} \\
&= \boxed{e^5 - e^{-2} + 4}
\end{aligned}
$$

CHOICE B

4. Determine the point(s) of intersection of $y = x^2 - 4x + 2$ and $y = 7$ as upper and lower limits of integration.

$$\begin{aligned} x^2 - 4x + 2 &= 7 \\ x^2 - 4x - 5 &= 0 \\ (x-5)(x+1) &= 0 \\ x = 5 \quad x &= -1 \end{aligned}$$

$$\begin{aligned} A &= \int_{-1}^{5} \left[7 - \left(x^2 - 4x + 2\right)\right] dx \\ &= \int_{-1}^{5} \left(-x^2 + 4x + 5\right) dx \\ &= \left[\frac{-x^3}{3} + 2x^2 + 5x\right]_{-1}^{5} \\ &= \left[\frac{-125}{3} + 50 + 25 - \left(\frac{1}{3} + 2 - 5\right)\right] \\ &= \boxed{36} \end{aligned}$$

CHOICE B

5. $A = \boxed{\int_b^1 [f(x) - g(x)]\, dx}$

CHOICE B

6. Determine the point of intersection of $y = \sin x$ and $y = \dfrac{x}{\pi} + 1$ as the lower limit of integration, $x = 0$ being the upper limit of integration. Using a graphing calculator suggests that the only intersection is $x = -\pi$.

$$\text{Check:}\ \sin(-\pi) \stackrel{?}{=} \frac{(-\pi)}{\pi} + 1$$
$$0 \stackrel{!}{=} 0$$

So the intersection is at $x = -\pi$.

$$\begin{aligned} A &= \int_{-\pi}^{0} \left(\frac{x}{\pi} + 1 - \sin x\right) dx \\ &= \left[\frac{x^2}{2\pi} + x + \cos x\right]_{-\pi}^{0} \\ &= \left(\frac{(0)^2}{2\pi} + (0) + \cos(0)\right) - \left(\frac{(-\pi)^2}{2\pi} + (-\pi) + \cos(-\pi)\right) \\ &= 1 - \left(\frac{\pi^2}{2\pi} - \pi - 1\right) \\ &= 1 - \frac{\pi}{2} + \pi + 1 \\ &= \boxed{2 + \frac{\pi}{2}} \end{aligned}$$

CHOICE C

7. Determine the point of intersection of $f(x) = e^x + 2$ and $g(x) = -2x + 3$ as the lower limit of integration, $x = 4$ being the upper limit of integration. Using a graphing calculator suggests that the only intersection is $x = 0$.

$$\text{Check: } e^{(0)} + 2 \stackrel{?}{=} -2(0) + 3$$
$$3 \stackrel{!}{=} 3$$

So the intersection is at $x = 0$.

$$\begin{aligned}
A &= \int_0^4 \left[e^x + 2 - (-2x + 3)\right] dx \\
&= \int_0^4 \left[e^x + 2x - 1\right] dx \\
&= \left[e^x + x^2 - x\right]_0^4 \\
&= \left[e^4 + 16 - 4 - \left(e^0\right)\right] \\
&= e^4 + 12 - 1 \\
&= \boxed{e^4 + 11}
\end{aligned}$$

CHOICE B

8. **(a)** Determine the point(s) of intersection of $f(x) = x^3 - 8x^2 + 15x + 2$ and $g(x) = 3x + 2$.

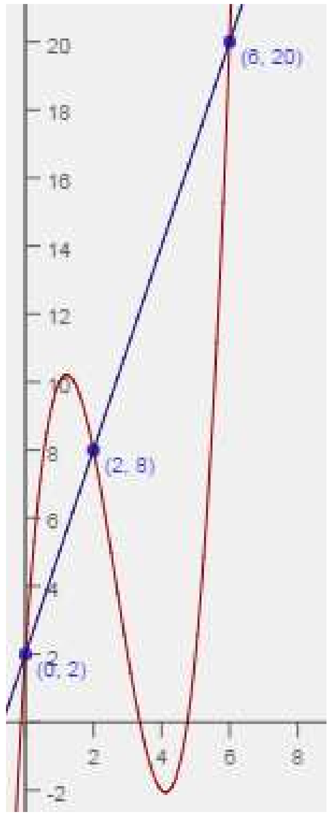
$$\begin{aligned}
x^3 - 8x^2 + 15x + 2 &= 3x + 2 \\
x^3 - 8x^2 + 15x + 2 - 3x - 2 &= 0 \\
x^3 - 8x^2 + 12x &= 0 \\
x\left(x^2 - 8x + 12\right) &= 0 \\
x(x - 6)(x - 2) &= 0 \\
x &= 0 \quad x = 6 \quad x = 2
\end{aligned}$$

Points of intersection at $(0, g(0)) = (0, 2)$ $(6, g(6)) = (6, 20)$ $(2, g(2)) = (2, 8)$

**(b)** Use the $x$-values of the intersection points as the limits of integration, checking which function is higher on each region (see graph).

$$\begin{aligned}
A &= \int_0^2 (f(x) - g(x))\, dx + \int_2^6 (g(x) - f(x))\, dx \\
&= \int_0^2 \left((x^3 - 8x^2 + 15x + 2) - (3x + 2)\right) dx + \int_2^6 (3x + 2) - \left(x^3 - 8x^2 + 15x + 2\right) dx \\
&= \int_0^2 (x^3 - 8x^2 + 12x)\, dx + \int_2^6 \left(-x^3 + 8x^2 - 12x\right) dx
\end{aligned}$$

**(c)**
$$\begin{aligned}
A &= \int_0^2 (x^3 - 8x^2 + 12x)\, dx + \int_2^6 \left(-x^3 + 8x^2 - 12x\right) dx \\
&= \left[\frac{x^4}{4} - \frac{8x^3}{3} + 6x^2\right]_0^2 + \left[\frac{-x^4}{4} + \frac{8x^3}{3} - 6x^2\right]_2^6 \\
&= \left[\frac{16}{4} - \frac{64}{3} + 24 - 0\right] + \left[\frac{-1296}{4} + \frac{1728}{3} - 216 - \left(\frac{-16}{4} + \frac{64}{3} - 24\right)\right] \\
&= \boxed{\frac{148}{3} \approx 49.333}
\end{aligned}$$




## 6.2 Volume of a Solid of Revolution: Disks and Washers

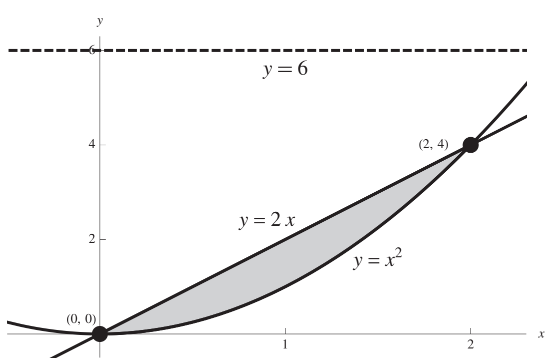
**Concepts and Vocabulary**

1. From the boxed formula in subsection 3 preceding Example 4, the formula is $V = \pi \int_a^b f(x)^2\, dx$.

2. False. The cross section is indeed the region contained between two concentric circles. The radii of those circles are $f(u_i)$ and $g(u_i)$, so the area between them is the difference of their areas, or $\pi f(u_i)^2 - \pi g(u_i)^2 = \pi[f(u_i)^2 - g(u_i)^2]$, not $\pi[f(u_i) - g(u_i)]^2$.

3. False. From Problem 1, or the boxed formula in subsection 3 preceding Example 4, the volume is $V = \pi \int_a^b [f(x)^2 - g(x)^2]\, dx$.

4. False. First of all, the two curves intersect at $x = 0$ and $x = 2$, so the integral must go from 0 to 2. See the diagram:



Second, for a given value of $x$, the outer radius is $6 - x^2$ and the inner radius is $6 - 2x$, so the formula is

$$V = \pi \int_0^2 [(6 - x^2)^2 - (6 - 2x)^2] \, dx.$$

## Skill Building

5. Using the disk method, the radius of each disk is $2\sqrt{x}$, so the volume is

$$V = \pi \int_1^4 \left(2\sqrt{x}\right)^2 \, dx = 4\pi \int_1^4 x \, dx = 4\pi \left[\frac{1}{2}x^2\right]_1^4 = \boxed{30\pi}.$$

6. Since we are revolving around the $y$-axis with the disk method, first solve the equation for $x$ to get $x = y^{1/4}$. Then the radius of each disk is $y^{1/4}$, so the volume is

$$V = \pi \int_0^1 \left(y^{1/4}\right)^2 \, dy = \pi \int_0^1 y^{1/2} \, dy = \pi \left[\frac{2}{3}y^{3/2}\right]_0^1 = \boxed{\frac{2}{3}\pi}.$$

7. Since we are revolving around the $y$-axis with the disk method, first solve the equation for $x$ to get $x = \frac{1}{y}$. Then the radius of each disk is $\frac{1}{y}$, so the volume is

$$V = \pi \int_1^4 \left(\frac{1}{y}\right)^2 \, dy = \pi \int_1^4 \frac{1}{y^2} \, dy = \pi \left[-\frac{1}{y}\right]_1^4 = \pi \left(1 - \frac{1}{4}\right) = \boxed{\frac{3}{4}\pi}.$$

8. Since we are revolving around the $y$-axis with the disk method, first solve the equation for $x$ to get $x = y^{3/2}$. Then the radius of each disk is $y^{3/2}$, so the volume is

$$V = \pi \int_1^4 \left(y^{3/2}\right)^2 \, dy = \pi \int_1^4 y^3 \, dy = \pi \left[\frac{1}{4}y^4\right]_1^4 = \boxed{\frac{255}{4}\pi}.$$

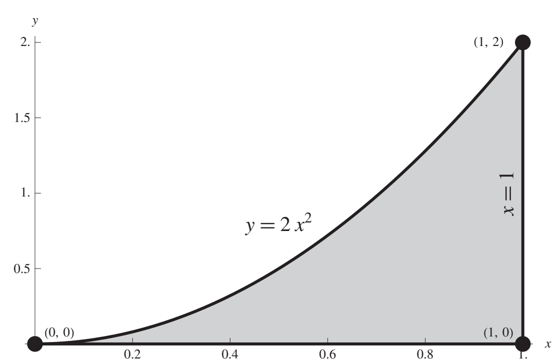
9. Using the washer method along the $x$-axis, each outer radius is $\sec x$ and the inner radius is 1, so that the volume is

$$V = \pi \int_{-1}^1 \left(\sec^2 x - 1^2\right) \, dx = \pi \int_{-1}^1 \left(\sec^2 x - 1\right) \, dx = \pi \left[\tan x - x\right]_{-1}^1 = \boxed{2\pi(\tan 1 - 1)}.$$

10. Since we are revolving around the $y$-axis, we solve $y = x^2$ for $x$, giving $x = \sqrt{y}$ (since $x \geq 0$). Then using the washer method, the volume is

$$V = \pi \int_0^4 \left(2^2 - (\sqrt{y})^2\right) \, dy = \pi \int_0^4 (4 - y) \, dy = \pi \left[4y - \frac{1}{2}y^2\right]_0^4 = \boxed{8\pi}.$$

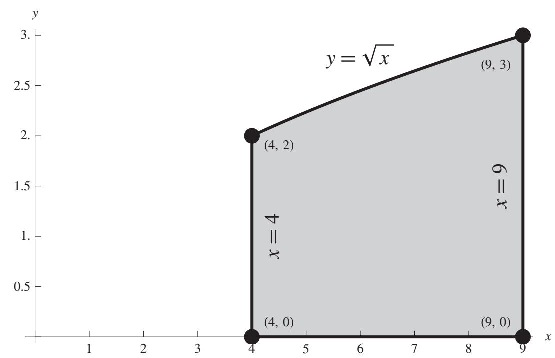
11. The region is shown below:



The radius of each disk is $2x^2$, so the volume is

$$V = \pi \int_0^1 \left(2x^2\right)^2 \, dx = \pi \int_0^1 4x^4 \, dx = \pi \left[\frac{4}{5}x^5\right]_0^1 = \boxed{\frac{4}{5}\pi}.$$

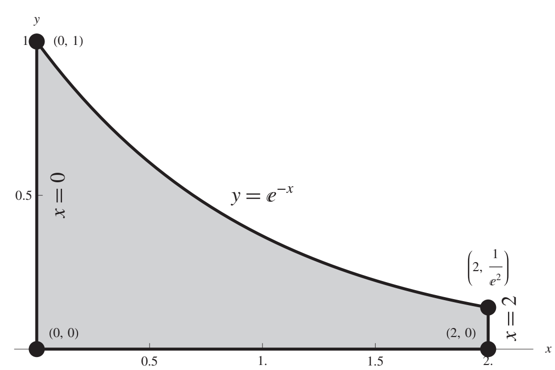
12. The region is shown below:




The radius of each disk is $\sqrt{x}$, so the volume is

$$V = \pi \int_4^9 \left(\sqrt{x}\right)^2 \, dx = \pi \int_4^9 x \, dx = \pi \left[\frac{1}{2}x^2\right]_4^9 = \boxed{\frac{65}{2}\pi}.$$

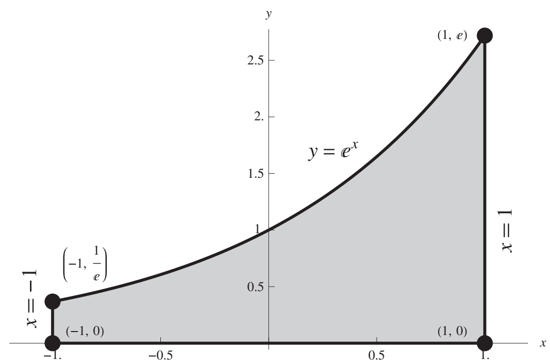
13. The region is shown below:




The radius of each disk is $e^{-x}$, so the volume is

$$V = \pi \int_0^2 \left(e^{-x}\right)^2 \, dx = \pi \int_0^2 e^{-2x} \, dx = \pi \left[-\frac{1}{2}e^{-2x}\right]_0^2 = \boxed{\frac{1}{2}\pi\left(1 - \frac{1}{e^4}\right)}.$$

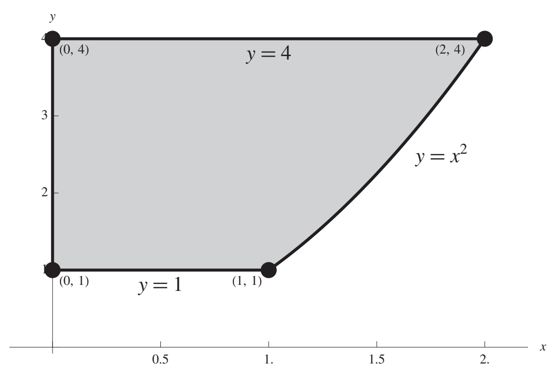
14. The region is shown below:




The radius of each disk is $e^x$, so the volume is

$$V = \pi \int_{-1}^{1} (e^x)^2 \, dx = \pi \int_{-1}^{1} e^{2x} \, dx = \pi \left[ \frac{1}{2} e^{2x} \right]_{-1}^{1} = \boxed{\frac{1}{2}\pi \left( e^2 - \frac{1}{e^2} \right)}.$$

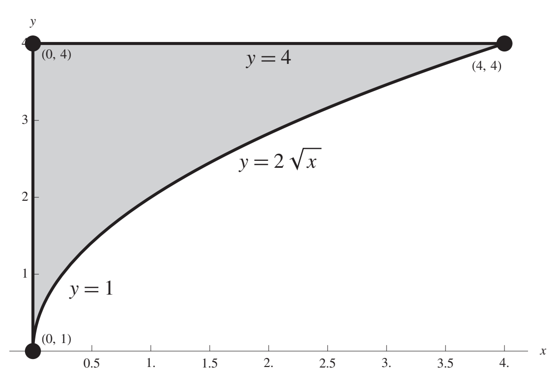
15. The region is shown below:




Solving for $x$ gives $x = \sqrt{y}$ (since $x \geq 0$), so the radius of each disk is $\sqrt{y}$ and the volume is

$$V = \pi \int_{1}^{4} (\sqrt{y})^2 \, dy = \pi \int_{1}^{4} y \, dy = \pi \left[ \frac{1}{2} y^2 \right]_{1}^{4} = \boxed{\frac{15}{2}\pi}.$$

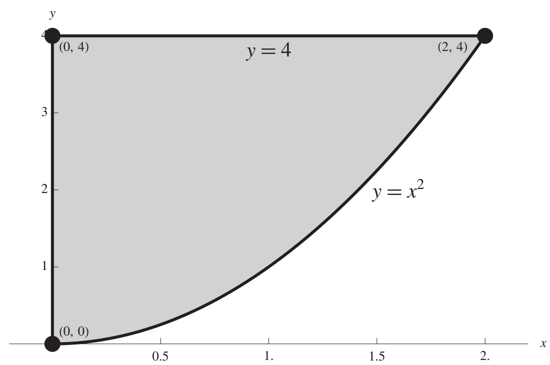
16. The region is shown below:



Solving for $x$ gives $x = \frac{y^2}{4}$, so the radius of each disk is $\frac{y^2}{4}$ and the volume is

$$V = \pi \int_0^4 \left(\frac{y^2}{4}\right)^2 dy = \pi \int_0^4 \frac{1}{16} y^4 \, dy = \pi \left[\frac{1}{80} y^5\right]_0^4 = \boxed{\frac{64}{5}\pi}.$$

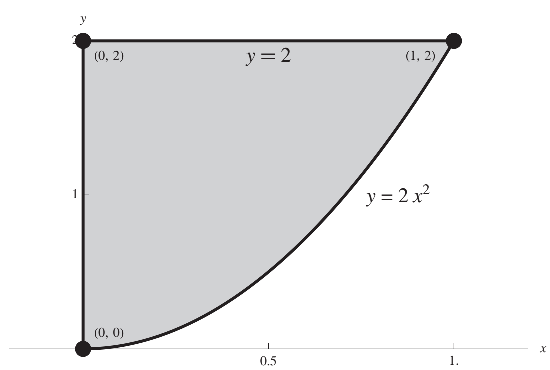
17. The region is shown below:




The outer radius of each washer is 4 and the inner radius is $x^2$, so the volume of the solid is

$$V = \pi \int_0^2 \left(4^2 - \left(x^2\right)^2\right) dx = \pi \int_0^2 \left(16 - x^4\right) dx = \pi \left[16x - \frac{1}{5}x^5\right]_0^2 = \boxed{\frac{128}{5}\pi}.$$

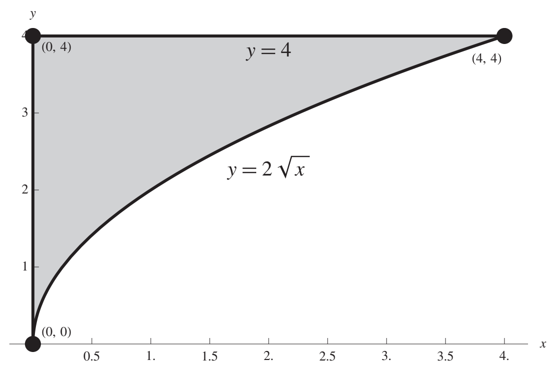
18. The region is shown below:




The outer radius of each washer is 2 and the inner radius is $2x^2$, so the volume of the solid is

$$V = \pi \int_0^1 \left(2^2 - \left(2x^2\right)^2\right) dx = \pi \int_0^1 \left(4 - 4x^4\right) dx = \pi \left[4x - \frac{4}{5}x^5\right]_0^1 = \boxed{\frac{16}{5}\pi}.$$

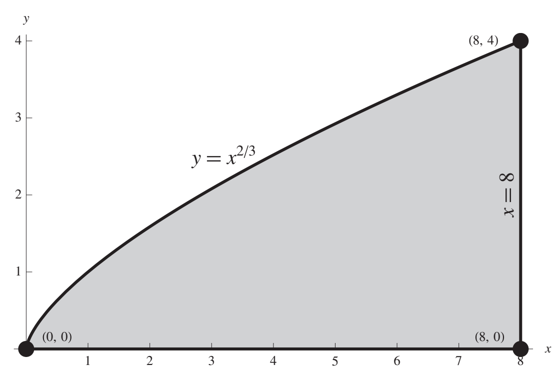
19. The region is shown below:




The outer radius of each washer is 4 and the inner radius is $2\sqrt{x}$, so the volume of the solid is

$$V = \pi \int_0^4 \left(4^2 - \left(2\sqrt{x}\right)^2\right)\, dx = \pi \int_0^4 (16 - 4x)\, dx = \pi \left[16x - 2x^2\right]_0^4 = \boxed{32\pi}.$$

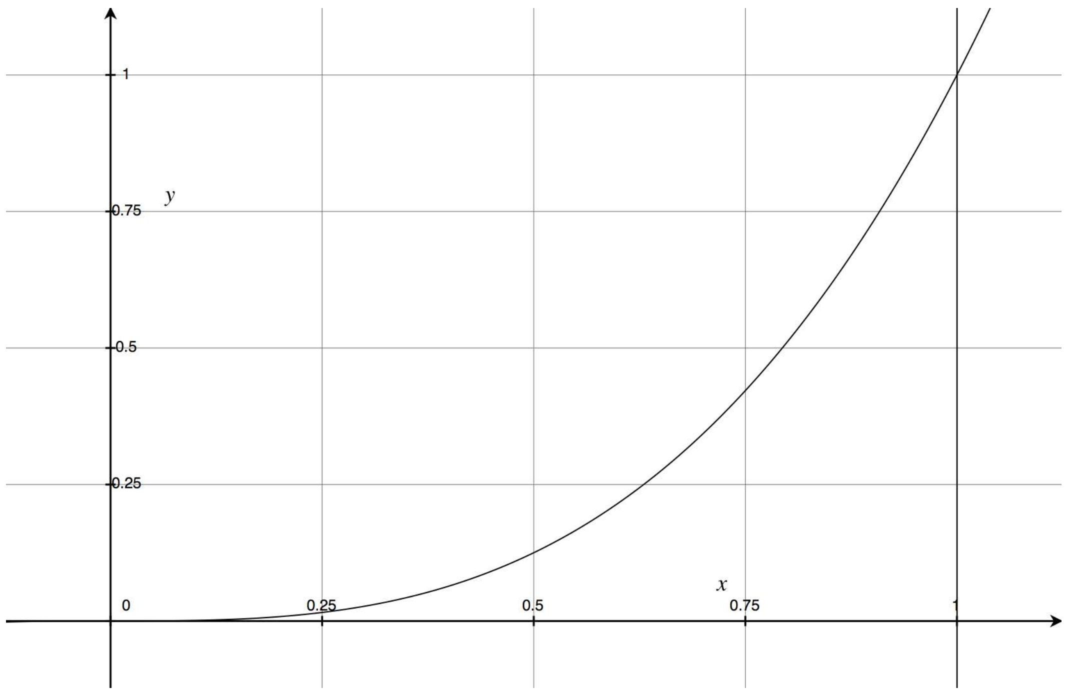
20. The region is shown below:




Solving $y = x^{2/3}$ for $x$ gives $x = y^{3/2}$, so the outer radius of each washer when revolved around the $y$-axis is 8 and the inner radius is $y^{3/2}$. Therefore the volume is

$$V = \pi \int_0^4 \left(8^2 - \left(y^{3/2}\right)^2\right)\, dy = \pi \int_0^4 \left(64 - y^3\right)\, dy = \pi \left[64y - \frac{1}{4}y^4\right]_0^4 = \boxed{192\pi}.$$

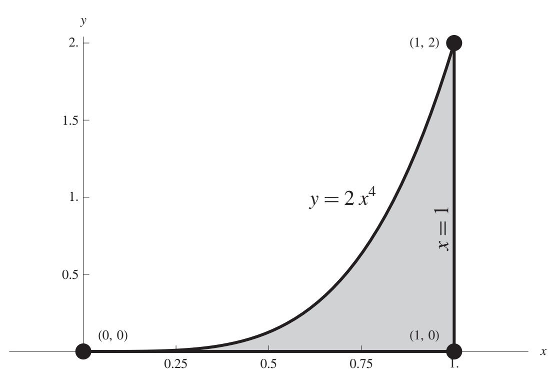
21. The region is shown below:




Solving $y = x^3$ for $x$ gives $x = y^{1/3}$, so the outer radius of each washer when revolved around the $y$-axis is 1 and the inner radius is $y^{1/3}$. Therefore the volume is

$$V = \pi \int_0^1 \left[1^2 - \left(y^{1/3}\right)^2\right] dy = \pi \int_0^1 \left(1 - y^{2/3}\right) dy = \pi \left[y - \frac{3}{5} y^{5/3}\right]_0^1$$
$$= \pi \left\{\left[(1) - \frac{3}{5}(1)^{5/3}\right] - 0\right\} = \boxed{\frac{2}{5}\pi}$$

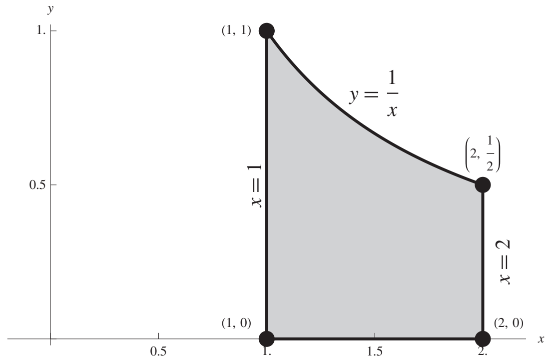
22. The region is shown below:




Solving $y = 2x^4$ for $x$ gives $x = \left(\frac{y}{2}\right)^{1/4}$ (since $x \geq 0$), so the outer radius of each washer when revolved around the $y$-axis is 1 and the inner radius is $\left(\frac{y}{2}\right)^{1/4}$. Therefore the volume is

$$V = \pi \int_0^2 \left(1^2 - \left(\left(\frac{y}{2}\right)^{1/4}\right)^2\right) dy = \pi \int_0^2 \left(1 - \frac{1}{\sqrt{2}} y^{1/2}\right) dy = \pi \left[y - \frac{\sqrt{2}}{3} y^{3/2}\right]_0^2 = \boxed{\frac{2}{3}\pi}.$$

23. The region is shown below:



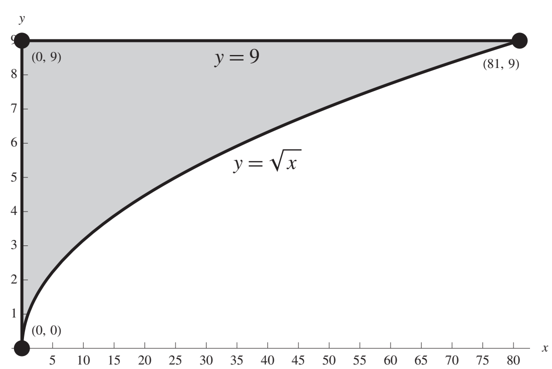

Using the disk method, the radius of each disk around the $x$-axis is $\frac{1}{x}$, so the volume is

$$V = \pi \int_1^2 \left(\frac{1}{x}\right)^2 dx = \pi \int_1^2 \frac{1}{x^2}\, dx = \pi \left[-\frac{1}{x}\right]_1^2 = \boxed{\frac{\pi}{2}}.$$

24. This is the same region from Problem 23; it is pictured there. To rotate around the $y$-axis, we use the washer method. Solving $y = \frac{1}{x}$ for $x$ gives $x = \frac{1}{y}$. Now, $x = \frac{1}{y}$ and $x = 2$ intersect when $y = \frac{1}{2}$. So from $y = 0$ to $y = \frac{1}{2}$, the outer radius is 2 while the inner radius is 1. From $y = \frac{1}{2}$ to $y = 1$ the outer radius is $\frac{1}{y}$ and the inner radius is 1. Therefore the volume is

$$\begin{aligned} V &= \pi \int_0^{1/2} (2^2 - 1^2)\, dy + \pi \int_{1/2}^1 \left(\left(\frac{1}{y}\right)^2 - 1^2\right) dy \\ &= \pi \int_0^{1/2} 3\, dy + \pi \int_{1/2}^1 \left(\frac{1}{y^2} - 1\right) dy \\ &= \pi \left[3y\right]_0^{1/2} + \pi \left[-\frac{1}{y} - y\right]_{1/2}^1 \\ &= \boxed{2\pi}. \end{aligned}$$

25. The region is shown below:



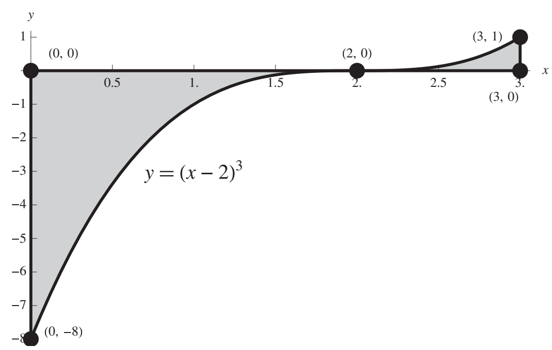

Using the disk method to revolve about the $y$-axis, first solve $y = \sqrt{x}$ for $x$ to get $x = y^2$. Then the radius of each disk is $y^2$, so the volume is

$$V = \pi \int_0^9 \left(y^2\right)^2 dy = \pi \int_0^9 y^4\, dy = \pi \left[\frac{1}{5}y^5\right]_0^9 = \boxed{\frac{59049}{5}\pi}.$$

26. This is the same region from Problem 25; it is pictured there. To rotate around the $x$-axis, we use the disk method. The outer radius of each disk is 9, while the inner radius is $\sqrt{x}$. Therefore the volume is

$$V = \pi \int_0^{81} \left(9^2 - \left(\sqrt{x}\right)^2\right)\, dx = \pi \int_0^{81} (81 - x)\, dx = \pi \left[81x - \frac{1}{2}x^2\right]_0^{81} = \boxed{\frac{6561}{2}\pi}.$$

27. The region is shown below:



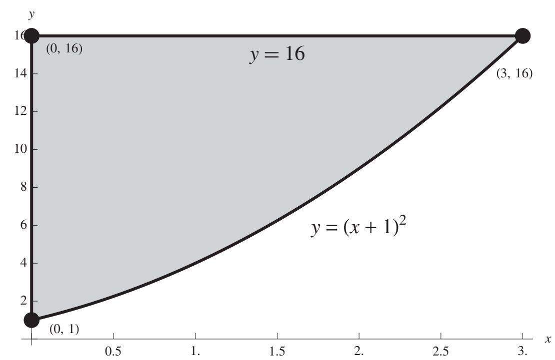

When revolving about the $x$-axis, we use the disk method. Note that the region lies both above and below the $x$-axis, so from $x = 0$ to $x = 2$, the radius will be $-(x-2)^3$, while from $x = 2$ to $x = 3$ it will be $(x-2)^3$. However, since we square the radius, the minus sign will disappear in the computation — that is, $((x-2)^3)^2 = (-(x-2)^3)^2$. Therefore the volume is

$$V = \pi \int_0^3 \left((x-2)^3\right)^2\, dx = \pi \int_0^3 (x-2)^6\, dx = \pi \left[\frac{1}{7}(x-2)^7\right]_0^3 = \boxed{\frac{129}{7}\pi}.$$

28. This is the same region from Problem 27; it is pictured there. To rotate around the $y$-axis, first solve $y = (x-2)^3$ for $x$ to get $x = 2 + y^{1/3}$. We use the disk method from $y = -8$ to $y = 0$, where the radius is $2 + y^{1/3}$. From $y = 0$ to $y = 1$ we use the washer method; the outer radius is 3 while the inner radius is $2 + y^{1/3}$. Therefore the volume is

$$\begin{aligned} V &= \pi \int_{-8}^{0} \left(2 + y^{1/3}\right)^2\, dy + \pi \int_0^1 \left(3^2 - \left(2 + y^{1/3}\right)^2\right)\, dy \\ &= \pi \int_{-8}^{0} \left(y^{2/3} + 4y^{1/3} + 4\right)\, dy + \pi \int_0^1 \left(5 - 4y^{1/3} - y^{2/3}\right)\, dy \\ &= \pi \left[\frac{3}{5}y^{5/3} + 3y^{4/3} + 4y\right]_{-8}^{0} + \pi \left[5y - 3y^{4/3} - \frac{3}{5}y^{5/3}\right]_0^1 \\ &= \boxed{\frac{23}{5}\pi}. \end{aligned}$$

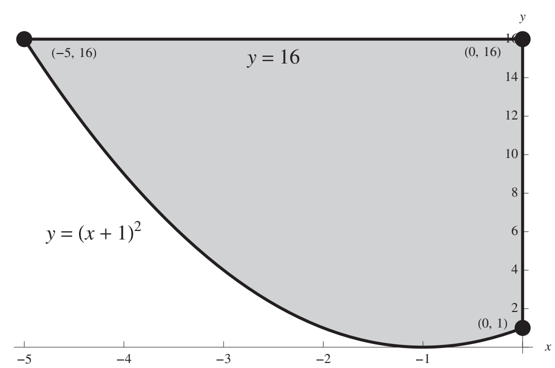
29. The region is shown below:




When revolving about the $y$-axis, we first solve $y = (x+1)^2$ for $x$ to get $x = \sqrt{y} - 1$. Using the disk method, the radius of each disk is $\sqrt{y} - 1$, so the volume is

$$V = \pi \int_1^{16} (\sqrt{y} - 1)^2 \, dy = \pi \int_1^{16} \left(y - 2y^{1/2} + 1\right) dy = \pi \left[\frac{1}{2}y^2 - \frac{4}{3}y^{3/2} + y\right]_1^{16} = \boxed{\frac{117}{2}\pi}.$$

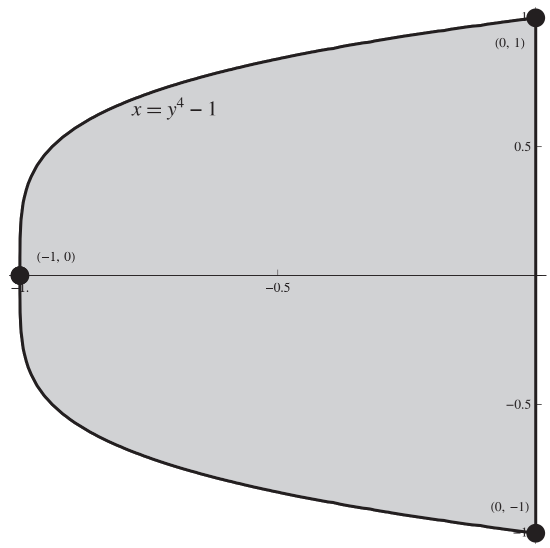
30. The region is shown below:




To revolve around the $x$-axis, we use the washer method; each outer radius is 16 while the inner radius is $(x+1)^2$. Therefore the volume is

$$V = \pi \int_{-5}^{0} \left(16^2 - \left((x+1)^2\right)^2\right) dx = \pi \int_{-5}^{0} \left(256 - (x+1)^4\right) dx = \pi \left[256x - \frac{1}{5}(x+1)^5\right]_{-5}^{0} = \boxed{1075\pi}.$$

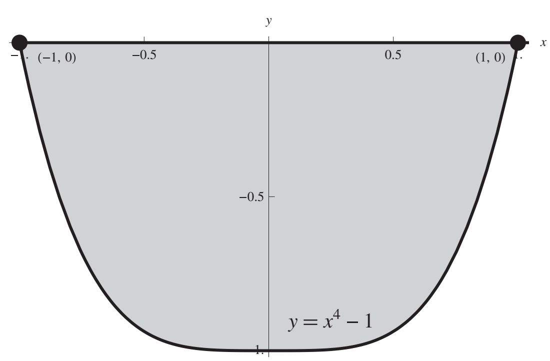
31. The region is shown below:




We use the disk method to revolve about the $y$-axis; the radius is $1 - y^4$. Therefore the volume is

$$V = \pi \int_{-1}^{1} \left(1 - y^4\right)^2 \, dy = \pi \int_{-1}^{1} \left(y^8 - 2y^4 + 1\right) \, dy = \pi \left[\frac{1}{9}y^9 - \frac{2}{5}y^5 + y\right]_{-1}^{1} = \boxed{\frac{64}{45}\pi}.$$

32. The region is shown below:

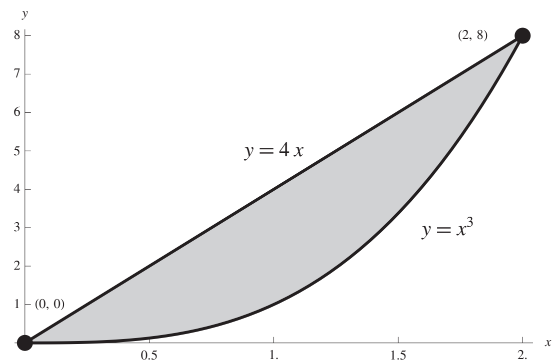



We use the disk method to revolve about the $x$-axis; the radius is $1 - x^4$ (since $x^4 - 1 < 0$). Therefore the volume is

$$V = \pi \int_{-1}^{1} \left(1 - x^4\right)^2 \, dx = \pi \int_{-1}^{1} \left(1 - 2x^4 + x^8\right) \, dx = \pi \left[x - \frac{2}{5}x^5 + \frac{1}{9}x^9\right]_{-1}^{1} = \boxed{\frac{64}{45}\pi}.$$

(Note that this is the same region as in Problem 31, rotated 90°, so we would expect to get the same answer.)

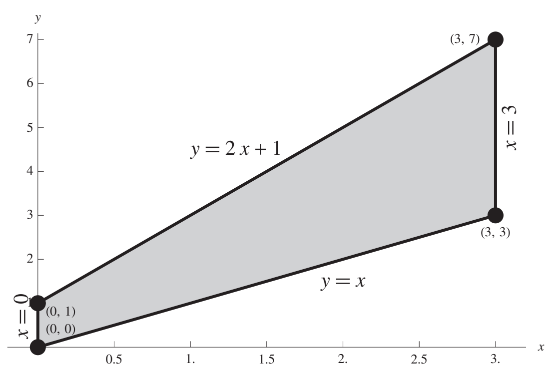
33. The region is shown below:




We use the washer method to revolve about the $x$-axis. Each outer radius is $4x$, and each inner radius is $x^3$, so the volume is

$$V = \pi \int_0^2 \left((4x)^2 - \left(x^3\right)^2\right)\,dx = \pi \int_0^2 \left(16x^2 - x^6\right)\,dx = \pi\left[\frac{16}{3}x^3 - \frac{1}{7}x^7\right]_0^2 = \boxed{\frac{512}{21}\pi}.$$

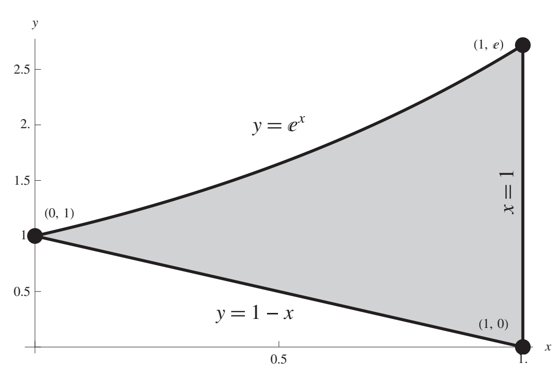
34. The region is shown below:




We use the washer method to revolve about the $x$-axis. Each outer radius is $2x+1$, and each inner radius is $x$, so the volume is

$$V = \pi \int_0^3 \left((2x+1)^2 - x^2\right)\,dx = \pi \int_0^3 \left(3x^2 + 4x + 1\right)dx = \pi\left[x^3 + 2x^2 + x\right]_0^3 = \boxed{48\pi}.$$

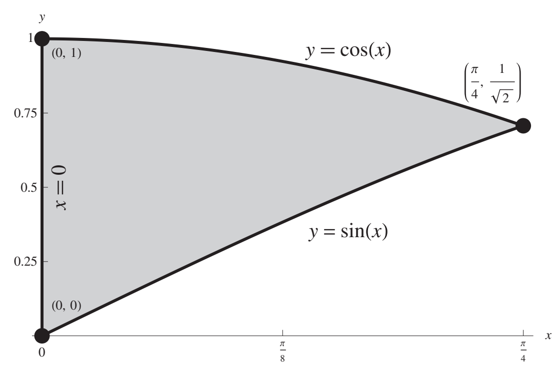
35. The region is shown below:



We use the washer method to revolve about the $x$-axis. Each outer radius is $e^x$, and each inner radius is $1-x$, so the volume is

$$\begin{aligned} V &= \pi \int_0^1 \left( (e^x)^2 - (1-x)^2 \right) dx \\ &= \pi \int_0^1 \left( e^{2x} - 1 + 2x - x^2 \right) dx \\ &= \pi \left[ \frac{1}{2}e^{2x} - x + x^2 - \frac{1}{3}x^3 \right]_0^1 \\ &= \boxed{\left( \frac{1}{2}e^2 - \frac{5}{6} \right) \pi}. \end{aligned}$$

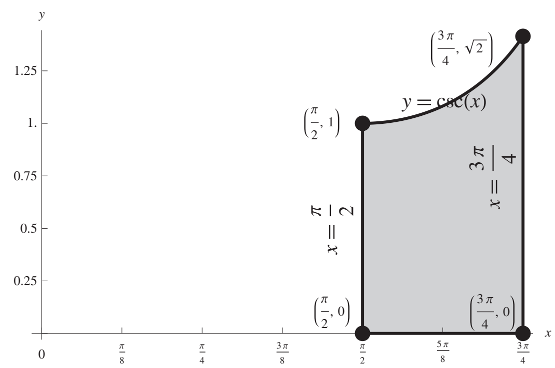
36. The region is shown below:




We use the washer method to revolve about the $x$-axis. Each outer radius is $\cos x$ and each inner radius is $\sin x$, so the volume is

$$\begin{aligned} V &= \pi \int_0^{\pi/4} \left( (\cos x)^2 - (\sin x)^2 \right) dx \\ &= \pi \int_0^{\pi/4} \left( \cos^2 x - \sin^2 x \right) dx \\ &= \pi \int_0^{\pi/4} \cos 2x \, dx \\ &= \pi \left[ \frac{1}{2} \sin 2x \right]_0^{\pi/4} \\ &= \boxed{\frac{\pi}{2}}. \end{aligned}$$

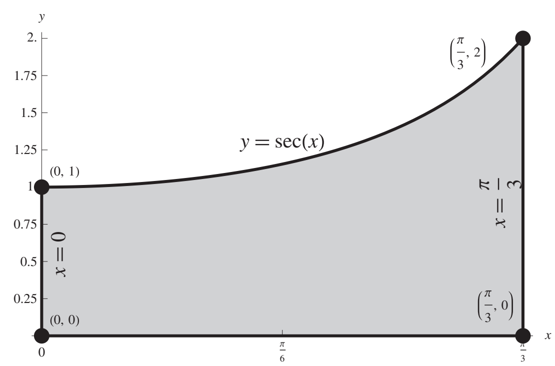
37. The region is shown below:




We use the disk method to revolve about the $x$-axis. Each radius is $\csc x$, so the volume is

$$V = \pi \int_{\pi/2}^{3\pi/4} (\csc x)^2 \; dx = \pi \int_{\pi/2}^{3\pi/4} \csc^2 x \, dx = \pi \left[-\cot x\right]_{\pi/2}^{3\pi/4} = \boxed{\pi}.$$

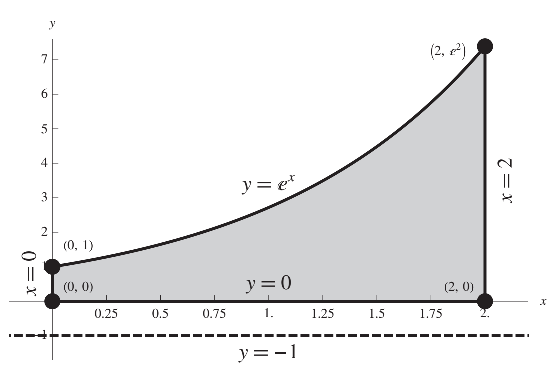
38. The region is shown below:




We use the disk method to revolve about the $x$-axis. Each radius is $\sec x$, so the volume is

$$V = \pi \int_{0}^{\pi/3} (\sec x)^2 \; dx = \pi \int_{0}^{\pi/3} \sec^2 x \, dx = \pi \left[\tan x\right]_{0}^{\pi/3} = \boxed{\pi\sqrt{3}}.$$

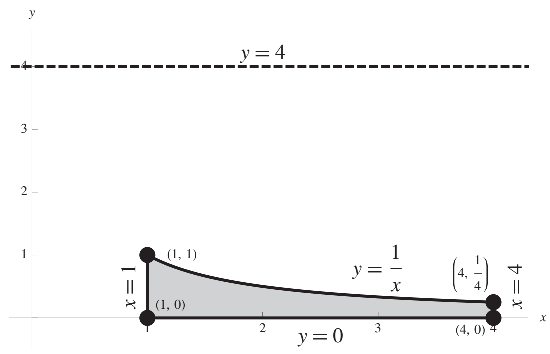
39. The region is shown below, with the line of revolution shown as a dashed line:



To revolve about $y = -1$, we use the washer method. Each outer radius is $e^x - (-1) = e^x + 1$, and each inner radius is $0 - (-1) = 1$. Therefore the volume is

$$V = \pi \int_0^2 \left( (e^x + 1)^2 - 1^2 \right) dx = \pi \int_0^2 \left( e^{2x} + 2e^x \right) dx = \pi \left[ \frac{1}{2} e^{2x} + 2e^x \right]_0^2 = \boxed{\pi \left( \frac{1}{2} e^4 + 2e^2 - \frac{5}{2} \right)}.$$

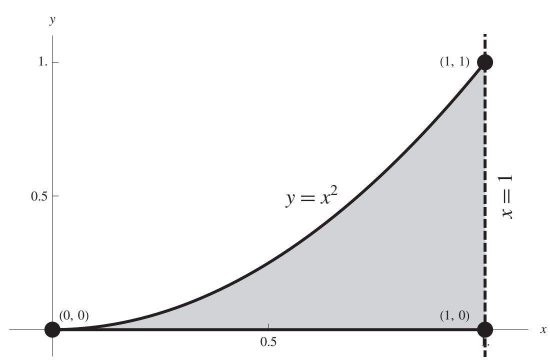
40. The region is shown below, with the line of revolution shown as a dashed line:




To revolve about $y = 4$, we use the washer method. Each outer radius is $4 - 0 = 4$, and each inner radius is $4 - \frac{1}{x}$. Therefore the volume is

$$V = \pi \int_1^4 \left( 4^2 - \left( 4 - \frac{1}{x} \right)^2 \right) dx = \pi \int_1^4 \left( \frac{8}{x} - \frac{1}{x^2} \right) dx = \pi \left[ 8 \ln x + \frac{1}{x} \right]_1^4 = \boxed{\left( 16 \ln 2 - \frac{3}{4} \right) \pi}.$$

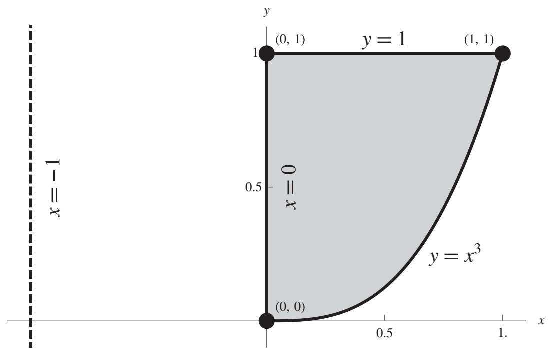
41. The region is shown below, with the line of revolution shown as a dashed line:




To revolve about $x = 1$, we first solve $y = x^2$ for $x$ to get $x = \sqrt{y}$. Using the disk method, each radius is $1 - \sqrt{y}$. Therefore the volume is

$$V = \pi \int_0^1 \left( 1 - \sqrt{y} \right)^2 dy = \pi \int_0^1 \left( 1 - 2y^{1/2} + y \right) dy = \pi \left[ y - \frac{4}{3} y^{3/2} + \frac{1}{2} y^2 \right]_0^1 = \boxed{\frac{\pi}{6}}.$$

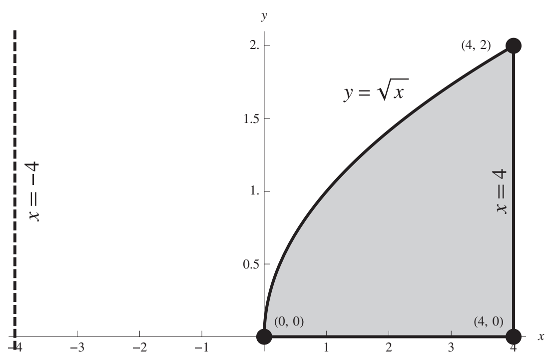
42. The region is shown below, with the line of revolution shown as a dashed line:




To revolve about $x = -1$, we first solve $y = x^3$ for $x$ to get $x = y^{1/3}$. Using the washer method, the outer radius is $y^{1/3} - (-1) = y^{1/3} + 1$, while the inner radius is $0 - (-1) = 1$. Therefore the volume is

$$V = \pi \int_0^1 \left( \left( y^{1/3} + 1 \right)^2 - 1^2 \right) dy = \pi \int_0^1 \left( y^{2/3} + 2y^{1/3} \right) dy = \pi \left[ \frac{3}{5} y^{5/3} + \frac{3}{2} y^{4/3} \right]_0^1 = \boxed{\frac{21}{10}\pi}.$$

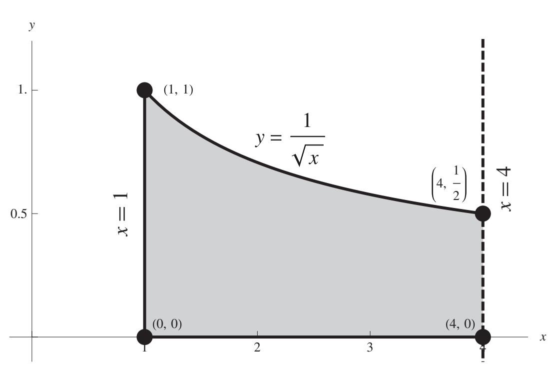
43. The region is shown below, with the line of revolution shown as a dashed line:




To revolve about $x = -4$, we first solve $y = \sqrt{x}$ for $x$ to get $x = y^2$. Using the washer method, the outer radius is $4 - (-4) = 8$ and the inner radius is $y^2 - (-4) = y^2 + 4$. Therefore the volume is

$$V = \pi \int_0^2 \left( 8^2 - (y^2 + 4)^2 \right) dy = \pi \int_0^2 \left( 48 - 8y^2 - y^4 \right) dy = \pi \left[ 48y - \frac{8}{3} y^3 - \frac{1}{5} y^5 \right]_0^2 = \boxed{\frac{1024}{15}\pi}.$$

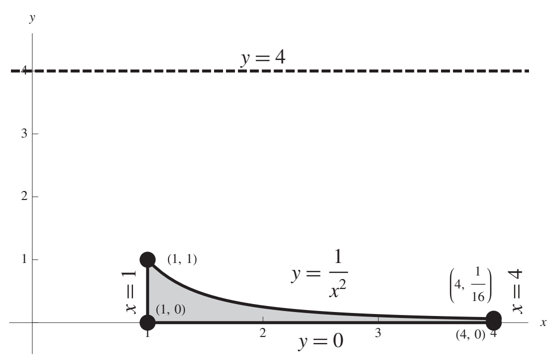
44. The region is shown below, with the line of revolution shown as a dashed line:



To revolve about $x = 4$, we first solve $y = \frac{1}{\sqrt{x}}$ for $x$ to get $x = \frac{1}{y^2}$. For $0 \le y \le \frac{1}{2}$, we can use the disk method with radius $4 - 1 = 3$. For $\frac{1}{2} \le y \le 1$, we use the washer method, with outer radius $4 - 1 = 3$ and inner radius $4 - \frac{1}{y^2}$. Therefore the volume is

$$\begin{aligned} V &= \pi \int_0^{1/2} 3^2\, dy + \pi \int_{1/2}^{1} \left(3^2 - \left(4 - \frac{1}{y^2}\right)^2\right) dy \\ &= \pi \int_0^{1/2} 9\, dy + \pi \int_{1/2}^{1} \left(-7 + \frac{8}{y^2} - \frac{1}{y^4}\right) dy \\ &= \pi \left[9y\right]_0^{1/2} + \pi \left[-7y - \frac{8}{y} + \frac{1}{3y^3}\right]_{1/2}^{1} \\ &= \boxed{\frac{20}{3}\pi}. \end{aligned}$$

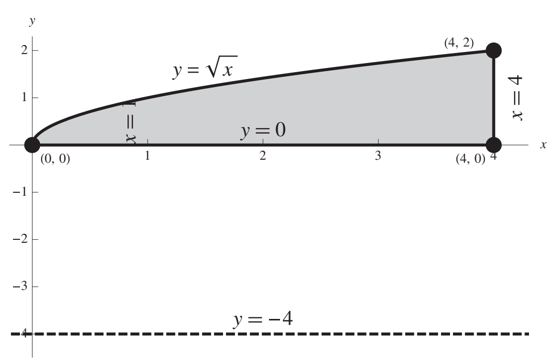
45. The region is shown below, with the line of revolution shown as a dashed line:




To revolve about $y = 4$, we use the washer method. The outer radius is $4 - 0 = 4$, while the inner radius is $4 - \frac{1}{x^2}$. Therefore the volume is

$$V = \pi \int_1^4 \left(4^2 - \left(4 - \frac{1}{x^2}\right)^2\right) dx = \pi \int_1^4 \left(\frac{8}{x^2} - \frac{1}{x^4}\right) dx = \pi \left[-\frac{8}{x} + \frac{1}{3x^3}\right]_1^4 = \boxed{\frac{363}{64}\pi}.$$

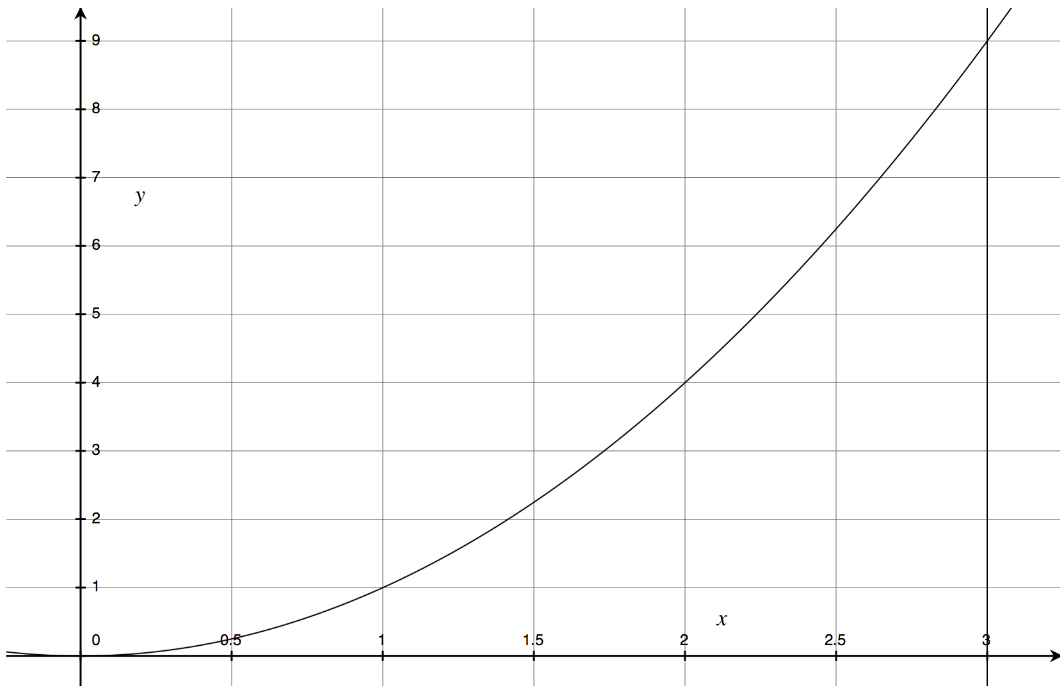
46. The region is shown below, with the line of revolution shown as a dashed line:




To revolve about $y = -4$, we use the washer method. The outer radius is $\sqrt{x} - (-4) = \sqrt{x} + 4$, while the inner radius is $0 - (-4) = 4$. Therefore the volume is

$$V = \pi \int_0^4 \left(\left(\sqrt{x} + 4\right)^2 - 4^2\right) dx = \pi \int_0^4 \left(8\sqrt{x} + x\right) dx = \pi \left[\frac{16}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^4 = \boxed{\frac{152}{3}\pi}.$$

47. The region is shown below:




**(a)** The radius of each disk when revolved around the $x$–axis is $y = x^2$. Therefore the volume is

$$V = \pi \int_0^3 y^2\,dx = \pi \int_0^3 \left(x^2\right)^2 dx = \pi \int_0^3 x^4\,dx = \pi \left[\frac{1}{5}x^5\right]_0^3 = \pi \left[\frac{1}{5}(3)^5 - 0\right] = \boxed{\frac{243}{5}\pi}$$

**(b)** The outer radius of each washer when revolved around the line $y = -1$ is $y - (-1) = x^2 + 1$, and the inner radius is $0 - (-1) = 1$. Therefore the volume is

$$V = \pi \int_0^3 \left[\left(x^2+1\right)^2 - 1^2\right] dx = \pi \int_0^3 \left[\left(x^4 + 2x^2 + 1\right) - 1\right] dx = \pi \int_0^3 \left(x^4 + 2x^2\right) dx$$
$$= \pi \left[\frac{1}{5}x^5 + \frac{2}{3}x^3\right]_0^3 = \pi \left\{\left[\frac{1}{5}(3)^5 + \frac{2}{3}(3)^3\right] - 0\right\} = \boxed{\frac{333}{5}\pi}$$

**(c)** The outer radius of each washer when revolved around the line $y = 10$ is $10 - 0 = 10$ and the inner radius is $10 - y = 10 - x^2$. Therefore the volume is

$$V = \pi \int_0^3 \left[10^2 - \left(10 - x^2\right)^2\right] dx = \pi \int_0^3 \left[100 - \left(100 - 20x^2 + x^4\right)\right] dx = \pi \int_0^3 \left(20x^2 - x^4\right) dx$$
$$= \pi \left[\frac{20}{3}x^3 - \frac{1}{5}x^5\right]_0^3 = \pi \left\{\left[\frac{20}{3}(3)^3 - \frac{1}{5}(3)^5\right] - 0\right\} = \boxed{\frac{657}{5}\pi}$$

**(d)** The outer radius of each washer when revolved around the line $y = a \geq 9$ is $a - 0 = a$ and the inner radius is $a - y = a - x^2$. Therefore the volume is

$$V = \pi \int_0^3 \left[a^2 - \left(a - x^2\right)^2\right] dx = \pi \int_0^3 \left[a^2 - \left(a^2 - 2ax^2 + x^4\right)\right] dx = \pi \int_0^3 \left(2ax^2 - x^4\right) dx$$

$$= \pi \left[\frac{2a}{3}x^3 - \frac{1}{5}x^5\right]_0^3 = \pi \left\{\left[\frac{2a}{3}(3)^3 - \frac{1}{5}(3)^5\right] - 0\right\} = \left(18a - \frac{243}{5}\right)\pi = \boxed{\frac{90a - 243}{5}\pi}$$

48. **(a)**

$$V = \pi \int_0^9 (3-0)^2\, dy - \pi \int_0^9 (x-0)^2\, dy$$
$$= \pi \int_0^9 9\, dy - \pi \int_0^9 x^2\, dy$$
$$= \pi \int_0^9 9\, dy - \int_0^9 y\, dy$$
$$= \pi[9y]_0^9 - \pi\left[\frac{y^2}{2}\right]_0^9$$
$$= \pi(81 - 0) - \pi\left(\frac{81}{2} - 0\right)$$
$$= 81\pi - \frac{81\pi}{2}$$
$$= \boxed{\frac{81}{2}\pi}$$

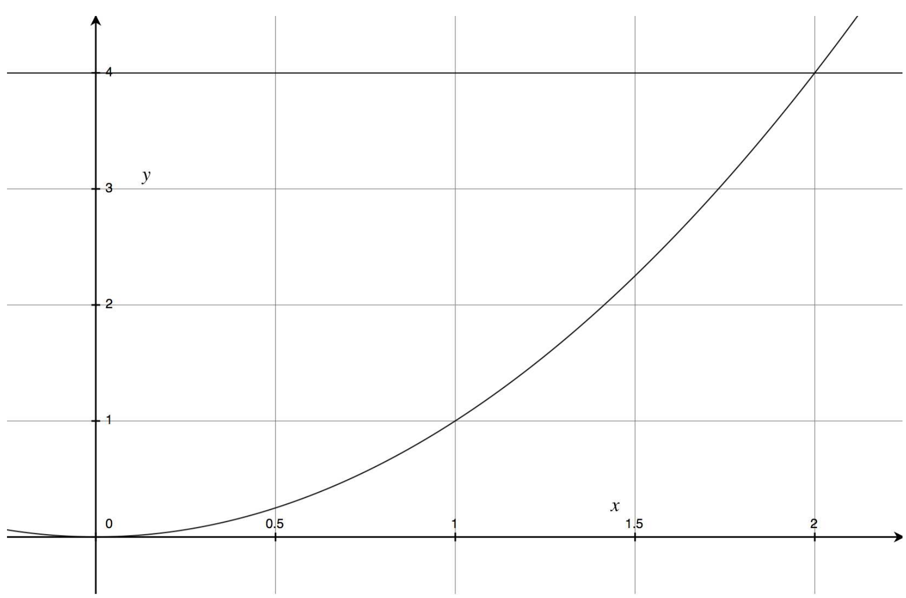
**(b)**

$$V = \pi \int_0^9 [3 - (-5)]^2\, dy - \pi \int_0^9 [x - (-5)]^2\, dy$$
$$= \pi \int_0^9 64 dy - \pi \int_0^9 (x+5)^2\, dy$$
$$= \pi[64y]_0^9 - \pi \int_0^9 \left(x^2 + 10x + 25\right) dy$$
$$= \pi[64(9)] - \pi \int_0^9 \left(y + 10\sqrt{x} + 25\right) dy$$
$$= 576\pi - \pi \int_0^9 y\, dy - 10\pi \int_0^9 \sqrt{y}\, dy - 25\pi \int_0^9 dy$$
$$= 576\pi - \left[\frac{\pi y^2}{2}\right]_0^9 - \left[10\pi\left(\frac{2}{3}\right)y^{3/2}\right]_0^9 - [25\pi y]_0^9$$
$$= 576\pi - \frac{81\pi}{2} - 10\pi\left(\frac{2}{3}\right)(9)^{3/2} - 25\pi(9)$$
$$= 576\pi - \frac{81\pi}{2} - 10\pi\left(\frac{2}{3}\right)(27) - 225\pi$$
$$= 576\pi - \frac{81\pi}{2} - 180\pi - 225\pi$$
$$= \boxed{\frac{261}{2}\pi}$$

**(c)**
$$\begin{aligned}
V &= \pi \int_0^9 (5-x)^2\,dy - \pi \int_0^9 (5-3)^2\,dy \\
&= \pi \int_0^9 \left(25 - 10x + x^2\right) dy - 4\pi \int_0^9 dy \\
&= \pi \int_0^9 (25 - 10\sqrt{y} + y)\,dy - 4\pi \int_0^9 dy \\
&= 25\pi \int_0^9 dy - 10\pi \int_0^9 y^{1/2}\,dy + \pi \int_0^9 y\,dy - 4\pi \int_0^9 dy \\
&= [25\pi]_0^9 - \frac{20\pi}{3}\left[y^{3/2}\right]_0^9 + \frac{\pi}{2}\left[y^2\right]_0^9 - 4\pi(9-0) \\
&= 225\pi - \frac{20\pi}{3}(27) + \frac{81\pi}{2} - 36\pi \\
&= 225\pi - 180\pi + \frac{81\pi}{2} - 36\pi \\
&= \boxed{\frac{99}{2}\pi}
\end{aligned}$$

**(d)**
$$\begin{aligned}
V &= \pi \int_0^9 (b-x)^2\,dy - \pi \int_0^9 (b-3)^2\,dy \\
&= \pi \int_0^9 (b-\sqrt{y})^2\,dy - \pi \int_0^9 (b-3)^2\,dy \\
&= \pi \int_0^9 \left(b^2 - 2b\sqrt{y} + y\right) dy - \pi \int_0^9 \left(b^2 - 6b + 9\right) dy \\
&= \pi \left[b^2 y - \frac{4b}{3} y^{3/2} + \frac{y^2}{2}\right]_0^9 - \pi\left[\left(b^2 - 6b + 9\right) y\right]_0^9 \\
&= 9b^2\pi - 36b\pi + \frac{81\pi}{2} - 9b^2\pi + 54b\pi - 81\pi \\
&= \boxed{\frac{36b - 81}{2}\pi}
\end{aligned}$$

49. The region is shown below:



**(a)** Solving $y = x^2$ for $x$ gives $x = y^{1/2}$. The radius of each disk when revolved around the $y$-axis is $x - 0 = y^{1/2} - 0 = y^{1/2}$. Therefore the volume is

$$V = \pi \int_0^4 \left(y^{1/2}\right)^2 dy = \pi \int_0^4 y\, dy = \pi \left[\frac{1}{2}y^2\right]_0^4 = \pi \left[\frac{1}{2}(4)^2 - 0\right] = \boxed{8\pi}$$

**(b)** The outer radius of each washer when revolving around the line $x = -5$ is $x - (-5) = y^{1/2} + 5$, and the inner radius is $0 - (-5) = 5$. Therefore the volume is

$$V = \pi \int_0^4 \left[\left(y^{1/2} + 5\right)^2 - 5^2\right] dy = \pi \int_0^4 \left[\left(y + 10y^{1/2} + 25\right) - 25\right] dy = \pi \int_0^4 \left(y + 10y^{1/2}\right) dy$$
$$= \left[\frac{1}{2}y^2 + \frac{20}{3}y^{3/2}\right]_0^4 = \pi \left\{\left[\frac{1}{2}(4)^2 + \frac{20}{3}(4)^{3/2}\right] - 0\right\} = \boxed{\frac{184}{3}\pi}$$

**(c)** The outer radius of each washer when revolving around the line $x = 5$ is $5 - 0 = 5$, and the inner radius is $5 - x = 5 - y^{1/2}$. Therefore the volume is

$$V = \pi \int_0^4 \left[5^2 - \left(5 - y^{1/2}\right)^2\right] dy = \pi \int_0^4 \left[25 - \left(25 - 10y^{1/2} + y\right)\right] dy = \pi \int_0^4 \left(10y^{1/2} - y\right) dy$$
$$= \left[\frac{20}{3}y^{3/2} - \frac{1}{2}y^2\right]_0^4 = \pi \left\{\left[\frac{20}{3}(4)^{3/2} - \frac{1}{2}(4)^2\right] - 0\right\} = \boxed{\frac{136}{3}\pi}$$

**(d)** The outer radius of each washer when revolving around the line $x = b \geq 2$ is $b - 0 = b$ and the inner radius is $b - x = b - y^{1/2}$. Therefore the volume is

$$V = \pi \int_0^4 \left[b^2 - \left(b - y^{1/2}\right)^2\right] dy = \pi \int_0^4 \left[b^2 - \left(b^2 - 2by^{1/2} + y\right)\right] dy = \pi \int_0^4 \left(2by^{1/2} - y\right) dy$$
$$= \left[\frac{4b}{3}y^{3/2} - \frac{1}{2}y^2\right]_0^4 = \pi \left\{\left[\frac{4b}{3}(4)^{3/2} - \frac{1}{2}(4)^2\right] - 0\right\} = \frac{32b}{3} - 8 = \boxed{\frac{32b - 24}{3}\pi}$$

50. **(a)** $V = \pi \int_0^2 (4 - 0)^2\, dx - \pi \int_0^2 \left(x^2\right)^2 dx$

$$= \pi \int_0^2 16\, dx - \pi \int_0^2 x^4\, dx$$
$$= 16\pi [x]_0^2 - \pi \left[\frac{x^5}{5}\right]_0^2$$
$$= 16\pi(2 - 0) - \pi\left(\frac{32}{5} - 0\right)$$
$$= 32\pi - \frac{32\pi}{5}$$
$$= \boxed{\frac{128}{5}\pi}$$

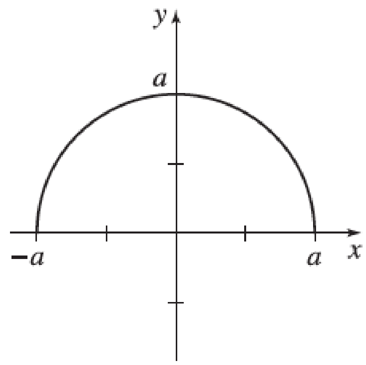
**(b)** $$V = \pi\int_0^2 [4-(-1)]^2\,dx - \pi\int_0^2 (y+1)^2\,dx$$
$$= \pi\int_0^2 25\,dx - \pi\int_0^2 \left(x^2+1\right)^2 dx$$
$$= [25\pi x]_0^2 - \pi\int_0^2 \left(x^4+2x^2+1\right) dx$$
$$= 50\pi - \pi\left[\frac{x^5}{5}+\frac{2x^3}{3}+x\right]_0^2$$
$$= 50\pi - \pi\left[\frac{32}{5}+\frac{16}{3}+2-0\right]$$
$$= 50\pi - \frac{206\pi}{15}$$
$$= \boxed{\frac{544}{15}\pi}$$

**(c)** $$V = \pi\int_0^2 (4-y)^2\,dx$$
$$= \pi\int_0^2 \left(16-8y+y^2\right) dx$$
$$= \pi\int_0^2 \left(16-8x^2+x^4\right) dx$$
$$= \pi\left[16x-\frac{8x^3}{3}+\frac{x^5}{5}\right]_0^2$$
$$= \pi\left(32-\frac{64}{3}+\frac{32}{5}-0\right)$$
$$= \boxed{\frac{256}{15}\pi}$$

**(d)** $$V = \pi\int_0^2 (a-y)^2\,dx - \pi\int_0^2 (a-4)^2\,dx$$
$$= \pi\int_0^2 (a^2-2ay+y^2)\,dx - \pi\int_0^2 (a^2-8a+16)\,dx$$
$$= \pi\int_0^2 \left(a^2-2ax^2+x^4\right) dx - \pi\int_0^2 (a^2-8a+16)\,dx$$
$$= \pi\left[a^2x-\frac{2ax^3}{3}+\frac{x^5}{5}-a^2x+8ax-16x\right]_0^2$$
$$= \pi\left(2a^2-\frac{16a}{3}+\frac{32}{5}-2a^2+16a-32\right)$$
$$= \boxed{\frac{160a-384}{15}\pi}$$

## Applications and Extensions

51. **(a)** The graph is an upper semicircle, centered at (0, 0) and with radius $a$.




**(b)** The radius of each disk when revolving around the $x$-axis is $y - 0 = \sqrt{a^2 - x^2}$. Therefore the volume is

$$V = \pi \int_{-a}^{a} \left(\sqrt{a^2 - x^2}\right)^2 dx$$

Because the function is an even function and is therefore symmetrical around the $y$-axis, we can make this slightly faster to calculate by integrating from 0 to $a$ and then doubling the result:

$$V = 2\left[\pi \int_0^a \left(\sqrt{a^2 - x^2}\right)^2 dx\right] = 2\pi \int_0^a (a^2 - x^2)dx = 2\pi \left[a^2 x - \frac{1}{3}x^3\right]_0^a$$

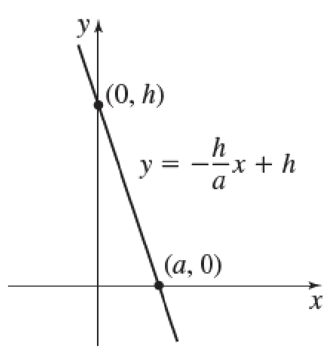
$$= 2\pi \left\{\left[a^2(a) - \frac{1}{3}(a)^3\right] - 0\right\} = 2\pi \cdot \frac{2}{3}a^3 = \boxed{\frac{4}{3}\pi a^3}$$

52. **(a)** $y - a = \dfrac{0 - h}{a - 0}(x - 0)$

$$= -\frac{h}{a}x$$

$$\boxed{y = -\frac{h}{a}x + h}$$

**(b)**



**(c)** $x = -\frac{a}{h}(y-h)$

$$\begin{aligned}
V &= \pi \int_0^h (x-0)^2\,dy \\
&= \pi \int_0^h x^2\,dy \\
&= \pi \int_0^h \left[-\frac{a}{h}(y-h)\right]^2 dy \\
&= \frac{\pi a^2}{h^2} \int_o^h (y-h)^2\,dy \\
&= \frac{\pi a^2}{h^2} \left[\frac{(y-h)^3}{3}\right]_0^h \\
&= \frac{\pi a^2}{3h^2} (y-h)_0^h \\
&= \frac{\pi a^2}{3h^2} \left[0-(-h)^3\right] \\
&= \frac{\pi a^2 h^3}{3h^2} \\
&= \boxed{\frac{a^2 h}{3}\pi}
\end{aligned}$$

53. **(a)** The radius of each disk when revolving around the $x$-axis is $y - 0 = \frac{1}{x^2+4}$. Therefore the volume is

$$V = \pi \int_0^1 \left(\frac{1}{x^2+4}\right)^2 dx = \boxed{\pi \int_0^1 \frac{1}{(x^2+4)^2}\,dx}$$

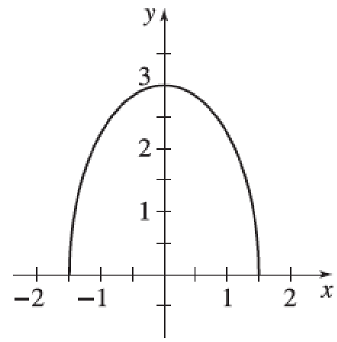
**(b)** Using an online algebra and calculus app, this equals

$$\boxed{\frac{2 + 5\tan^{-1}\left(\frac{1}{2}\right)}{80}\pi \approx 0.170}$$

54. **(a)**
$$\begin{aligned}
V &= \pi \int_1^e (y-0)^2\,dx \\
&= \boxed{\pi \int_1^e (\ln x)^2\,dx}
\end{aligned}$$

**(b)** $\boxed{(e-2)\pi \approx 2.256}$

55. **(a)** The ellipse is centered at (0, 0) and has semi-major axis length 3 vertically and semi-minor axis length $\frac{3}{2}$ horizontally.



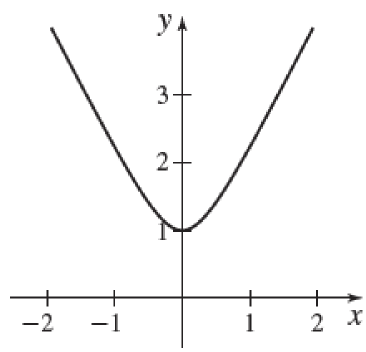

**(b)** The radius of each disk when revolving around the $x$-axis is $y - 0 = \sqrt{9-4x^2}$. Therefore the volume is

$$V = \pi \int_{-3/2}^{3/2} \left(\sqrt{9-4x^2}\right)^2 dx$$

Because the function is an even function and is therefore symmetrical around the $y$-axis, we can make this slightly faster to calculate by integrating from 0 to $\frac{3}{2}$ and doubling the result:

$$V = 2\left[\pi \int_0^{3/2} \left(\sqrt{9-4x^2}\right)^2 dx\right] = 2\pi \int_0^{3/2} \left(9-4x^2\right) dx = 2\pi \left[9x - \frac{4}{3}x^3\right]_0^{3/2}$$
$$= 2\pi \left\{\left[9\left(\frac{3}{2}\right) - \frac{4}{3}\left(\frac{3}{2}\right)^3\right] - 0\right\} = 2\pi \cdot 9 = \boxed{18\pi}$$

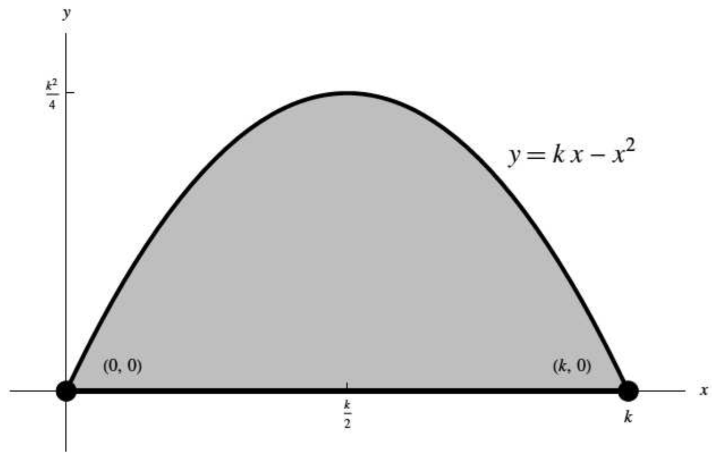
56. **(a)**




**(b)** $V = \pi \int_{-1}^{1} (y-0)^2 \, dx$

$$= \pi \int_{-1}^{1} y^2 \, dx$$
$$= \pi \int_{-1}^{1} \left(4x^2+1\right) dx$$
$$= \pi \left[\frac{4x^3}{3} + x\right]_{-1}^{1}$$
$$= \pi \left[\left(\frac{4}{3}+1\right) - \left(\frac{-4}{3} - 1\right)\right]$$
$$= \boxed{\frac{14}{3}\pi}$$

57. The graph of $y = kx - x^2$ intersects the $x$-axis at $x = 0$ and again at $x = k$. A graph of $kx - x^2$, with the region in question shaded, is below:




**(a)** When the region is revolved around the $x$-axis, we use the disk method; the radius is $kx - x^2$, so the volume is

$$V_x = \pi \int_0^k \left(kx - x^2\right)^2 dx = \pi \int_0^k \left(k^2x^2 - 2kx^3 + x^4\right) dx = \pi \left[\frac{1}{3}k^2x^3 - \frac{1}{2}kx^4 + \frac{1}{5}x^5\right]_0^k = \boxed{\frac{1}{30}\pi k^5}.$$

**(b)** This is just the area between the function and the $x$-axis, so the area is

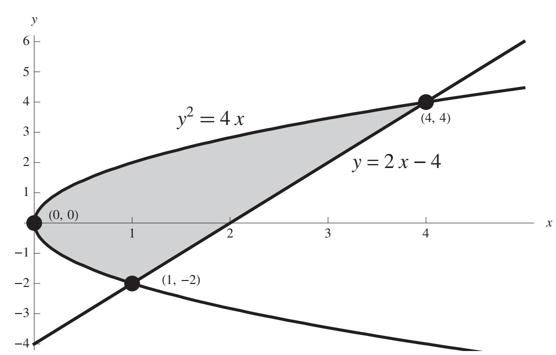
$$A = \int_0^k \left(kx - x^2\right) dx = \left[\frac{1}{2}kx^2 - \frac{1}{3}x^3\right]_0^k = \left[\frac{1}{2}k(k)^2 - \frac{1}{3}(k)^3\right] - 0 = \boxed{\frac{k^3}{6}}$$

58. **(a)** The graphs of $y = 2x + b$ and $y^2 = 4x$ intersect when $(2x + b)^2 = 4x$. Expanding this equation gives $4x^2 + (4b - 4)x + b^2 = 0$; this quadratic equation has roots

$$x = \frac{4 - 4b \pm \sqrt{(4b-4)^2 - 16b^2}}{8} = \frac{1}{2}(1 - b \pm \sqrt{1 - 2b}).$$

So there are two distinct points of intersection when the roots of this quadratic equation are real and distinct, which occurs when $1 - 2b > 0$, or when $b < \frac{1}{2}$.

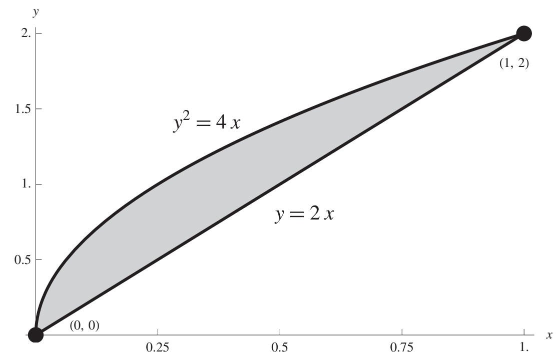
**(b)** When $b = -4$, the region is shown below:




To find the area of this region, we partition along the $y$-axis. Solving these two equations for $x$ gives $x = \frac{y^2}{4}$ and $x = \frac{y+4}{2} = \frac{1}{2}y + 2$, so the area of the region is

$$\int_{-2}^{4} \left(\frac{1}{2}y + 2 - \frac{y^2}{4}\right) dy = \left[\frac{1}{4}y^2 + 2y - \frac{1}{12}y^3\right]_{-2}^{4} = \boxed{9}.$$

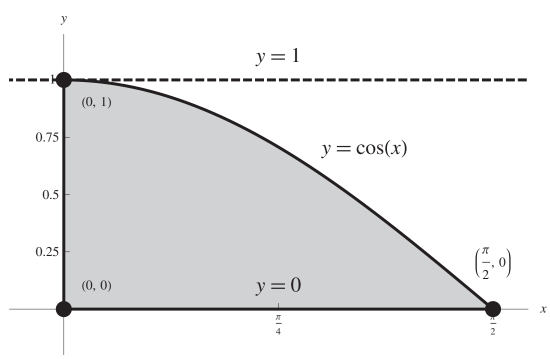
**(c)** When $b = 0$, the region is shown below:




Revolving this region about the $x$-axis, we use the washer method; the outer radius is $\sqrt{4x}$ and the inner radius is $2x$, so that the volume is

$$V = \pi \int_0^1 \left( \left( \sqrt{4x} \right)^2 - (2x)^2 \right) dx = \pi \int_0^1 \left( 4x - 4x^2 \right) dx = \pi \left[ 2x^2 - \frac{4}{3}x^3 \right]_0^1 = \boxed{\frac{2}{3}\pi}.$$

59. The region is shown below, with the line of revolution shown as a dashed line:




Revolving about the line $y = 1$, we use the washer method. The outer radius is $1 - 0 = 1$, and the inner radius is $1 - \cos x$, so the volume is

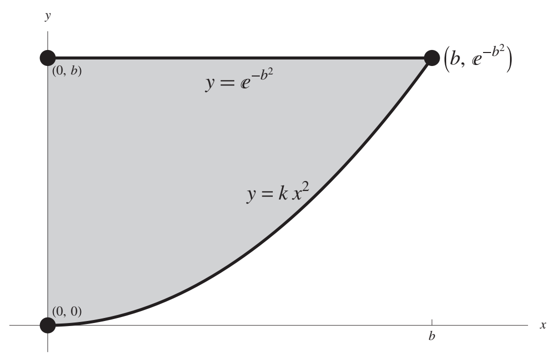
$$\begin{aligned} V &= \pi \int_0^{\pi/2} \left( 1^2 - (1 - \cos x)^2 \right) dx = \pi \int_0^{\pi/2} \left( 2\cos x - \cos^2 x \right) dx \\ &= \pi \int_0^{\pi/2} \left( 2\cos x - \left( \frac{1 + \cos 2x}{2} \right) \right) dx \\ &= \pi \int_0^{\pi/2} \left( 2\cos x - \frac{1}{2} - \frac{1}{2}\cos 2x \right) dx \\ &= \pi \left[ 2\sin x - \frac{1}{2}x - \frac{1}{4}\sin 2x \right]_0^{\pi/2} \\ &= \pi \left( 2 - \frac{\pi}{4} \right) = \boxed{2\pi - \frac{\pi^2}{4}}. \end{aligned}$$

60. A plot of this region is identical to the plot for Problem 59, except that the line of rotation is $y = -1$ instead of $y = 1$. Again we use the washer method; this time the outer radius is $\cos x - (-1) = 1 + \cos x$ and the inner radius is $0 - (-1) = 1$. Therefore the volume is

$$\begin{aligned} V &= \pi \int_0^{\pi/2} \left((1+\cos x)^2 - 1^2\right) dx = \pi \int_0^{\pi/2} \left(2\cos x + \cos^2 x\right) dx \\ &= \pi \int_0^{\pi/2} \left(2\cos x + \frac{1+\cos 2x}{2}\right) dx = \pi \int_0^{\pi/2} \left(2\cos x + \frac{1}{2} + \frac{1}{2}\cos 2x\right) dx \\ &= \pi \left[2\sin x + \frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\pi/2} \\ &= \pi\left(2 + \frac{\pi}{4}\right) = \boxed{2\pi + \frac{\pi^2}{4}}. \end{aligned}$$

## Challenge Problems

61. $P(x)$ together with the line $y = e^{-b^2}$ is below:




**(a)** No matter what $k$ is, $P(0) = 0$, so that $(0, 0)$ is always on the graph. Evaluating at $b$, we also have $P(b) = kb^2 = e^{-b^2}$, so that $k = \frac{1}{b^2}e^{-b^2}$. Therefore $P(x) = \frac{1}{b^2}e^{-b^2}x^2$.

**(b)** The line $y = e^{-b^2}$ intersects the curve at $x = b$. Also, for $0 \le x \le b$, we have $\frac{x^2}{b^2} \le 1$ and so $P(x) = \frac{x^2}{b^2}e^{-b^2} \le e^{-b^2}$, so that the region is bounded above by $y = e^{-b^2}$ and below by $P(x)$. The left edge of the region is at $x = 0$ and the right edge is at $x = b$. Using the shell method, we get for the volume

$$\begin{aligned} V = 2\pi \int_0^b x\left(e^{-b^2} - \frac{1}{b^2}e^{-b^2}x^2\right) dx &= 2\pi e^{-b^2} \int_0^b \left(x - \frac{1}{b^2}x^3\right) dx \\ &= 2\pi e^{-b^2}\left[\frac{1}{2}x^2 - \frac{1}{4b^2}x^4\right]_0^b = \boxed{\frac{1}{2}\pi\frac{b^2}{e^{b^2}}}. \end{aligned}$$

**(c)** Regarding the volume $V$ as a function of $b$, it reaches its maximum either at a critical point or at $b = 0$, which is the left end of its interval. However, we need not consider $b = 0$ since we are assuming $b > 0$. Now,

$$V'(b) = \pi b e^{-b^2} + \frac{1}{2}\pi b^2 e^{-b^2} \cdot (-2b) = \pi e^{-b^2}(b - b^3),$$

so the only critical points are when $b - b^3 = 0$, so for $b = 0$ or $b = \pm 1$. Since we are assuming $b > 0$, the only relevant critical point is $b = 1$. Then

$$V(1) = \frac{1}{2}\pi \cdot 1^2 e^{-1^2} = \frac{1}{2}\pi e^{-1/2} \approx 0.578.$$

Since $\lim_{b\to\infty} V(b) = 0$ as the exponential term dominates, it follows that $\boxed{b = 1}$ is in fact a global maximum.

62. Since $\cosh x \geq 1$ for all values of $x$, it follows that

$$a\cosh\left(\frac{x}{a}\right) + b - a \geq a + b - a = b,$$

so assuming $b > 0$, we see that $y \geq 0$ for any $x$. Then using the disk method, the radius of each disk is $a\cosh\left(\frac{x}{a}\right) + b - a$, so the volume is

$$\begin{aligned}
V &= \pi \int_0^1 \left(a\cosh\left(\frac{x}{a}\right) + b - a\right)^2 dx \\
&= \pi \int_0^1 \left(a^2\cosh^2\left(\frac{x}{a}\right) + 2(b-a)a\cosh\left(\frac{x}{a}\right) + (b-a)^2\right) dx \\
&= \pi \int_0^1 \left(a^2\left(\frac{1+\cosh 2\left(\frac{x}{a}\right)}{2}\right) + 2(b-a)a\cosh\left(\frac{x}{a}\right) + (b-a)^2\right) dx \\
&= \pi \int_0^1 \left(\frac{a^2}{2}\cosh 2\left(\frac{x}{a}\right) + \frac{a^2}{2} + 2(b-a)a\cosh\left(\frac{x}{a}\right) + (b-a)^2\right) dx \\
&= \pi\left[\frac{a^3}{4}\sinh 2\left(\frac{x}{a}\right) + \frac{a^2}{2}x + 2(b-a)a^2\sinh\left(\frac{x}{a}\right) + (b-a)^2 x\right]_0^1 \\
&= \boxed{\pi\left(\frac{a^3}{4}\sinh\frac{2}{a} + \frac{a^2}{2} + 2(b-a)a^2\sinh\frac{1}{a} + (b-a)^2\right)}.
\end{aligned}$$

## AP® Practice Problems

1. $$\begin{aligned}
V &= \pi \int_0^3 (e^x)^2\, dx \\
&= \pi \int_0^3 \left(e^{2x}\right) dx \\
&= \frac{\pi}{2}\left[e^{2x}\right]_0^3 \\
&= \boxed{\frac{\pi}{2}\left(e^6 - 1\right)}
\end{aligned}$$

   CHOICE B

2. $$\begin{aligned}
V &= \pi \int_{\pi/4}^{3\pi/4} (\csc x - 0)^2\, dx \\
&= \pi \int_{\pi/4}^{3\pi/4} \left(\csc^2 x\right) dx \\
&= -\pi[\cot x]_{\pi/4}^{3\pi/4} \\
&= -\pi\left(\cot\frac{3\pi}{4} - \cot\frac{\pi}{4}\right) \\
&= -\pi(-1-1) \\
&= \boxed{2\pi}
\end{aligned}$$

   CHOICE D

3. $$\begin{aligned} V &= \pi \int_0^8 (2-0)^2\,dy - \pi \int_0^8 (x-0)^2\,dy \\ &= \pi \int_0^8 4dy - \pi \int_0^8 x^2\,dy \\ &= \pi \int_0^8 4dy - \int_0^8 y^{2/3}\,dy \\ &= 4\pi[y]_0^8 - \frac{3\pi}{5}\left[y^{5/3}\right]_0^8 \\ &= 4\pi(8-0) - \frac{3\pi}{5}\left(8^{5/3}-0\right) \\ &= 32\pi - \frac{96\pi}{5} \\ &= \boxed{\frac{64\pi}{5}} \end{aligned}$$

CHOICE B

4. Using the washer method, the outer radius is $\frac{\pi}{6} - 0 = \frac{\pi}{6}$ and the inner radius (of the hole) is $x - 0 = x$.

Since $y = \sin x$, therefore

if $0 \le x \le \frac{\pi}{6}$

then $\sin 0 \le \sin x \le \sin \frac{\pi}{6}$

or $0 \le y \le \frac{1}{2}$

so the limits of integration are 0 and $\frac{1}{2}$.

Also, $x = \arcsin y$, so

$$\begin{aligned} V &= \pi \int_0^{1/2} \left[\left(\frac{\pi}{6}\right)^2 - x^2\right] dy \\ &= \boxed{\pi \int_0^{1/2} \left[\left(\frac{\pi}{6}\right)^2 - (\arcsin y)^2\right] dy} \end{aligned}$$

CHOICE B

5. $V = \pi \int_{-1}^{1} (x-0)^2 \, dy$

$= 2 \cdot \left[ \pi \int_{0}^{1} (x-0)^2 \, dy \right]$, since the function is symmetrical about the $x$-axis

$= 2\pi \int_{0}^{1} \left( \sqrt{4-4y^2} \right)^2 dy$

$= 2\pi \int_{0}^{1} \left(4-4y^2\right) dy$

$= 2\pi \left[ 4y - \frac{4y^3}{3} \right]_0^1$

$= 2\pi \left( 4 - \frac{4}{3} - 0 \right)$

$= \boxed{\frac{16}{3}\pi}$

CHOICE D

6. $V = \pi \int_{0}^{2} (e^x - 0)^2 \, dx - \pi \int_{0}^{2} (1-0)^2 \, dx$

$= \pi \int_{0}^{2} e^{2x} dx - \pi \int_{0}^{2} dx$

$= \frac{\pi}{2} \left[e^{2x}\right]_0^2 - \pi x|_0^2$

$= \frac{\pi}{2} \left(e^4 - e^0\right) - 2\pi$

$= \boxed{\frac{\pi}{2} \left(e^4 - 5\right)}$

CHOICE A

7. $y = \sqrt{x}$

$x = y^2$

$V = \pi \int_{0}^{2} (4-0)^2 \, dy - \pi \int_{0}^{2} (4-x)^2 \, dy$

$= \pi \int_{0}^{2} (4-0)^2 \, dy - \pi \int_{0}^{2} \left(4-y^2\right)^2 dy$

$= \pi \int_{0}^{2} 16 dy - \pi \int_{0}^{2} \left(y^4 - 8y^2 + 16\right) dy$

$= \pi \int_{0}^{2} \left(-y^4 + 8y^2\right) dy$

$= \pi \left[ \frac{-y^5}{5} + \frac{-8y^3}{3} \right]_0^2$

$= \pi \left[ \frac{-32}{5} + \frac{64}{3} - 0 \right]$

$= \boxed{\frac{224}{15}\pi}$

CHOICE B

8. $$V = \pi \int_0^2 (5-y)^2\,dx - \pi \int_0^2 (5-4)^2\,dx$$
$$= \pi \int_0^2 \left(5-x^2\right)^2 dx - \pi \int_0^2 dx$$
$$= \boxed{\pi \int_0^2 \left[\left(5-x^2\right)^2 - 1\right] dx}$$

CHOICE C

9. Determine the point of intersection of $y = x+2$ and $y = 2x$, for the upper limit on the integral.

$$x + 2 = 2x$$
$$x = 2$$
$$V = \pi \int_0^2 [(x+2)-(-1)]^2\,dx - \pi \int_0^2 [2x-(-1)]^2\,dx$$
$$= \pi \int_0^2 (x+3)^2\,dx - \pi \int_0^2 (2x+1)^2\,dx$$
$$= \pi \int_0^2 \left(x^2+6x+9\right) dx - \pi \int_0^2 \left(4x^2+4x+1\right) dx$$
$$= \pi \int_0^2 \left(-3x^2+2x+8\right) dx$$
$$= \pi \left[-x^3+x^2+8x\right]_0^2$$
$$= \pi(-8+4+16-0)$$
$$= \boxed{12\pi}$$

CHOICE B

10. **(a)** To determine the point(s) of intersection of $f(x)$ and $g(x)$, let $f(x) = g(x)$:

$$-x^2+4x = 3x^2-12x+12$$
$$4x^2-16x+12 = 0$$
$$x^2-4x+3 = 0$$
$$(x-3)(x-1) = 0$$
$$x = 3 \quad \text{or} \quad x = 1$$

The points of intersection are $(3, f(3)) = \boxed{(3,3)}$ and $(1, f(1)) = \boxed{(1,3)}$.

**(b)** $$A = \int_1^3 \left[\left(-x^2+4x\right) - \left(3x^2-12x+12\right)\right] dx$$
$$= \int_1^3 \left(-4x^2+16x-12\right) dx$$
$$= \left[\frac{-4x^3}{3} + 8x^2 - 12x\right]_1^3$$
$$= [-4(9)+72-36] - \left(\frac{-4}{3}+8-12\right)$$
$$= \boxed{\frac{16}{3}}$$

**(c)** $V = \pi \int_1^3 \left[ \left( -x^2 + 4x \right)^2 - \left( 3x^2 - 12x + 12 \right)^2 \right] dx$

**(d)** $V = \pi \int_1^3 \left[ (f(x) - (-2))^2 - (g(x) - (-2))^2 \right] dx$

$= \pi \int_1^3 \left[ \left( -x^2 + 4x + 2 \right)^2 - \left( 3x^2 - 12x + 14 \right)^2 \right] dx$

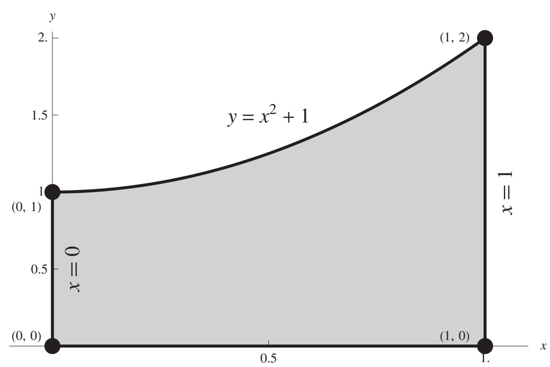
## 6.3 Volume of a Solid of Revolution: Cylindrical Shells

### Concepts and Vocabulary

1. False. When using the method of shells, the integration occurs along the axis perpendicular to the axis of revolution.
2. False. It is given by $\pi R^2 h - \pi r^2 h = \pi (R^2 - r^2) h$.
3. False. Sections 6.2 and 6.3 present methods for determining such volumes under rotation about either the $x$ or $y$-axis.
4. True. See the discussion in the text. The $x$ represents the radius of each cylinder, $f(x)$ represents the height, and $dx$ represents the thickness.

### Skill Building

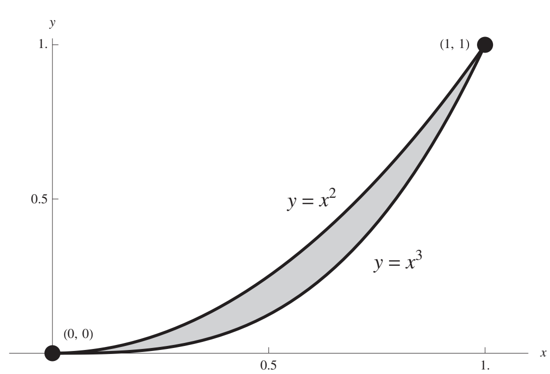
5. The region is shown below:




When revolving about the $y$-axis, using the shell method, each cylinder has a radius of $x$ and a height of $x^2 + 1$, so the volume is

$$V = 2\pi \int_0^1 x(x^2 + 1)\, dx = 2\pi \int_0^1 \left( x^3 + x \right) dx = 2\pi \left[ \frac{1}{4}x^4 + \frac{1}{2}x^2 \right]_0^1 = \boxed{\frac{3}{2}\pi}.$$

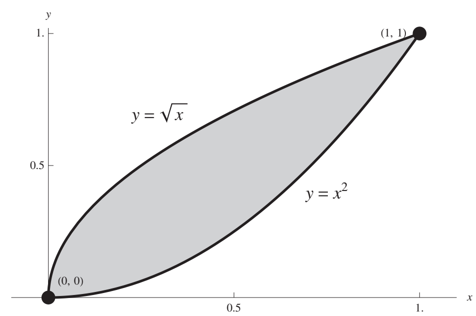
6. The region is shown below:



When revolving about the $y$-axis using the shell method, each cylinder has a radius of $x$ and a height of $x^2 - x^3$, so the volume is

$$V = 2\pi \int_0^1 x(x^2 - x^3)\,dx = 2\pi \int_0^1 \left(x^3 - x^4\right)\,dx = 2\pi \left[\frac{1}{4}x^4 - \frac{1}{5}x^5\right]_0^1 = \boxed{\frac{1}{10}\pi}.$$

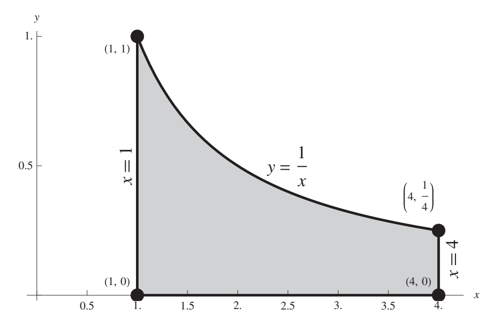
7. The region is shown below:




When revolving about the $y$-axis using the shell method, each cylinder has a radius of $x$ and a height of $\sqrt{x} - x^2$, so the volume is

$$V = 2\pi \int_0^1 x\left(\sqrt{x} - x^2\right)\,dx = 2\pi \int_0^1 \left(x^{3/2} - x^3\right)\,dx = 2\pi \left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^1 = \boxed{\frac{3}{10}\pi}.$$

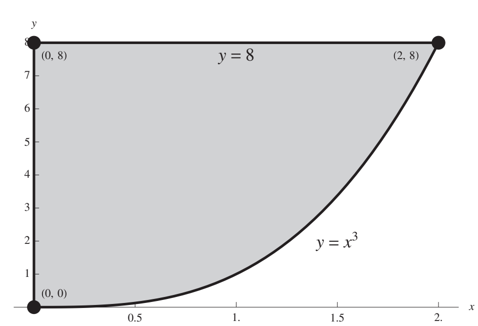
8. The region is shown below:




When revolving about the $y$-axis using the shell method, each cylinder has a radius of $x$ and a height of $\frac{1}{x}$, so the volume is

$$V = 2\pi \int_1^4 x \cdot \frac{1}{x}\,dx = 2\pi \int_1^4 1\,dx = \boxed{6\pi}.$$

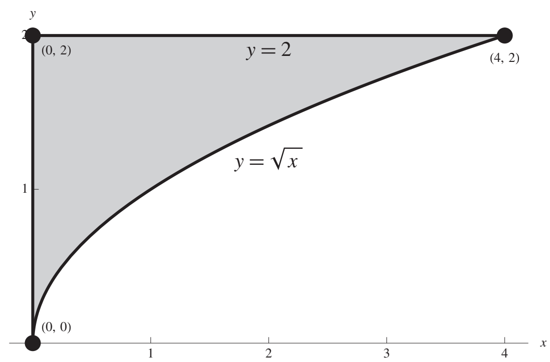
9. The region is shown below:



When revolving about the $x$-axis using the shell method, we first solve $y = x^3$ for $x$, giving $x = y^{1/3}$. Then each cylinder has a radius of $y$ and a height of $y^{1/3}$, so the volume is

$$V = 2\pi \int_0^8 y \cdot y^{1/3}\, dy = 2\pi \int_0^8 y^{4/3}\, dy = 2\pi \left[\frac{3}{7} y^{7/3}\right]_0^8 = \boxed{\frac{768}{7}\pi}.$$

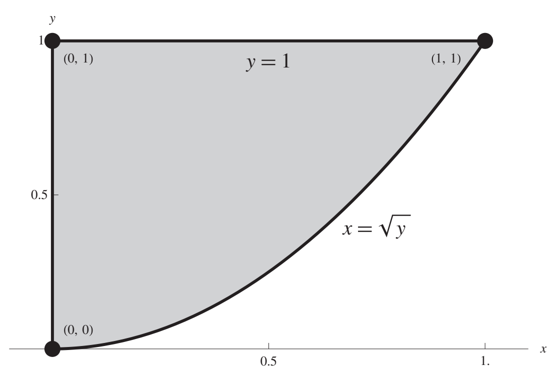
10. The region is shown below:




When revolving about the $x$-axis using the shell method, we first solve $y = \sqrt{x}$ for $x$, giving $x = y^2$. Then each cylinder has a radius of $y$ and a height of $y^2$, so the volume is

$$V = 2\pi \int_0^2 y \cdot y^2\, dy = 2\pi \int_0^2 y^3\, dy = 2\pi \left[\frac{1}{4} y^4\right]_0^2 = \boxed{8\pi}.$$

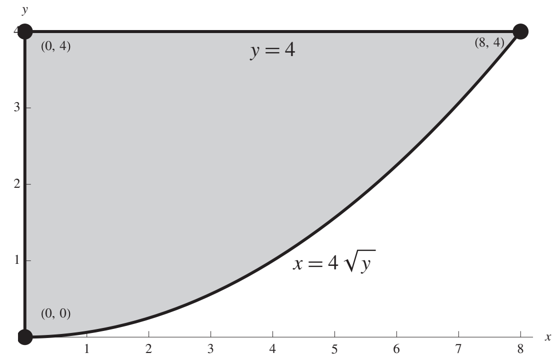
11. The region is shown below:




When revolving about the $x$-axis using the shell method, each cylinder has a radius of $y$ and a height of $\sqrt{y}$, so the volume is

$$V = 2\pi \int_0^1 y \cdot \sqrt{y}\, dy = 2\pi \int_0^1 y^{3/2}\, dy = 2\pi \left[\frac{2}{5} y^{5/2}\right]_0^1 = \boxed{\frac{4}{5}\pi}.$$

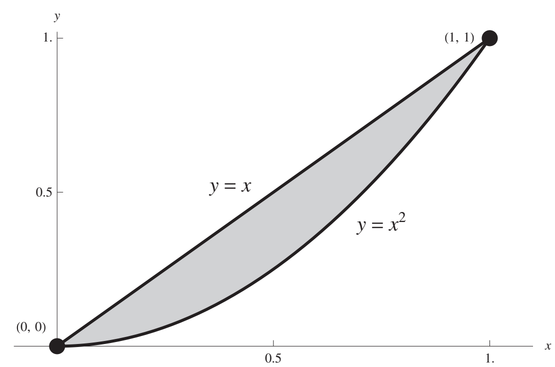
12. The region is shown below:




When revolving about the $x$-axis using the shell method, each cylinder has a radius of $y$ and a height of $4\sqrt{y}$, so the volume is

$$V = 2\pi \int_0^4 y \cdot 4\sqrt{y}\,dy = 8\pi \int_0^4 y^{3/2}\,dy = 8\pi \left[\frac{2}{5}y^{5/2}\right]_0^4 = \boxed{\frac{512}{5}\pi}.$$

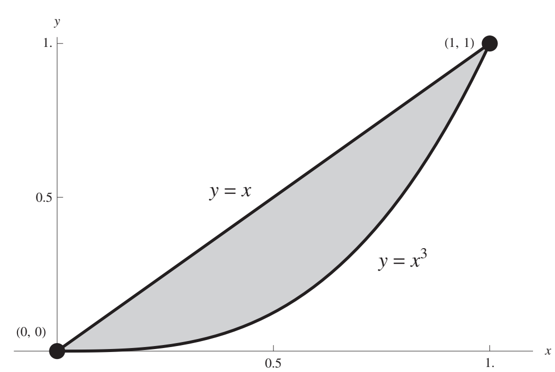
13. The region is shown below:




When revolving about the $x$-axis using the shell method, we first solve the two equations for $x$, giving $x = y$ and $x = \sqrt{y}$. Then each shell has a radius of $y$ and a height of $\sqrt{y} - y$, so that the volume is

$$V = 2\pi \int_0^1 y\left(\sqrt{y} - y\right)\,dy = 2\pi \int_0^1 \left(y^{3/2} - y^2\right)\,dy = 2\pi \left[\frac{2}{5}y^{5/2} - \frac{1}{3}y^3\right]_0^1 = \boxed{\frac{2}{15}\pi}.$$

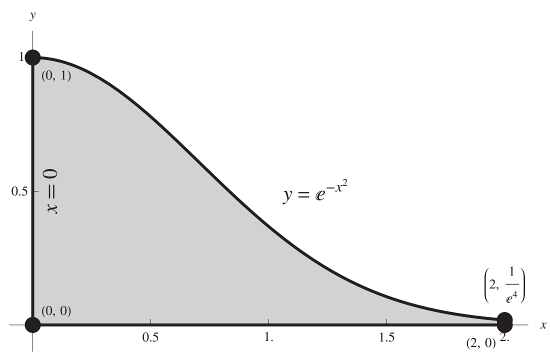
14. The region is shown below:



When revolving about the $x$-axis using the shell method, we first solve the two equations for $x$, giving $x = y$ and $x = y^{1/3}$. Then each shell has a radius of $y$ and a height of $y^{1/3} - y$, so that the volume is

$$V = 2\pi \int_0^1 y\left(y^{1/3} - y\right)\,dy = 2\pi \int_0^1 \left(y^{4/3} - y^2\right)\,dy = 2\pi\left[\frac{3}{7}y^{7/3} - \frac{1}{3}y^3\right]_0^1 = \boxed{\frac{4}{21}\pi}.$$

15. The region is shown below:



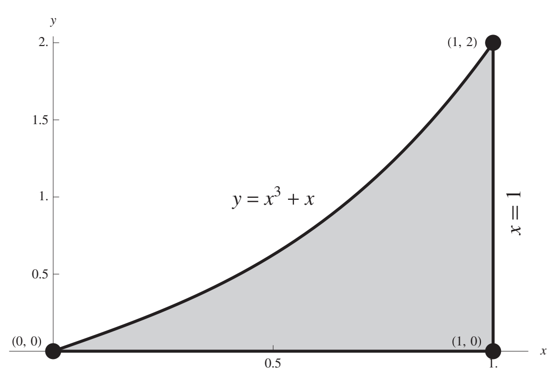

When revolving about the $y$-axis using the shell method, each shell has a radius of $x$ and a height of $e^{-x^2}$, so that the volume is

$$V = 2\pi \int_0^2 xe^{-x^2}\,dx.$$

Now use the substitution $u = -x^2$, so that $du = -2x\,dx$. Then $x = 0$ corresponds to $u = 0$, and $x = 2$ to $u = -4$, so we get

$$V = -\frac{1}{2}\cdot 2\pi \int_0^{-4} e^u\,du = -\pi\left[e^u\right]_0^{-4} = \boxed{\pi\left(1 - \frac{1}{e^4}\right)}.$$

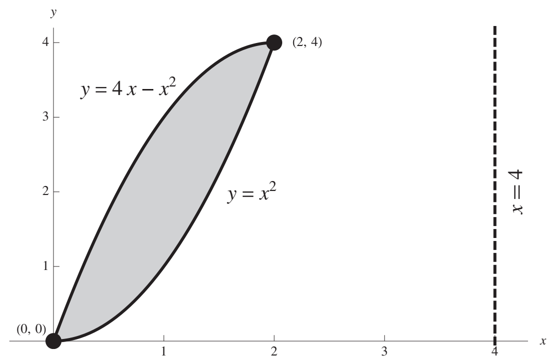
16. The region is shown below:




When revolving about the $y$-axis using the shell method, each shell has a radius of $x$ and a height of $x^3 + x$, so that the volume is

$$V = 2\pi \int_0^1 x\left(x^3 + x\right)\,dx = 2\pi \int_0^1 \left(x^4 + x^2\right)\,dx = 2\pi\left[\frac{1}{5}x^5 + \frac{1}{3}x^3\right]_0^1 = \boxed{\frac{16}{15}\pi}.$$

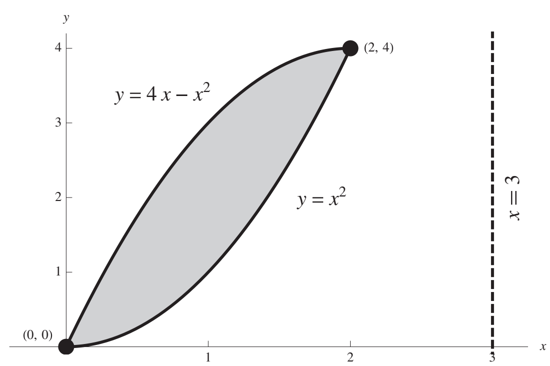
17. The region is shown below:




When revolving about $x = 4$ using the shell method, each shell has a radius of $4 - x$ (the distance from $x$ to 4) and a height of $4x - x^2 - x^2 = 4x - 2x^2$. Therefore the volume is

$$V = 2\pi \int_0^2 (4-x)(4x-2x^2)\,dx = 2\pi \int_0^2 (2x^3-12x^2+16x)\,dx = 2\pi \left[\frac{1}{2}x^4 - 4x^3 + 8x^2\right]_0^2 = \boxed{16\pi}.$$

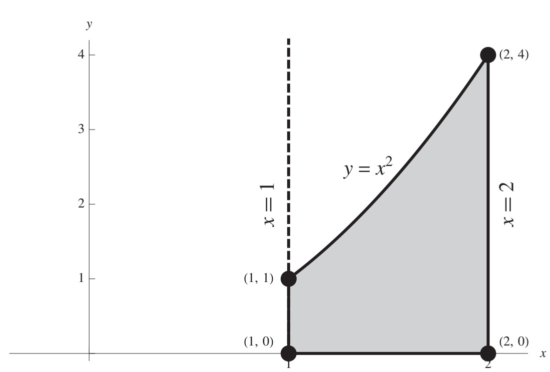
18. The region is shown below:




When revolving about $x = 3$ using the shell method, each shell has a radius of $3 - x$ (the distance from $x$ to 3) and a height of $4x - x^2 - x^2 = 4x - 2x^2$. Therefore the volume is

$$V = 2\pi \int_0^2 (3-x)(4x-2x^2)\,dx = 2\pi \int_0^2 (2x^3-10x^2+12x)\,dx = 2\pi \left[\frac{1}{2}x^4 - \frac{10}{3}x^3 + 6x^2\right]_0^2 = \boxed{\frac{32}{3}\pi}.$$

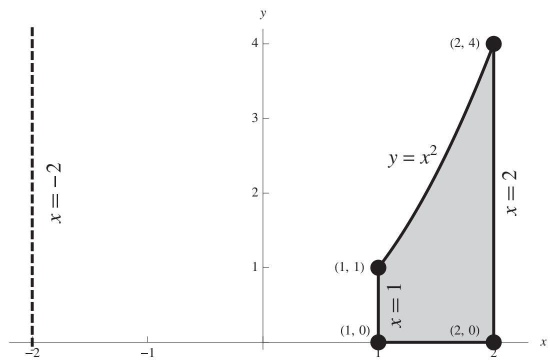
19. The region is shown below:



When revolving about $x = 1$ using the shell method, each shell has a radius of $x - 1$ (the distance from $x$ to 1) and a height of $x^2$. Therefore the volume is

$$V = 2\pi \int_1^2 (x-1)x^2\,dx = 2\pi \int_1^2 (x^3 - x^2)\,dx = 2\pi \left[\frac{1}{4}x^4 - \frac{1}{3}x^3\right]_1^2 = \boxed{\frac{17}{6}\pi}.$$

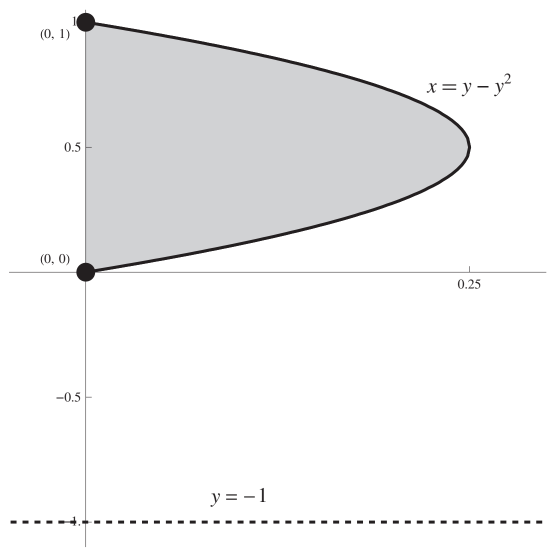
20. The region is shown below:




When revolving about $x = -2$ using the shell method, each shell has a radius of $x - (-2) = x + 3$ (the distance from $x$ to $-2$) and a height of $x^2$. Therefore the volume is

$$V = 2\pi \int_1^2 (x+3)x^2\,dx = 2\pi \int_1^2 (x^3 + 3x^2)\,dx = 2\pi \left[\frac{1}{4}x^4 + x^3\right]_1^2 = \boxed{\frac{43}{2}\pi}.$$

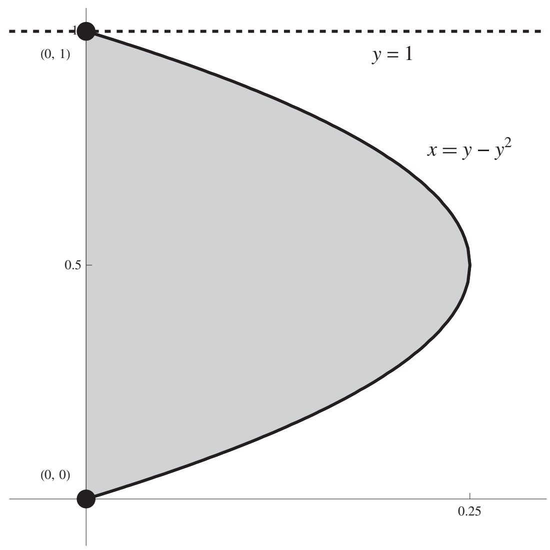
21. The region is shown below:




When revolving about $y = -1$ using the shell method, each shell has a radius of $y - (-1) = y + 1$ (the distance from $y$ to $-1$) and a height of $y - y^2$. Therefore the volume is

$$V = 2\pi \int_0^1 (y+1)(y - y^2)\,dy = 2\pi \int_0^1 (-y^3 + y)\,dy = 2\pi \left[-\frac{1}{4}y^4 + \frac{1}{2}y^2\right]_0^1 = \boxed{\frac{\pi}{2}}.$$

22. The region is shown below:




When revolving about $y = 1$ using the shell method, each shell has a radius of $1 - y$ (the distance from $y$ to 1) and a height of $y - y^2$. Therefore the volume is

$$V = 2\pi \int_0^1 (1-y)(y-y^2)\,dy = 2\pi \int_0^1 (y^3 - 2y^2 + y)\,dy = 2\pi \left[\frac{1}{4}y^4 - \frac{2}{3}y^3 + \frac{1}{2}y^2\right]_0^1 = \boxed{\frac{\pi}{6}}.$$

23. **(a)** To rotate about the $x$-axis, use the disk method; the radius of each disk is $2x^2 - x^3$. Therefore the volume is

$$V = \pi \int_0^2 (2x^2 - x^3)^2\,dx = \pi \int_0^2 (4x^4 - 4x^5 + x^6)\,dx = \pi \left[\frac{4}{5}x^5 - \frac{2}{3}x^6 + \frac{1}{7}x^7\right]_0^2 = \boxed{\frac{128}{105}\pi}.$$

**(b)** To rotate about the $y$-axis, use the shell method; the radius of each shell is $x$ and its height is $2x^2 - x^3$, so the volume is

$$V = 2\pi \int_0^2 x(2x^2 - x^3)\,dx = 2\pi \int_0^2 (2x^3 - x^4)\,dx = 2\pi \left[\frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^2 = \boxed{\frac{16}{5}\pi}.$$

**(c)** In each case, we chose to integrate by partitioning along the $x$-axis, since partitioning along the $y$-axis would entail solving $y = 2x^2 - x^3$ for $x$ and then using one solution as the lower bound and the other as the upper bound.

24. **(a)** To rotate about the $x$-axis, use the washer method; the outer radius of each disk is $-x^2 + 9$ and the inner radius is $2x + 1$. Therefore the volume is

$$\begin{aligned} V &= \pi \int_0^2 \left((-x^2+9)^2 - (2x+1)^2\right)\,dx \\ &= \pi \int_0^2 \left(x^4 - 22x^2 - 4x + 80\right)\,dx \\ &= \pi \left[\frac{1}{5}x^5 - \frac{22}{3}x^3 - 2x^2 + 80x\right]_0^2 \\ &= \boxed{\frac{1496}{15}\pi}. \end{aligned}$$

**(b)** To rotate about the $y$-axis, use the shell method. The radius of each shell is $x$, and its height is $-x^2+9-(2x+1) = -x^2-2x+8$. Therefore the volume is

$$V = 2\pi \int_0^2 x(-x^2-2x+8)\,dx = 2\pi \int_0^2 (-x^3-2x^2+8x)\,dx = 2\pi \left[-\frac{1}{4}x^4 - \frac{2}{3}x^3 + 4x^2\right]_0^2 = \boxed{\frac{40}{3}\pi}.$$

**(c)** In each case, we chose to integrate by partitioning along the $x$-axis, since partitioning along the $y$-axis would entail splitting the integral into two separate integrals, one from $y=0$ to $y=5$ and the other from $y=5$ to $y=9$.

25. **(a)** To rotate about the $x$-axis, use the disk method; the radius of each disk is $x^2+1$, so the volume is

$$V = \pi \int_0^2 \left(x^2+1\right)^2\,dx = \pi \int_0^2 \left(x^4+2x^2+1\right)\,dx = \pi \left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right]_0^2 = \boxed{\frac{206}{15}\pi}.$$

**(b)** To rotate about the $y$-axis, use the shell method; the radius of each shell is $x$ and its height is $x^2+1$, so the volume is

$$V = 2\pi \int_0^2 x(x^2+1)\,dx = 2\pi \int_0^2 (x^3+x)\,dx = 2\pi \left[\frac{1}{4}x^4 + \frac{1}{2}x^2\right]_0^2 = \boxed{12\pi}.$$

**(c)** In each case, we chose to integrate by partitioning along the $x$-axis, since partitioning along the $y$-axis would entail splitting the integral into two separate integrals, one from $y=0$ to $y=1$ and the other from $y=1$ to $y=5$.

26. **(a)** To rotate about the $x$-axis, use the disk method; the radius of each disk is $4x-x^2$, so the volume is

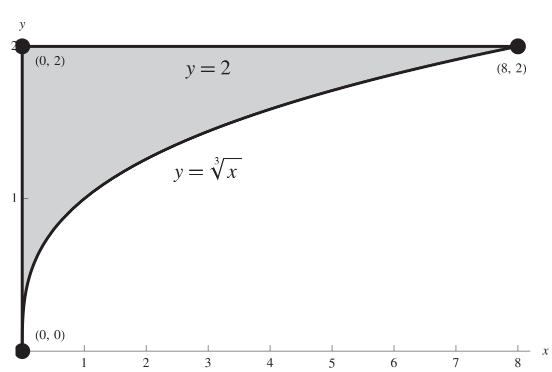
$$V = \pi \int_0^4 (4x-x^2)^2\,dx = \pi \int_0^4 \left(16x^2-8x^3+x^4\right)\,dx = \pi \left[\frac{16}{3}x^3 - 2x^4 + \frac{1}{5}x^5\right]_0^4 = \boxed{\frac{512}{15}\pi}.$$

**(b)** To rotate about the $y$-axis, use the shell method; the radius of each shell is $x$ and its height is $4x-x^2$, so the volume is

$$V = 2\pi \int_0^4 x(4x-x^2)\,dx = 2\pi \int_0^4 (4x^2-x^3)\,dx = 2\pi \left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_0^4 = \boxed{\frac{128}{3}\pi}.$$

**(c)** In each case, we chose to integrate by partitioning along the $x$-axis, since partitioning along the $y$-axis would entail solving $y=4x-x^2$ for $x$ and then using one solution as the lower bound and the other as the upper bound.

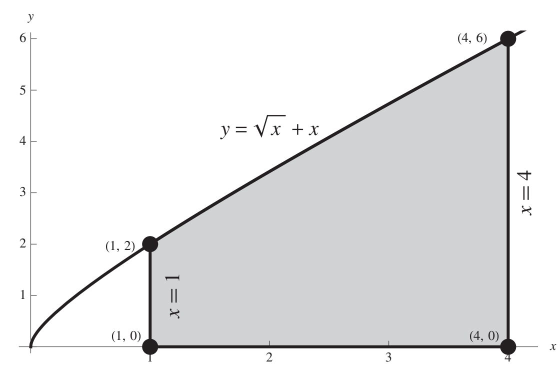
27. The region is shown below:



This could be done using shells along the $y$-axis or washers along the $x$-axis. Since $y = x^{1/3}$ becomes $x = y^3$, which is somewhat simpler, we choose the shell method. Then the radius of each shell is $y$ and its height is $y^3$, so the volume is

$$V = 2\pi \int_0^2 y \cdot y^3\, dy = 2\pi \int_0^2 y^4\, dy = 2\pi \left[\frac{1}{5}y^5\right]_0^2 = \boxed{\frac{64}{5}\pi}.$$

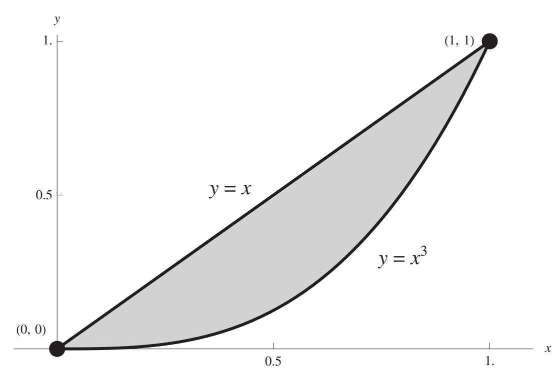
28. The region is shown below:




If we chose disks/washers along the $y$-axis, we would have to solve the equation for $x$, and we would have to split the integral up into two separate integrals. So use shells along the $x$-axis. The radius of each shell is $x$, and the height is $\sqrt{x} + x$, so the volume is

$$V = 2\pi \int_1^4 x(\sqrt{x} + x)\, dx = 2\pi \int_1^4 \left(x^{3/2} + x^2\right) dx = 2\pi \left[\frac{2}{5}x^{5/2} + \frac{1}{3}x^3\right]_1^4 = \boxed{\frac{334}{5}\pi}.$$

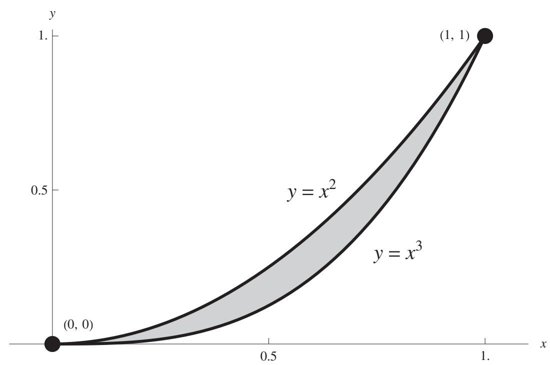
29. The region is shown below:




Either disks/washers along the $y$-axis or shells along the $x$-axis could be used. Since we are given $y$ in terms of $x$, we choose shells. The radius of each shell is $x$, and the height is $x - x^3$, so the volume is

$$V = 2\pi \int_0^1 x(x - x^3)\, dx = 2\pi \int_0^1 (x^2 - x^4)\, dx = 2\pi \left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \boxed{\frac{4}{15}\pi}.$$

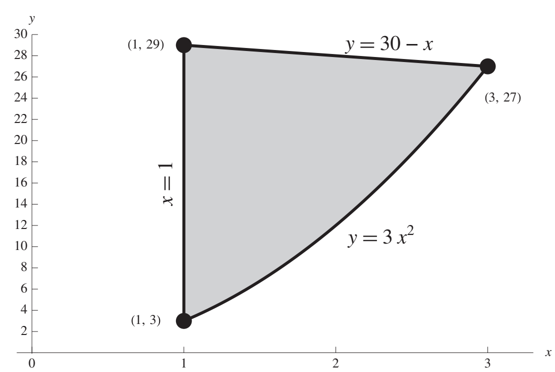
30. The region is shown below:




Either disks/washers along the $x$-axis or shells along the $y$-axis could be used. Since we are given $y$ in terms of $x$, we choose disks/washers. The outer radius of each washer is $x^2$ and the inner radius is $x^3$, so the volume is

$$V = \pi \int_0^1 \left( \left(x^2\right)^2 - \left(x^3\right)^2 \right) \, dx = \pi \int_0^1 \left(x^4 - x^6\right) \, dx = \pi \left[ \frac{1}{5}x^5 - \frac{1}{7}x^7 \right]_0^1 = \boxed{\frac{2}{35}\pi}.$$

31. The region is shown below:




If we use disks along the $y$-axis, we will need two separate integrals, since the outer radius changes equations at $y = 27$. So use shells along the $x$-axis. The radius of each shell is $x$, and the height is $30 - x - 3x^2$, so the volume is

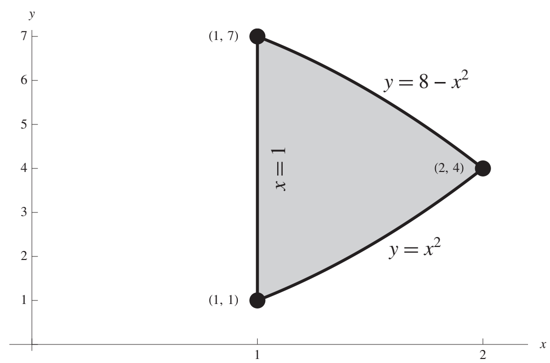
$$V = 2\pi \int_1^3 x\left(30 - x - 3x^2\right) \, dx = 2\pi \int_1^3 \left(30x - x^2 - 3x^3\right) \, dx = 2\pi \left[ 15x^2 - \frac{1}{3}x^3 - \frac{3}{4}x^4 \right]_1^3$$

$$= \boxed{\frac{308}{3}\pi}.$$

32. This is the same region as in Problem 31, but revolved about the $x$-axis. A similar argument applies: if we use shells along the $y$-axis, we would require two separate integrals, so we use the washer method along the $x$-axis. The outer radius of each washer is $30 - x$ and the inner radius is $3x^2$, so the volume is

$$\begin{aligned} V &= \pi \int_1^3 \left( (30-x)^2 - \left(3x^2\right)^2 \right)\, dx \\ &= \pi \int_1^3 \left( -9x^4 + x^2 - 60x + 900 \right)\, dx \\ &= \pi \left[ -\frac{9}{5}x^5 + \frac{1}{3}x^3 - 30x^2 + 900x \right]_1^3 \\ &= \boxed{\frac{16996}{15}\pi}. \end{aligned}$$

33. The region is shown below:



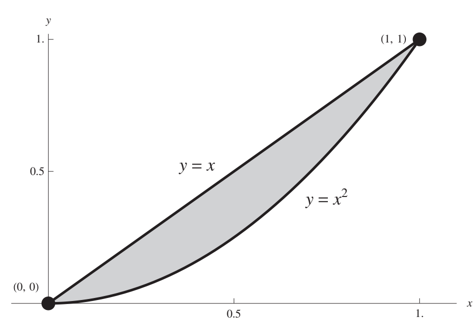

If we use shells along the $y$-axis, we will need two separate integrals, since the outer radius changes equations at $y = 4$. So use washers along the $x$-axis. The outer radius is $8 - x^2$ and the inner radius is $x^2$, so the volume is

$$V = \pi \int_1^2 \left( \left(8 - x^2\right)^2 - \left(x^2\right)^2 \right)\, dx = \pi \int_1^2 \left(64 - 16x^2\right)\, dx = \pi \left[ 64x - \frac{16}{3}x^3 \right]_1^2 = \boxed{\frac{80}{3}\pi}.$$

34. This is the same region as in Problem 33, but revolved about the $y$-axis. A similar argument applies: if we use disks along the $y$-axis, we will need two separate integrals, since the outer radius changes equations at $y = 4$. So use shells along the $x$-axis. The radius of each shell is $x$, and the height is $8 - x^2 - x^2 = 8 - 2x^2$, so the volume is

$$V = 2\pi \int_1^2 x\left(8 - 2x^2\right)\, dx = 2\pi \int_1^2 \left(8x - 2x^3\right)\, dx = 2\pi \left[ 4x^2 - \frac{1}{2}x^4 \right]_1^2 = \boxed{9\pi}.$$

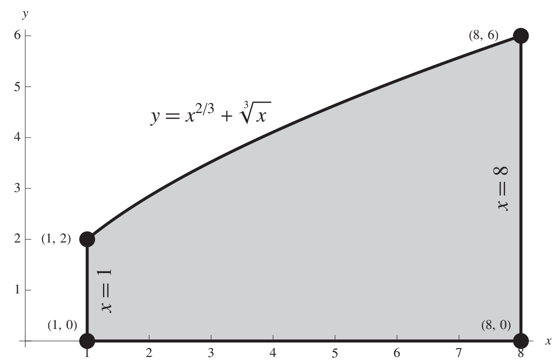
35. The region is shown below:



This can be done either using washers along the $y$-axis or shells along the $x$-axis. Since we are given $y$ in terms of $x$, we choose shells. The radius of each shell is $x$ and the height is $x - x^2$, so the volume is

$$V = 2\pi \int_0^1 x\left(x - x^2\right)\, dx = 2\pi \int_0^1 \left(x^2 - x^3\right)\, dx = 2\pi \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \boxed{\frac{\pi}{6}}.$$

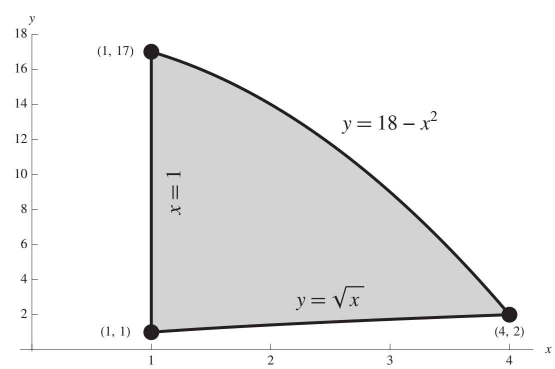
36. The region is shown below:




If we use washers along the $y$-axis, we will have to solve this equation for $x$ in terms of $y$, and also split it into two integrals. So using shells along the $x$-axis is much easier. The radius of each shell is $x$, and the height is $x^{2/3} + x^{1/3}$, so the volume is

$$V = 2\pi \int_1^8 x\left(x^{2/3} + x^{1/3}\right)\, dx = 2\pi \int_1^8 \left(x^{5/3} + x^{4/3}\right)\, dx = 2\pi \left[\frac{3}{8}x^{8/3} + \frac{3}{7}x^{7/3}\right]_1^8 = \boxed{\frac{8403}{28}\pi}.$$

37. The region is shown below:




Revolving about the $y$-axis, if we use washers along the $y$-axis we will require two separate integrals. So use shells along the $x$-axis. The radius of each shell is $x$, and the height is $18 - x^2 - \sqrt{x}$, so the volume is

$$V = 2\pi \int_1^4 x(18 - x^2 - \sqrt{x})\, dx = 2\pi \int_1^4 \left(18x - x^3 - x^{3/2}\right)\, dx = 2\pi \left[9x^2 - \frac{1}{4}x^4 - \frac{2}{5}x^{5/2}\right]_1^4$$

$$= \boxed{\frac{1177}{10}\pi}.$$

38. This is the same region as in Problem 37, but revolved about the $x$-axis. A similar argument applies: if we use shells along the $y$-axis, we will need two separate integrals, since the radius changes equations at $y = 2$. So use washers along the $x$-axis. The outer radius is $18 - x^2$, and the inner radius is $\sqrt{x}$. Therefore the volume is

$$\begin{aligned} V &= \pi \int_1^4 \left( \left(18 - x^2\right)^2 - \left(\sqrt{x}\right)^2 \right) dx \\ &= \pi \int_1^4 \left(x^4 - 36x^2 - x + 324\right) dx \\ &= \pi \left[ \frac{1}{5}x^5 - 12x^3 - \frac{1}{2}x^2 + 324x \right]_1^4 \\ &= \boxed{\frac{4131}{10}\pi}. \end{aligned}$$

## Applications and Extensions

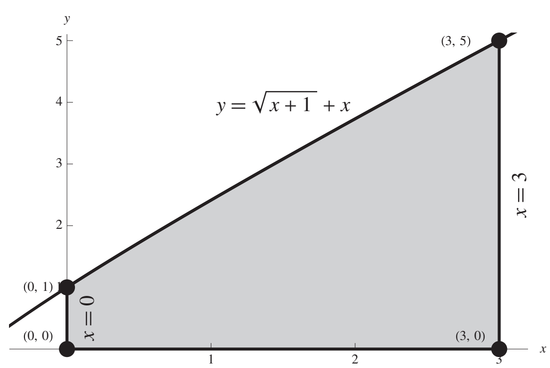
39. Use the shell method along the $x$-axis. The radius of each shell is $x$ and the height is $\frac{1}{(x^2+1)^2}$, so the volume is

$$V = 2\pi \int_0^1 \frac{x}{(x^2+1)^2}\, dx.$$

Now use the substitution $u = x^2 + 1$, so that $du = 2x\, dx$. Then $x = 0$ corresponds to $u = 1$, and $x = 1$ to $u = 2$, so that we get

$$V = 2\pi \cdot \frac{1}{2} \int_1^2 \frac{1}{u^2}\, du = \pi \left[ -\frac{1}{u} \right]_1^2 = \boxed{\frac{\pi}{2}}.$$

40. The region is shown below:



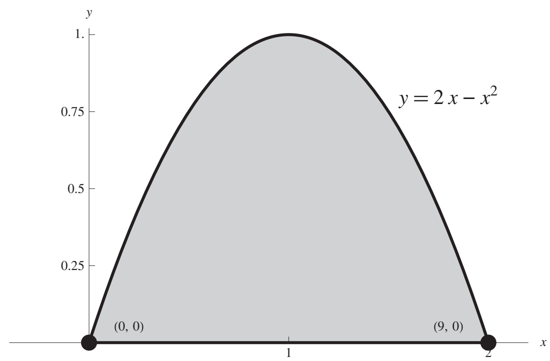

The disk/washer method along the $y$-axis requires solving the equation for $x$, which looks difficult. So we use the shell method along the $x$-axis. The radius of each shell is $x$, and the height is $\sqrt{x+1} + x$, so the volume is

$$V = 2\pi \int_0^3 x\left(\sqrt{x+1} + x\right) dx = 2\pi \left( \int_0^3 x\sqrt{x+1}\, dx + \int_0^3 x^2\, dx \right).$$

For the first integral, use the substitution $u = x + 1$, so that $x = u - 1$ and $du = dx$. Then $x = 0$ corresponds to $u = 1$ while $x = 3$ corresponds to $u = 4$. So we get

$$\begin{aligned} V &= 2\pi \left( \int_1^4 (u-1)\sqrt{u}\, du + \int_0^3 x^2\, dx \right) \\ &= 2\pi \left( \int_1^4 \left( u^{3/2} - u^{1/2} \right) du + \int_0^3 x^2\, dx \right) \\ &= 2\pi \left( \left[ \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_1^4 + \left[ \frac{1}{3}x^3 \right]_0^3 \right) \\ &= \boxed{\frac{502}{15}\pi}. \end{aligned}$$

41. The region is shown below:




**(a)** To revolve about the $x$-axis, use the disk method. The radius of each disk is $2x - x^2$, so the volume is

$$V = \pi \int_0^2 \left(2x - x^2\right)^2 dx = \pi \int_0^2 \left(4x^2 - 4x^3 + x^4\right) dx = \pi \left[ \frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5 \right]_0^2 = \boxed{\frac{16}{15}\pi}.$$

**(b)** To revolve about the $y$-axis, use the shell method along the $x$-axis. The radius of each shell is $x$ and its height is $2x - x^2$, so the volume is

$$V = 2\pi \int_0^2 x\left(2x - x^2\right) dx = 2\pi \int_0^2 \left(2x^2 - x^3\right) dx = 2\pi \left[ \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_0^2 = \boxed{\frac{8}{3}\pi}.$$

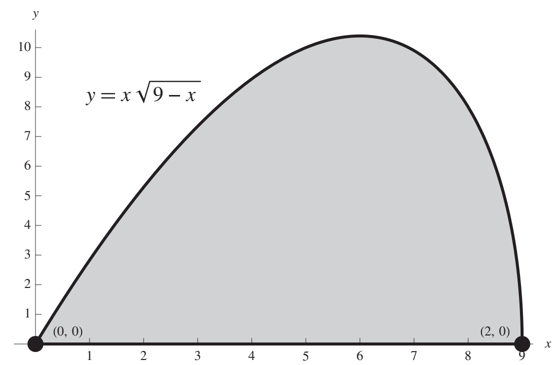
**(c)** To revolve about the line $x = 3$, use the shell method along the $x$-axis. The radius of each shell is $3 - x$ (the distance from $x$ to 3), and the height is $2x - x^2$, so the volume is

$$V = 2\pi \int_0^2 (3-x)(2x-x^2)\, dx = 2\pi \int_0^2 \left(x^3 - 5x^2 + 6x\right) dx = 2\pi \left[ \frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2 \right]_0^2 = \boxed{\frac{16}{3}\pi}.$$

**(d)** To revolve about the line $y = 1$, use the washer method along the $x$-axis. The outer radius of each shell is $1 - 0 = 1$, and the inner radius is $1 - (2x - x^2) = x^2 - 2x + 1 = (x-1)^2$. Therefore the volume is

$$V = \pi \int_0^2 \left(1^2 - \left((x-1)^2\right)^2\right) dx = \pi \int_0^2 \left(1 - (x-1)^4\right) dx = \pi \left[ x - \frac{1}{5}(x-1)^5 \right]_0^2 = \boxed{\frac{8}{5}\pi}.$$

42. The region is shown below:




**(a)** To revolve about the $x$-axis, use the disk method. The radius of each disk is $x\sqrt{9-x}$, so the volume is

$$V = \pi \int_0^9 \left(x\sqrt{9-x}\right)^2 \, dx = \pi \int_0^9 \left(9x^2 - x^3\right) \, dx = \pi \left[3x^3 - \frac{1}{4}x^4\right]_0^9 = \boxed{\frac{2187}{4}\pi}.$$

**(b)** To revolve about the $y$-axis, use the shell method along the $x$-axis. The radius of each shell is $x$ and its height is $x\sqrt{9-x}$, so the volume is

$$V = 2\pi \int_0^9 x\left(x\sqrt{9-x}\right) \, dx = 2\pi \int_0^9 x^2\sqrt{9-x} \, dx.$$

Now use the substitution $u = 9 - x$, so that $x = 9 - u$ and $du = -dx$. Then $x = 0$ corresponds to $u = 9$ and $x = 9$ corresponds to $u = 0$, so that we get

$$\begin{aligned} V &= -2\pi \int_9^0 (9-u)^2 \sqrt{u} \, du \\ &= 2\pi \int_0^9 \left(u^{5/2} - 18u^{3/2} + 81u^{1/2}\right) \, du \\ &= 2\pi \left[\frac{2}{7}u^{7/2} - \frac{36}{5}u^{5/2} + 54u^{3/2}\right]_0^9 \\ &= \boxed{\frac{23328}{35}\pi}. \end{aligned}$$

**(c)** To revolve about the line $y = -3$, use the washer method along the $x$-axis. The outer radius of each shell is $x\sqrt{9-x} - (-3) = x\sqrt{9-x} + 3$, and the inner radius is $0 - (-3) = 3$. Therefore the volume is

$$\begin{aligned} V &= \pi \int_0^9 \left((x\sqrt{9-x} + 3)^2 - 3^2\right) \, dx \\ &= \pi \int_0^9 \left(x^2(9-x) + 6x\sqrt{9-x}\right) \, dx \\ &= \pi \left(\int_0^9 \left(9x^2 - x^3\right) \, dx + 6\int_0^9 x\sqrt{9-x} \, dx\right). \end{aligned}$$

For the second integral, use the substitution $u = 9 - x$, so that $x = 9 - u$ and $du = -dx$. Then $x = 0$ corresponds to $u = 9$ and $x = 9$ corresponds to $u = 0$, so that we get

$$\begin{aligned} V &= \pi \left( \int_0^9 \left(9x^2 - x^3\right) dx - 6 \int_9^0 (9-u)\sqrt{u}\, du \right) \\ &= \pi \left( \int_0^9 \left(9x^2 - x^3\right) dx + 6 \int_0^9 \left(9u^{1/2} - u^{3/2}\right) du \right) \\ &= \pi \left( \left[3x^3 - \frac{1}{4}x^4\right]_0^9 + 6\left[6u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^9 \right) \\ &= \boxed{\frac{18711}{20}\pi}. \end{aligned}$$

**(d)** To revolve about the line $x = -2$, use the shell method along the $x$-axis. The radius of each shell is $x - (-2) = x + 2$ (the distance from $x$ to $-2$), and the height is $x\sqrt{9-x}$, so the volume is

$$V = 2\pi \int_0^9 (x+2)x\sqrt{9-x}\, dx = 2\pi \int_0^9 \left(x^2 + 2x\right)\sqrt{9-x}\, dx.$$

Again we use the substitution $u = 9 - x$, so that $x = 9 - u$, $x^2 + 2x = u^2 - 20u + 99$, and $du = -dx$. Then $x = 0$ corresponds to $u = 9$ and $x = 9$ corresponds to $u = 0$, so that we get

$$\begin{aligned} V &= -2\pi \int_9^0 \left(u^2 - 20u + 99\right)\sqrt{u}\, du \\ &= 2\pi \int_0^9 \left(u^{5/2} - 20u^{3/2} + 99u^{1/2}\right) du \\ &= 2\pi \left[\frac{2}{7}u^{7/2} - 8u^{5/2} + 66u^{3/2}\right]_0^9 \\ &= \boxed{\frac{6480}{7}\pi}. \end{aligned}$$

43. Using the disk method to compute the volume gives

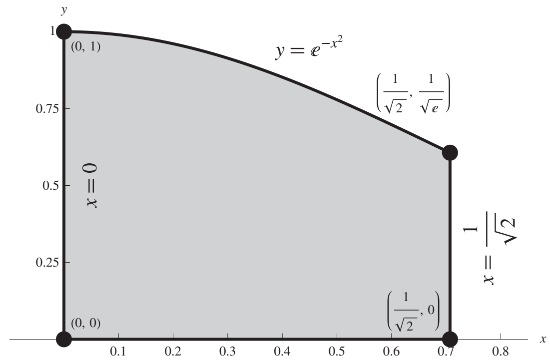
$$\pi \int_0^k f(x)^2\, dx = \frac{1}{5}k^5 + k^4 + \frac{4}{3}k^3.$$

Assuming that the antiderivative of $f(x)^2$, evaluated at 0 is zero, we see that $\frac{1}{5}k^5 + k^4 + \frac{4}{3}k^3$ is $\pi$ times the antiderivative of $f(x)^2$ evaluated at $x = k$. Therefore $f(x)^2$ has as an antiderivative $\frac{1}{\pi}\left(\frac{1}{5}x^5 + x^4 + \frac{4}{3}x^3\right)$, so that $f(x)^2 = \frac{1}{\pi}\left(x^4 + 4x^3 + 4x^2\right) = \frac{1}{\pi}x^2(x^2 + 4x + 4) = \frac{1}{\pi}\left(x(x+2)\right)^2$. Since we are given that $f(x) \geq 0$, taking square roots gives $f(x) = \frac{1}{\sqrt{\pi}}x(x+2) = \boxed{\frac{1}{\sqrt{\pi}}\left(x^2 + 2x\right)}$.

44. We first determine the inflection point. With $f(x) = e^{-x^2}$, we have $f'(x) = -2xe^{-x^2}$, so that

$$f''(x) = -2x\left(-2xe^{-x^2}\right) - 2e^{-x^2} = (4x^2 - 2)e^{-x^2}.$$

Then $f''(x) = 0$ when $x = \pm\frac{1}{\sqrt{2}}$; since we want the positive value, we get $x = \frac{1}{\sqrt{2}}$. Therefore the region in question is the following:



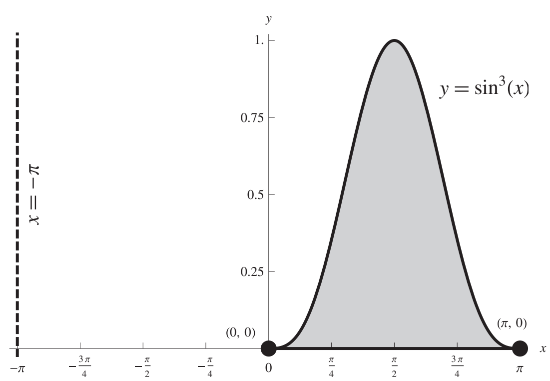

To revolve this about the $y$-axis, it appears easiest to use the shell method along the $x$-axis. The radius of each shell is $x$ and its height is $e^{-x^2}$, so we get for the volume

$$V = 2\pi \int_0^{1/\sqrt{2}} xe^{-x^2}\, dx.$$

Use the substitution $u = -x^2$, so that $du = -2x\, dx$. Then $x = 0$ corresponds to $u = 0$ while $x = \frac{1}{\sqrt{2}}$ corresponds to $u = -\frac{1}{2}$. So we get for the integral

$$V = 2\pi \cdot \left(-\frac{1}{2}\right) \int_0^{-1/2} e^u\, du = -\pi \left[e^u\right]_0^{-1/2} = \boxed{\left(1 - \frac{1}{e^{1/2}}\right)\pi}.$$

45. The region is shown below:




We should use the shell method, as the washer method will involve solving $y = \sin^3 x$ for $x$ and then integrating. With the shell method, the radius of each shell is $x - (-\pi) = x + \pi$, and the height is $\sin^3 x$. Therefore the volume is (using a CAS)

$$V = 2\pi \int_0^{\pi} (x + \pi) \sin^3 x\, dx \approx 12.566\pi \approx 39.478 \approx 4\pi^2.$$

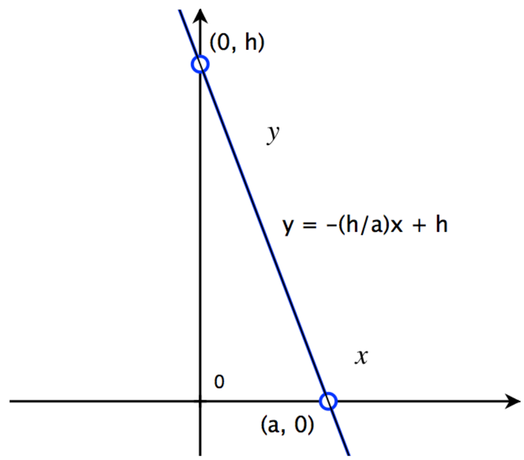
46. **(a)**

$$m = \frac{0 - h}{a - 0} = \frac{-h}{a}$$

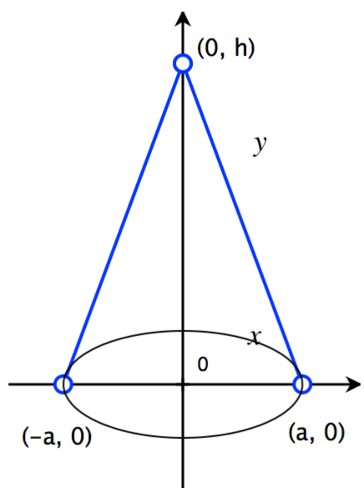
$$y - h = \frac{-h}{a}(x)$$

$$\boxed{y = -\frac{h}{a}x + h}$$

**(b)**




**(c)**




**(d)**
$$
\begin{aligned}
2\pi \int_0^a x\left(\frac{-hx}{a} + h\right) dx &= 2\pi \int_0^a \left(\frac{-hx^2}{a} + hx\right) dx \\
&= 2\pi \left[\frac{-hx^3}{3a} + \frac{hx^2}{2}\right]_0^a \\
&= 2\pi \left[\frac{-ha^3}{3a} + \frac{ha^2}{2} - 0\right] \\
&= 2\pi \left[\frac{-ha^3}{3a} + \frac{ha^2}{2}\right] \\
&= 2\pi \left[\frac{-ha^2}{3} + \frac{ha^2}{2}\right] \\
&= 2\pi \left[\frac{ha^2}{6}\right] \\
&= \boxed{\frac{\pi h a^2}{3}}
\end{aligned}
$$

47. **(a)** By the Shell Method, the radius of each shell is $x$, the height is $y = \cos x$, and the radius of the base of the solid is from $x = 0$ to $x = \frac{\pi}{2}$, so

$$V = \boxed{2\pi \int_0^{\pi/2} x \cos x \, dx}$$

By the Disk Method, the radius of each disk is $x = \cos^{-1} y$, and the height of the solid is from $y = 0$ to $y = 1$, so

$$V = \boxed{\pi \int_0^1 \left(\cos^{-1} y\right)^2 dy}$$

**(b)** Using technology, $V = \boxed{\pi(\pi - 1) \approx 3.586}$

48. **(a)** $V = \pi \int_0^1 \left[\left(\frac{\pi}{2}\right)^2 - \left(\sin^{-1} y\right)^2\right] dy$ by the Washer Method

$V = 2\pi \int_0^{\pi/2} x \sin x \, dx$ by the Shell Method

**(b)** $V = \boxed{2\pi \approx 6.283}$

49. Consider the region $A$. Suppose that it is wholly contained between the vertical lines $x = a > 0$ and $x = b > a$. Integrate it using the shell method. At any $x \geq 0$, $A$ consists of one or more line segments. Define $g(x)$ to be the total length of all of those segments at $x$. Then using the shell method, the volume of the solid of revolution about the $y$-axis is

$$V = 2\pi \int_a^b x g(x) \, dx.$$

When revolving $A$ about the line $x = -k$, the heights at $x$ are the same, $g(x)$, but the radius in each case is now $x + k$. So using the shell method in this case gives

$$V_{x=-k} = 2\pi \int_a^b (x+k) g(x) \, dx = 2\pi \int_a^b x g(x) \, dx + 2\pi \int_a^b k g(x) \, dx = V + 2\pi k \int_a^b g(x) \, dx = V + 2\pi k A,$$

as desired.

50. For this parabola, the latus rectum is the line $x = 2$; we wish to revolve the area to the left of this line about that line.

**(a)** Using the disk method, we must solve the equation for $x$ in terms of $y$, giving $x = \frac{y^2}{8}$. Then the radius of each disk is $2 - \frac{y^2}{8}$, so that the volume of the solid is

$$V = \pi \int_{-4}^4 \left(2 - \frac{y^2}{8}\right)^2 dy = \pi \int_{-4}^4 \left(4 - \frac{y^2}{2} + \frac{y^4}{64}\right) dy = \pi \left[4y - \frac{y^3}{6} + \frac{y^5}{320}\right]_{-4}^4 = \boxed{\frac{256}{15}\pi}.$$

**(b)** Using the shell method, the radius of each shell is $2 - x$, and the height of each shell is then $\sqrt{8x} - \left(-\sqrt{8x}\right) = 2\sqrt{8x} = 4x^{1/2}\sqrt{2}$, so that the volume is

$$\begin{aligned} V &= 2\pi \int_0^2 (2 - x)\left(4x^{1/2}\sqrt{2}\right) dx \\ &= 8\sqrt{2}\,\pi \int_0^2 \left(2x^{1/2} - x^{3/2}\right) dx \\ &= 8\sqrt{2}\,\pi \left[\frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^2 \\ &= \boxed{\frac{256}{15}\pi}. \end{aligned}$$

### Challenge Problem

51. **(a)** From the diagram, $f(x)$ is constant and equal to $f(a)$ for $0 \leq x \leq a$, and also $g(x)$ is constant and equal to $g(a)$ for $0 \leq x \leq a$. So using the shell method, we get two separate integrals, one on $[0, a]$ and one on $[a, b]$. For the first, the radius of each shell is $x$ and the height is $f(x) - g(x) = f(a) - g(a)$, while for the second, the radius is $x$ and the height is $f(x) - g(x)$. Therefore the volume of the solid is

$$\begin{aligned} V &= 2\pi \left( \int_0^a x[f(a) - g(a)]\, dx + \int_a^b x[f(x) - g(x)]\, dx \right) \\ &= 2\pi \left( \left[ \frac{1}{2} x^2 [f(a) - g(a)] \right]_0^a + \int_a^b x[f(x) - g(x)]\, dx \right) \\ &= \pi a^2 [f(a) - g(a)] + 2\pi \int_a^b x[f(x) - g(x)]\, dx. \end{aligned}$$

**(b)** Using the disk method, we also get two separate integrals, one from $y = g(a)$ to $y = g(b) = f(b)$ and one from $y = f(b)$ to $y = f(a)$. For the first integral, the radius of the disk is $g^{-1}(y)$ and for the second it is $f^{-1}(y)$. Therefore the volume is the sum of these integrals, or

$$V = \pi \int_{g(a)}^{g(b)} \left[ g^{-1}(y) \right]^2 \, dy + \pi \int_{f(b)}^{f(a)} \left[ f^{-1}(y) \right]^2 \, dy.$$

### AP® Practice Problems

1. By the Shell Method,

$$\begin{aligned} V &= 2\pi \int x f(x)\, dx \\ &= 2\pi \int_0^3 x\left(-x^2 + 3x\right) dx \\ &= 2\pi \int_0^3 \left(-x^3 + 3x^2\right) dx \\ &= 2\pi \left[ \frac{-x^4}{4} + x^3 \right]_0^3 \\ &= 2\pi \left[ \frac{-(3)^4}{4} + 3^3 - 0 \right] \\ &= \frac{27\pi}{2} \\ &= \boxed{13.5\pi} \end{aligned}$$

By the Disk Method:

First, solve to get $x$ as a function of $y$:

$$y = -x^2 + 3x$$
$$y = -\left[x^2 - 3x + \left(\frac{3}{2}\right)^2\right] + \frac{9}{4}$$
$$y = -\left(x - \frac{3}{2}\right)^2 + \frac{9}{4}$$
$$y - \frac{9}{4} = -\left(x - \frac{3}{2}\right)^2$$
$$x - \frac{3}{2} = \pm\sqrt{\frac{9}{4} - y}$$
$$x = \frac{3}{2} \pm \sqrt{\frac{9-4y}{4}}$$
$$x = \frac{3 \pm \sqrt{9-4y}}{2}$$

Then

$$\begin{aligned}
V &= \pi \int_0^{9/4} \left(\frac{3+\sqrt{9-4y}}{2} - 0\right)^2 dy - \pi \int_0^{9/4} \left(\frac{3-\sqrt{9-4y}}{2} - 0\right)^2 dy \\
&= \pi \int_0^{9/4} \frac{9 + 6\sqrt{9-4y} + 9 - 4y - \left(9 - 6\sqrt{9-4y} + 9 - 4y\right)}{4}\, dy \\
&= \pi \int_0^{9/4} \frac{12\sqrt{9-4y}}{4}\, dy \\
&= 3\pi \int_0^{9/4} (9-4y)^{1/2}\, dy \\
&= \left(\frac{-\pi}{2}\right) \left[(9-4y)^{3/2}\right]_0^{9/4} \\
&= \left(\frac{-\pi}{2}\right) \left(0 - (9)^{3/2}\right) \\
&= \left(\frac{-\pi}{2}\right) (-27) \\
&= \frac{27\pi}{2} \\
&= \boxed{13.5\pi}
\end{aligned}$$

CHOICE B

2. By the Shell Method,

$$\begin{aligned}
V &= 2\pi \int_0^2 (x+3)\left(x^2+1\right) dx \\
&= 2\pi \int_0^2 \left(x^3+3x^2+x+3\right) dx \\
&= 2\pi \int_0^2 \left(x^3+3x^2+x+3\right) dx \\
&= 2\pi \left[\frac{x^4}{4}+x^3+\frac{x^2}{2}+3x\right]_0^2 \\
&= 2\pi(4+8+2+6-0) \\
&= \boxed{40\pi}
\end{aligned}$$

By the Disk Method:

First, solve to get $x$ as a function of $y$:

$$\begin{aligned}
y &= x^2+1 \\
x^2 &= y-1 \\
x &= \pm\sqrt{y-1}
\end{aligned}$$

Then

$$\begin{aligned}
V &= \pi \int_1^5 (x-(-3))^2\, dy - \pi \int_1^5 (0-(-3))^2\, dy \\
&= \pi \int_1^5 (x+3)^2\, dy - 9\pi \int_1^5 dy \\
&= \pi \int_1^5 \left(3+\sqrt{y-1}\right)^2 dy - 9\pi \int_1^5 dy \\
&= \pi \int_1^5 \left(9+6\sqrt{y-1}+y-1\right) dy - 9\pi \int_1^5 dy \\
&= \pi \int_1^5 \left(6\sqrt{y-1}+y-1\right) dy \\
&= \pi \left[4(y-1)^{3/2}+\frac{y^2}{2}-y\right]_1^5 \\
&= \pi \left[4(4)^{3/2}+\frac{25}{2}-5-\left(0+\frac{1}{2}-1\right)\right] \\
&= \boxed{40\pi}
\end{aligned}$$

CHOICE D

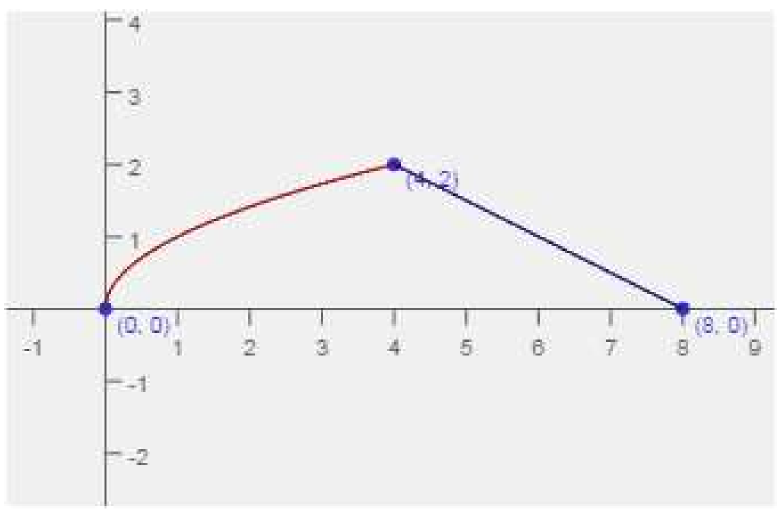
3. Determine the point of intersection of $y = 4 - \frac{x}{2}$ and $y = \sqrt{x}$.

$$\begin{aligned} 4 - \frac{x}{2} &= \sqrt{x} \\ 8 - x &= 2\sqrt{x} \\ (8-x)^2 &= \left(2\sqrt{x}\right)^2 \\ 64 - 16x + x^2 &= 4x \\ x^2 - 20x + 64 &= 0 \\ (x-4)(x-16) &= 0 \\ x &= 4 \end{aligned}$$

(Testing $x = 16$ in the original equation shows that it is an extraneous solution.)

$y = \sqrt{x}$ intersects the $x$-axis at $x = 0$.

$y = 4 - \frac{x}{2}$ intersects the $x$-axis at $x = 8$.




$$\begin{aligned} V &= \pi \int_0^4 \left(\sqrt{x}\right)^2 dx + \pi \int_4^8 \left(4 - \frac{x}{2}\right)^2 dx \\ &= \pi \int_0^4 x\ dx + \pi \int_4^8 \left(16 - 4x + \frac{x^2}{4}\right) dx \\ &= \left[\frac{\pi x^2}{2}\right]_0^4 + \pi \left[16x - 2x^2 + \frac{x^3}{12}\right]_4^8 \\ &= 8\pi + \pi \left[128 - 128 + \frac{512}{12} - \left(64 - 32 + \frac{64}{12}\right)\right] \\ &= 8\pi + \pi \left[\frac{448}{12} - 32\right] \\ &= \boxed{\frac{40\pi}{3}} \end{aligned}$$

CHOICE B

4. By the Shell Method,

$$\begin{aligned} V &= 2\pi \int_0^2 x e^{x^2}\, dx \\ &= \pi e^{x^2}\Big|_0^2 \\ &= \pi\left(e^4 - e^0\right) \\ &= \boxed{\pi\left(e^4 - 1\right)} \end{aligned}$$

(By the Washer/Disk Method: Here, the volume would have to be divided into two pieces, the "top" done by the Washer Method and integrating from 1 to $e^4$ and the "bottom" done by the Disk Method and integrating from 0 to 1. This would be considerably more complicated than the simple result of the Shell Method, so the Shell Method is the best one to use.)

CHOICE A

5. **(a)** $\boxed{V = \pi \int_0^2 \left(-2x^3 + 4x^2\right)^2 dx}$

**(b)** $\boxed{V = 2\pi \int_0^2 x\left(-2x^3 + 4x^2\right) dx}$

## 6.4 Volume of a Solid: Slicing

### Concepts and Vocabulary

1. Answers vary. When computing the volume of a solid whose cross sections have a regular geometric shape whose area we can compute, we can integrate over the area of each slice of the solid along one of its axes.

2. False. For instance, Example 3 uses the method of slicing to compute the volume of a square pyramid, which is not a solid of revolution. The slicing method can be used effectively whenever the cross sections have a regular geometric shape whose area we can compute.

### Skill Building

3. The equation of the circle is $x^2 + y^2 = 4$, so that for a given value of $x$, $y$ can range from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$. Therefore, for each value of $x$ from $-2$ to 2, the cross section is a square of side $2\sqrt{4-x^2}$, so that its area is $4(4-x^2)$. Therefore the volume of the solid is

$$V = \int_{-2}^{2} 4\left(4-x^2\right) dx = 4\int_{-2}^{2}\left(4-x^2\right) dx = 4\left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = \boxed{\frac{128}{3}}.$$

4. The equation of the circle is $x^2 + y^2 = 4$, so that for a given value of $x$, $y$ can range from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$. Therefore, for each value of $x$ from $-2$ to 2, the cross section is an isosceles right triangle one of whose legs is $2\sqrt{4-x^2}$, so that its area is $\frac{1}{2}\left(2\sqrt{4-x^2}\right)^2 = 2(4-x^2)$. Therefore the volume of the solid is

$$V = \int_{-2}^{2} 2\left(4-x^2\right) dx = 2\int_{-2}^{2}\left(4-x^2\right) dx = 2\left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = \boxed{\frac{64}{3}}.$$

5. **(a)** From the figure, these two curves intersect at (0, 0) and at (4, 2), so we will integrate from 0 to 4. For each value of $x$, the cross section is a semicircle of diameter $\sqrt{x}-\frac{1}{8}x^2$, so its area is

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{\sqrt{x}-\frac{1}{8}x^2}{2}\right)^2 = \frac{\pi}{512}\left(8\sqrt{x}-x^2\right)^2$$
$$= \frac{\pi}{512}\left(64x - 16x^{5/2} + x^4\right)$$

Therefore the volume is

$$V = \int_0^4 \frac{\pi}{512}\left(64x - 16x^{5/2} + x^4\right)dx = \frac{\pi}{512}\left[32x^2 - \frac{32}{7}x^{7/2} + \frac{1}{5}x^5\right]_0^4 = \boxed{\frac{9}{35}\pi}$$

**(b)** For each value of $x$, the cross section is an equilateral triangle with base $B = \sqrt{x} - \frac{1}{8}x^2$ and height $H = \frac{\sqrt{3}}{2}B = \frac{\sqrt{3}}{2}\left(\sqrt{x}-\frac{1}{8}x^2\right)$, so its area is

$$A = \frac{1}{2}BH = \frac{1}{2}\left(\sqrt{x}-\frac{1}{8}x^2\right)\cdot\frac{\sqrt{3}}{2}\left(\sqrt{x}-\frac{1}{8}x^2\right) = \frac{\sqrt{3}}{4}\left(\sqrt{x}-\frac{1}{8}x^2\right)^2$$
$$= \frac{\sqrt{3}}{4}\left(x - \frac{1}{4}x^{5/2} + \frac{1}{64}x^4\right).$$

Therefore the volume is

$$V = \int_0^4 \frac{\sqrt{3}}{4}\left(x - \frac{1}{4}x^{5/2} + \frac{1}{64}x^4\right)dx = \frac{\sqrt{3}}{4}\left[\frac{1}{2}x^2 - \frac{1}{4}\cdot\frac{2}{7}x^{7/2} + \frac{1}{64}\cdot\frac{1}{5}x^5\right]_0^4$$
$$= \frac{\sqrt{3}}{4}\left[\left(\frac{1}{2}\cdot 4^2 - \frac{1}{14}\cdot 4^{7/2} + \frac{1}{320}\cdot 4^5\right) - 0\right] = \frac{\sqrt{3}}{4}\cdot\frac{72}{35} = \boxed{\frac{18\sqrt{3}}{35}}$$

6. **(a)** Determine the point(s) of intersection of $y = \sqrt{x}$ and $y = \frac{1}{8}x^2$:

$$\begin{aligned}
\sqrt{x} &= \frac{x^2}{8}\\
x &= \frac{x^4}{64}\\
64x &= x^4\\
x^4 - 64x &= 0\\
x\left(x^3 - 64\right) &= 0\\
x(x-4)\left(x^2 + x + 16\right) &= 0\\
x &= 0,\ \text{so } y = \sqrt{0} = 0\\
\text{or } x &= 4,\ \text{so } y = \sqrt{4} = 2
\end{aligned}$$

Then

$$\begin{aligned}
V &= \int_0^2 \frac{1}{2}\cdot\pi\left(\frac{\sqrt{8y}-y^2}{2}\right)^2 dy \\
&= \frac{\pi}{2}\int_0^2 \left(\frac{8y-2y^2\sqrt{8y}+y^4}{4}\right) dy \\
&= \frac{\pi}{8}\int_0^2 \left(8y-4\sqrt{2}y^{5/2}+y^4\right) dy \\
&= \frac{\pi}{8}\left[4y^2-\frac{8\sqrt{2}}{7}(y)^{7/2}+\frac{y^5}{5}\right]_0^2 \\
&= \frac{\pi}{8}\left[16-\frac{8\sqrt{2}}{7}(2)^{7/2}+\frac{32}{5}-0\right] \\
&= \frac{\pi}{8}\left[16-\frac{128}{7}+\frac{32}{5}\right] \\
&= \frac{\pi}{8}\left[\frac{560-640+224}{35}\right] \\
&= \frac{144\pi}{8(35)} \\
&= \boxed{\frac{18\pi}{35}\approx 1.615}
\end{aligned}$$

**(b)**

$$\begin{aligned}
V &= \int_0^2 \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\left(\sqrt{8y}-y^2\right)^2 dy \\
&= \frac{\sqrt{3}}{4}\int_0^2 \left(8y-2y^2\sqrt{8y}+y^4\right) dy \\
&= \frac{\sqrt{3}}{4}\int_0^2 \left(8y-4\sqrt{2}y^{5/2}+y^4\right) dy \\
&= \frac{\sqrt{3}}{4}\left[4y^2-\frac{8\sqrt{2}}{7}(y)^{7/2}+\frac{y^5}{5}\right]_0^2 \\
&= \frac{\sqrt{3}}{4}\left[16-\frac{8\sqrt{2}}{7}(2)^{7/2}+\frac{32}{5}-0\right] \\
&= \frac{\sqrt{3}}{4}\left[16-\frac{128}{7}+\frac{32}{5}\right] \\
&= \frac{\sqrt{3}}{4}\left[\frac{560-640+224}{35}\right] \\
&= \boxed{\frac{36\sqrt{3}}{35}\approx 1.781}
\end{aligned}$$

7. **(a)** For each value of $x$, the cross section is a square with side $s = y - 0 = x^2$, so its area is

$$A = s^2 = \left(x^2\right)^2 = x^4.$$

Therefore the volume is

$$V = \int_0^4 x^4\,dx = \left.\frac{x^5}{5}\right]_0^2 = \frac{2^5}{5} - 0 = \boxed{\frac{32}{5}}$$

**(b)** For each value of $x$, the cross section is a semicircle with diameter $D = x^2$ and therefore radius $r = \frac{D}{2} = \frac{x^2}{2}$, so its area is

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \cdot \left(\frac{x^2}{2}\right)^2 = \frac{\pi}{8}x^4.$$

Therefore the volume is

$$V = \frac{\pi}{8}\int_0^2 x^4\,dx = \frac{\pi}{8} \cdot \frac{32}{5} = \boxed{\frac{4}{5}\pi}$$

using the integral from Part (a).

**(c)** For each value of $x$, the cross section is a triangle of base $B = x^2$ and height $H = \frac{\sqrt{3}}{2}x^2$, so its area is

$$A = \frac{1}{2}BH = \frac{1}{2}x^2 \cdot \frac{\sqrt{3}}{2}x^2 = \frac{\sqrt{3}}{4}x^4.$$

Therefore the volume is

$$V = \frac{\sqrt{3}}{4}\int_0^2 x^4\,dx = \frac{\sqrt{3}}{4} \cdot \frac{32}{5} = \boxed{\frac{8\sqrt{3}}{5}}$$

using the integral from Part (a).

8. **(a)** 
$$\begin{aligned}
V &= \int_0^4 (2 - \sqrt{y})^2\,dy \\
&= \int_0^4 (4 - 4\sqrt{y} + y)\,dy \\
&= 4\int_0^4 dy - 4\int_0^4 \sqrt{y}\,dy + \int_0^4 y\,dy \\
&= [4y]_0^4 - \left[\frac{8}{3}y^{3/2}\right]_0^4 + \left[\frac{y^2}{2}\right]_0^4 \\
&= 16 - \frac{8}{3}(4)^{3/2} + \frac{16}{2} \\
&= 16 - \frac{64}{3} + 8 \\
&= \boxed{\frac{8}{3}}
\end{aligned}$$

**(b)**
$$
\begin{aligned}
V &= \int_0^4 \frac{1}{2}\cdot\pi\left(\frac{2-\sqrt{y}}{2}\right)^2 dy\\
&= \frac{\pi}{2}\int_0^4 \frac{\left(4-4\sqrt{y}+y\right)}{4}\,dy\\
&= \frac{\pi}{8}\int_0^4 (4-4\sqrt{y}+y)\,dy\\
&= \frac{4\pi}{8}\int_0^4 dy - \frac{4\pi}{8}\int_0^4 \sqrt{y}\;dy + \frac{\pi}{8}\int_0^4 y\,dy\\
&= \left[\frac{\pi y}{2}\right]_0^4 - \left[\frac{\pi}{3}y^{3/2}\right]_0^4 + \left[\frac{\pi y^2}{16}\right]_0^4\\
&= 2\pi - \frac{8\pi}{3} + \pi\\
&= \boxed{\frac{\pi}{3}}
\end{aligned}
$$

**(c)**
$$
\begin{aligned}
V &= \int_0^4 \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\,(2-\sqrt{y})^2\,dy\\
&= \frac{\sqrt{3}}{4}\int_0^4 (4-4\sqrt{y}+y)\,dy\\
&= \sqrt{3}\int_0^4 dy - \sqrt{3}\int_0^4 \sqrt{y}\;dy + \frac{\sqrt{3}}{4}\int_0^4 y\,dy\\
&= \frac{\sqrt{3}}{4}\left([4y]_0^4 - \left[\frac{8}{3}y^{3/2}\right]_0^4 + \left[\frac{y^2}{2}\right]_0^4\right)\\
&= \frac{\sqrt{3}}{4}\left[16 - \frac{8}{3}(4)^{3/2} + \frac{16}{2}\right]\\
&= \frac{\sqrt{3}}{4}\left[16 - \frac{64}{3} + 8\right]\\
&= \frac{\sqrt{3}}{4}\cdot\frac{8}{3}\\
&= \boxed{\frac{2\sqrt{3}}{3}}
\end{aligned}
$$

9. Find the intersections of the two curves, for the limits of integration:

$$
\begin{aligned}
x^2 &= 3\sqrt{3x}\\
\left(x^2\right)^2 &= \left(3\sqrt{3x}\right)^2\\
x^4 &= 27x\\
x^4 - 27x &= 0\\
x\left(x^3-27\right) &= 0\\
x\,(x-3)\left(x^2+3x+9\right) &= 0\\
x &= 0,\ \text{so } y = (0)^2 = 0\\
\text{or } x &= 3,\ \text{so } y = (3)^2 = 9
\end{aligned}
$$

(Algebra or technology shows that $x^2+3x+9$ is always positive.)

Solve each equation for $x$ as a function of $y$:

$$3\sqrt{3x} = y$$
$$\left(3\sqrt{3x}\right)^2 = y^2$$
$$27x = y^2$$
$$x = \frac{1}{27}y^2$$

---

$$x^2 = y$$
$$x = \sqrt{y}$$

---

(Positive $\sqrt{y}$ because the region is in the first quadrant.)

**(a)** For each value of $y$, the cross section is a square with side $s = \sqrt{y} - \frac{1}{27}y^2$, so its area is

$$A = s^2 = \left(\sqrt{y} - \frac{1}{27}y^2\right)^2$$

Therefore the volume is

$$V = \int_0^9 \left(\sqrt{y} - \frac{1}{27}y^2\right)^2 dy = \int_0^9 \left(y - \frac{2}{27}y^{5/2} + \frac{1}{729}y^4\right) dy$$
$$= \left[\frac{1}{2}y^2 - \frac{2}{27}\cdot\frac{2}{7}y^{7/2} + \frac{1}{729}\cdot\frac{1}{5}y^5\right]_0^9 = \left[\left(\frac{1}{2}\cdot 4^2 - \frac{4}{189}\cdot 4^{7/2} + \frac{1}{3645}\cdot 4^5\right) - 0\right] = \boxed{\frac{729}{70}}$$

**(b)** For each value of $y$, the cross section is a semicircle with diameter $D = \sqrt{y} - \frac{y^2}{27}$ and therefore radius $r = \frac{D}{2} = \frac{1}{2}\left(\sqrt{y} - \frac{1}{27}y^2\right)$, so its area is

$$A = \frac{1}{2}\pi r^2 = \frac{\pi}{2}\cdot\left[\frac{1}{2}\left(\sqrt{y} - \frac{1}{27}y^2\right)\right]^2 = \frac{\pi}{8}\cdot\left(\sqrt{y} - \frac{1}{27}y^2\right)^2$$

Therefore the volume is

$$V = \frac{\pi}{8}\int_0^9 \left(\sqrt{y} - \frac{1}{27}y^2\right)^2 dy = \frac{\pi}{8}\cdot\frac{729}{70} = \boxed{\frac{729}{560}\pi}$$

using the integral from Part (a).

**(c)** For each value of $y$, the cross section is an equilateral triangle with base $B = \sqrt{y} - \frac{1}{27}y^2$ and height $H = \frac{\sqrt{3}}{2}B = \frac{\sqrt{3}}{2}\left(\sqrt{y} - \frac{1}{27}y^2\right)$, so its area is

$$A = \frac{1}{2}BH = \frac{1}{2}\left(\sqrt{y} - \frac{1}{27}y^2\right)\cdot\frac{\sqrt{3}}{2}\left(\sqrt{y} - \frac{1}{27}y^2\right) = \frac{\sqrt{3}}{4}\left(\sqrt{y} - \frac{1}{27}y^2\right)^2.$$

Therefore the volume is

$$V = \frac{\sqrt{3}}{4}\int_0^9 \left(\sqrt{y} - \frac{1}{27}y^2\right)^2 dy = \frac{\sqrt{3}}{4}\cdot\frac{729}{70} = \boxed{\frac{729\sqrt{3}}{280}}$$

using the integral from Part (a).

10. **(a)**
$$\begin{aligned} V &= \int_0^3 \left(3\sqrt{3x} - x^2\right)^2 dx \\ &= \int_0^3 \left(27x - 6x^2\sqrt{3x} + x^4\right) dx \\ &= \int_0^3 \left(27x - \left(6\sqrt{3}\right) x^{5/2} + x^4\right) dx \\ &= \left[\frac{27x^2}{2} - \frac{\left(12\sqrt{3}\right) x^{7/2}}{7} + \frac{x^5}{5}\right]_0^3 \\ &= \frac{243}{2} - \frac{972}{7} + \frac{243}{5} - 0 \\ &= \boxed{\frac{2187}{70}} \end{aligned}$$

**(b)**
$$\begin{aligned} V &= \int_0^3 \frac{1}{2} \cdot \pi \left(\frac{3\sqrt{3x} - x^2}{2}\right)^2 dx \\ &= \frac{\pi}{8} \int_0^3 \left(27x - 6x^2\sqrt{3x} + x^4\right) dx \\ &= \frac{\pi}{8} \int_0^3 \left(27x - \left(6\sqrt{3}\right) x^{5/2} + x^4\right) dx \\ &= \frac{\pi}{8} \left[\frac{27x^2}{2} - \frac{\left(12\sqrt{3}\right) x^{7/2}}{7} + \frac{x^5}{5}\right]_0^3 \\ &= \frac{\pi}{8} \left(\frac{243}{2} - \frac{972}{7} + \frac{243}{5} - 0\right) \\ &= \boxed{\frac{2187\pi}{560}} \end{aligned}$$

**(c)**
$$\begin{aligned} V &= \int_0^3 \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \left(3\sqrt{3x} - x^2\right)^2 dx \\ &= \frac{\sqrt{3}}{4} \int_0^3 \left(27x - 6x^2\sqrt{3x} + x^4\right) dx \\ &= \frac{\sqrt{3}}{4} \int_0^3 \left(27x - \left(6\sqrt{3}\right) x^{5/2} + x^4\right) dx \\ &= \frac{\sqrt{3}}{4} \left[\frac{27x^2}{2} - \frac{\left(12\sqrt{3}\right) x^{7/2}}{7} + \frac{x^5}{5}\right]_0^3 \\ &= \frac{\sqrt{3}}{4} \left(\frac{243}{2} - \frac{972}{7} + \frac{243}{5} - 0\right) \\ &= \boxed{\frac{2187\sqrt{3}}{280}} \end{aligned}$$

11. Place the origin at the tip of the pyramid, with the $x$-axis along the axis of the pyramid. Then at $x$, the cross section is a square with side $2x$, so that its area is $(2h)^2 = 4h^2\,\text{m}^2$. Therefore the volume of the pyramid is

$$V = \int_0^{40} 4x^2\,dx = \left.\frac{4x^3}{3}\right]_0^{40} = \frac{4\cdot 40^3}{3} - 0 = \boxed{\frac{256{,}000}{3}\,\text{m}^3 = 85{,}333.\bar{3}\,\text{m}^3}$$

12. $$\begin{aligned} V &= \int_0^{20} 2x^2\,dx \\ &= \left[\frac{2x^3}{3}\right]_0^{20} \\ &= \frac{2\,(8000)}{3} \\ &= \boxed{\frac{16{,}000}{3}\,\text{m}^3} \end{aligned}$$

**Applications and Extensions**

13. Place the center of the sphere at the origin. Then $x$ ranges from $-R$ to $R$, so that for each $x$, the possible values of $y$ are from $-\sqrt{R^2-x^2}$ to $\sqrt{R^2-x^2}$. Therefore the cross section at $x$ is a circle with radius $\sqrt{R^2-x^2}$, so its area is $\pi\left(\sqrt{R^2-x^2}\right)^2 = \pi(R^2-x^2)$. So the total volume of the sphere is

$$V = \int_{-R}^{R} \pi(R^2-x^2)\,dx = \pi\left[R^2x - \frac{1}{3}x^3\right]_{-R}^{R} = \pi\cdot\frac{4}{3}R^3 = \boxed{\frac{4}{3}\pi R^3}.$$

14. **(a)** Determine the point(s) of intersection of $y=\sqrt{x}$ and $y=\frac{1}{8}x^2$:

$$\begin{aligned} \sqrt{x} &= \frac{x^2}{8} \\ x &= \frac{x^4}{64} \\ 64x &= x^4 \\ x^4 - 64x &= 0 \\ x\left(x^3-64\right) &= 0 \\ x\,(x-4)\left(x^2+x+16\right) &= 0 \\ x &= 0 \text{ or } x = 4 \end{aligned}$$

Then

$$\begin{aligned} V &= \int_0^4 \frac{1}{2}\cdot 3\left(\sqrt{x}-\frac{x^2}{8}\right)^2 dx \\ &= \frac{3}{2}\int_0^4 \left(x - \frac{x^2\sqrt{x}}{4} + \frac{x^4}{64}\right) dx \\ &= \frac{3}{2}\int_0^4 \left(x - \frac{x^{5/2}}{4} + \frac{x^4}{64}\right) dx \\ &= \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^{7/2}}{14} + \frac{x^5}{320}\right]_0^4 \\ &= \frac{3}{2}\left[8 - \frac{64}{7} + \frac{16}{5} - 0\right] \\ &= \boxed{\frac{108}{35}} \end{aligned}$$

**(b)** Solve each equation for $x$ as a function of $y$:

$$\begin{aligned} y &= \sqrt{x} \\ x &= y^2 \end{aligned}$$

$$\begin{aligned} y &= \frac{x^2}{8} \\ x^2 &= 8y \\ x &= \sqrt{8y} \end{aligned}$$

As determined above in 14(a), $y = \sqrt{x}$ and $y = \frac{1}{8}x^2$ intersect at $x = 0$ and $x = 4$. The points of intersection are (0, 0) and (4, 2).
Then

$$\begin{aligned} V &= \int_0^2 \frac{1}{2} \cdot 3 \left(\sqrt{8y} - y^2\right)^2 dy \\ &= \frac{3}{2} \int_0^2 \left(8y - 4y^2\sqrt{2y} + y^4\right) dy \\ &= \frac{3}{2} \int_0^2 \left(8y - 2^{5/2}y^{5/2} + y^4\right) dy \\ &= \frac{3}{2} \left[4y^2 - \frac{2^{7/2}}{7}y^{7/2} + \frac{y^5}{5}\right]_0^2 \\ &= \frac{3}{2} \left[16 - \frac{128}{7} + \frac{32}{5}\right] \\ &= \boxed{\frac{216}{35}} \end{aligned}$$

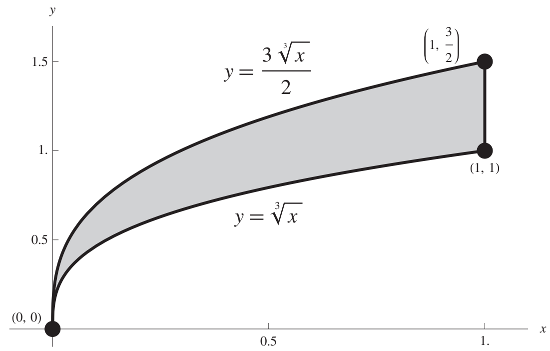
15. For each value of $x$, the cross section is a circle with diameter $D = \frac{4}{3}x^{1/3} - x^{1/2}$, and therefore radius $r = \frac{D}{2} = \frac{1}{2}\left(\frac{4}{3}x^{1/3} - x^{1/2}\right) = \frac{2}{3}x^{1/3} - \frac{1}{2}x^{1/2}$. Therefore its area is

$$A = \pi r^2 = \pi \left(\frac{2}{3}x^{1/3} - \frac{1}{2}x^{1/2}\right)^2 = \pi \left(\frac{4}{9}x^{2/3} - \frac{2}{3}x^{5/6} + \frac{1}{4}x\right).$$

Therefore the volume is

$$\begin{aligned} V &= \pi \int_0^1 \left(\frac{4}{9}x^{2/3} - \frac{2}{3}x^{5/6} + \frac{1}{4}x\right) dx = \pi \left[\frac{4}{9} \cdot \frac{3}{5}x^{5/3} - \frac{2}{3} \cdot \frac{6}{11}x^{11/6} + \frac{1}{4} \cdot \frac{1}{2}x^2\right]_0^1 \\ &= \pi \left[\left(\frac{4}{15} \cdot 1^{5/3} - \frac{4}{11} \cdot 1^{11/6} + \frac{1}{8} \cdot 1^2\right) - 0\right] = \boxed{\frac{37}{1320}\pi} \end{aligned}$$

16. The region is shown below:




For each $x$, the cross section is a circle whose diameter is $\frac{3}{2}x^{1/3} - x^{1/3} = \frac{1}{2}x^{1/3}$, so the area of that circle is

$$A = \pi r^2 = \pi\left(\frac{\frac{1}{2}x^{1/3}}{2}\right)^2 = \frac{1}{16}\pi x^{2/3}.$$

Therefore the volume is

$$V = \int_0^1 \left(\frac{1}{16}\pi x^{2/3}\right)\,dx = \frac{1}{16}\pi\left[\frac{3}{5}x^{5/3}\right]_0^1 = \boxed{\frac{3}{80}\pi}.$$

17. Because of symmetry around the $x$-axis, for each value of $x$ the radius of the cross section is $r = e^{-x^2} - 0 = e^{-x^2}$. Therefore its area is

$$A = \pi r^2 = \pi\left(e^{-x^2}\right)^2 = \pi e^{-2x^2}$$

By symmetry around the $y$-axis, the volume from $x = -1$ to $x = 1$ is twice the volume from $x = 0$ to $x = 1$:

$$V = \pi\int_{-1}^1 e^{-2x^2}\,dx = 2\pi\int_0^1 e^{-2x^2}$$

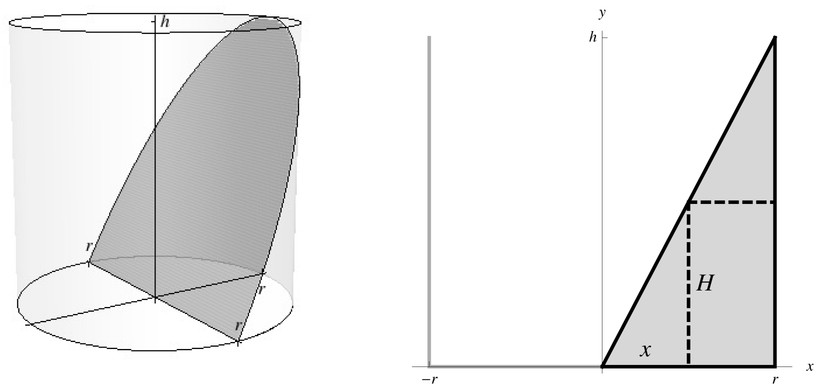
Using technology, the volume is $\boxed{\approx 3.758}$.

18. Place the center of the top of the hemispherical bowl at the origin, with the $x$-axis pointing down (so the bottom of the bowl is at $(R, 0)$). Then $x$ ranges from $R - h$ to $R$, and for each $x$, the possible values of $y$ are from $-\sqrt{R^2 - x^2}$ to $\sqrt{R^2 - x^2}$. Therefore the cross section at $x$ is a circle with radius $\sqrt{R^2 - x^2}$, so its area is $\pi\left(\sqrt{R^2 - x^2}\right)^2 = \pi(R^2 - x^2)$. So the total volume of the contents is

$$\begin{aligned} V &= \int_{R-h}^R \pi(R^2 - x^2)\,dx \\ &= \pi\left[R^2x - \frac{1}{3}x^3\right]_{R-h}^R \\ &= \pi\left(R^3 - \frac{1}{3}R^3 - R^2(R-h) + \frac{1}{3}(R-h)^3\right) \\ &= \pi\left(R^2h - R^2h + Rh^2 - \frac{1}{3}h^3\right) \\ &= \boxed{\pi h^2\left(R - \frac{h}{3}\right)}. \end{aligned}$$

Note that if $h = 0$ (the bowl is empty), this gives 0, while if $h = R$ (the bowl is completely full), it gives $\frac{2}{3}\pi R^3$, which is indeed half the volume of a sphere of radius $R$.

19. A diagram of the water in the glass, in a three-dimensional view on the left and a two-dimensional projection on the right, is shown below:




For a given value of $x$, the length of the water line along the base is $2\sqrt{r^2 - x^2}$, since the base is a circle of radius $r$. To determine its height, use similar triangles (look at the two-dimensional picture): if $H$ is the height, then $\frac{H}{x} = \frac{h}{r}$, so that $H = \frac{h}{r}x$. So the cross section at $x$ is a rectangle with dimensions $2\sqrt{r^2 - x^2} \times \frac{h}{r}x$. Therefore the volume of this amount of water is

$$V = \boxed{\int_0^r \frac{h}{r}x \cdot 2\sqrt{r^2 - x^2}\,dx}.$$

20. Place the origin at the point where the dotted line in the diagram meets the sharp edge of the solid, with the $x$-axis along that sharp edge. Note that since the cylinder has diameter 10, the wedge penetrates halfway through the cylinder, so that the $x$-axis coordinates of the wedge range from $-5$ to 5. Now, for each value of $x$, the distance from the $x$-axis to the outer edge of the wedge is $\sqrt{25 - x^2}$, and the height of the wedge at that point is $\sqrt{25 - x^2} \cdot \tan 30° = \frac{\sqrt{3}}{3}\sqrt{25 - x^2}$. The cross section is therefore a right triangle whose area is

$$A = \frac{1}{2}bh = \frac{1}{2}\sqrt{25 - x^2} \cdot \frac{\sqrt{3}}{3}\sqrt{25 - x^2} = \frac{\sqrt{3}}{6}(25 - x^2).$$

So the volume of the wedge is

$$V = \int_{-5}^{5} \frac{\sqrt{3}}{6}(25 - x^2)\,dx = \frac{\sqrt{3}}{6}\left[25x - \frac{1}{3}x^3\right]_{-5}^{5} = \boxed{\frac{250\sqrt{3}}{9}\ \text{m}^3}.$$

## Challenge Problems

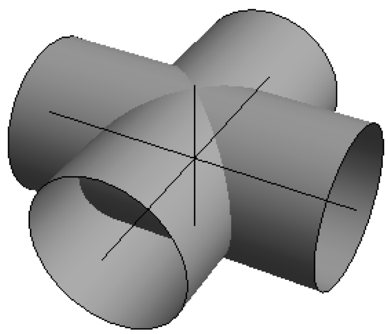
21. The volume removed is almost a cylinder with radius 2, except for the ends, which are spherical caps. Let's first determine the $x$-coordinate of the point where the bore reaches the edge of the sphere (which is the "edge" in the front of the diagram). We have $x^2 + y^2 = 25$, since the sphere has radius 5. But at this point, $y = 2$, so that $x^2 = 21$ and $x = \sqrt{21}$. Then the volume of metal removed consists of three pieces: a cylinder of height $2\sqrt{21}$ and radius 2, and two spherical "caps" of height $5 - \sqrt{21}$ taken from a sphere of radius 5. We can compute the volume of those spherical caps using Problem 14 with $R = 5$ and $h = 5 - \sqrt{21}$, giving a volume of

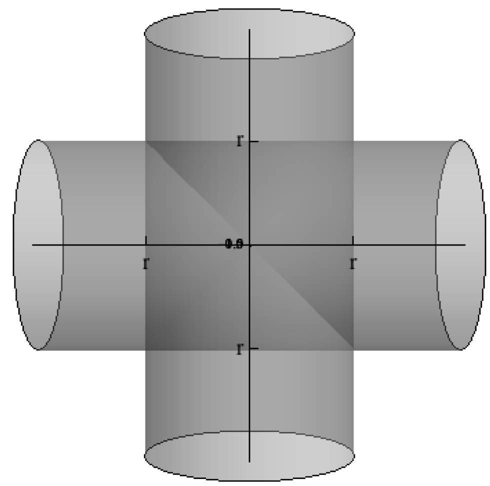
$$\pi h^2\left(R - \frac{h}{3}\right) = \pi(5 - \sqrt{21})^2\left(5 - \frac{5 - \sqrt{21}}{3}\right) = \pi\left(\frac{250}{3} - 18\sqrt{21}\right).$$

Then the total volume of the bore is

$$\begin{aligned} V &= \text{ volume of cylinder } + 2 \text{ volume of cap} \\ &= \pi \cdot 2^2 \cdot 2\sqrt{21} + 2\pi\left(\frac{250}{3} - 18\sqrt{21}\right) = \boxed{\pi\left(\frac{500}{3} - 28\sqrt{21}\right)}. \end{aligned}$$

22. Place the pipes so that their axes lie along the $x$ and $y$-axes. Two plots of the pipes are below; the one on the right is the view from above.

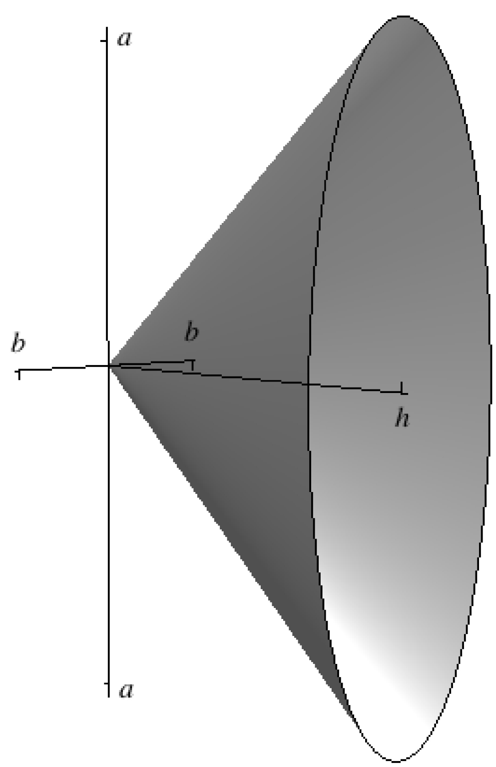





Consider a horizontal slice through the pipes at a distance $R$ from the $xy$-plane. The cross section of one pipe is the lines $x = \pm\sqrt{r^2 - R^2}$ and the cross section of the other is the lines $y = \pm\sqrt{r^2 - R^2}$. Therefore the slice representing the cross sectional area contained in both pipes at this level is a square of side $2\sqrt{r^2 - R^2}$ (see the right-hand diagram). The possible distance $R$ range from $-r$ to $r$, so the volume of the solid bounded by both pipes is

$$V = \int_{-r}^{r} \left(2\sqrt{r^2 - R^2}\right)^2 dR = 4\int_{-r}^{r} (r^2 - R^2)\, dR = 4\left[r^2 R - \frac{1}{3}R^3\right]_{-r}^{r} = \boxed{\frac{16}{3}r^3}.$$

23. Place the origin at the tip of the cone, with the positive $x$-axis pointing in the direction of the base, so that the base of the cone is an ellipse at $x = h$:



Then for a given value of $x$, similar triangles tell us that the cross section at $x$ is an ellipse with major axis $2a\frac{x}{h}$ and minor axis $2b\frac{x}{h}$. We must integrate over the area of these ellipses. Now by the hint, the area of each ellipse is

$$\pi \cdot \left(a\frac{x}{h}\right) \cdot \left(b\frac{x}{h}\right) = \pi\frac{ab}{h^2}x^2,$$

so the volume is

$$V = \int_0^h \left(\pi\frac{ab}{h^2}x^2\right) dx = \pi\frac{ab}{h^2}\left[\frac{1}{3}x^3\right]_0^h = \boxed{\frac{1}{3}\pi abh}.$$

24. The parallelepiped is made up of a number of slices, each of which is a parallelogram with sides $a$ and $b$ where the angle between $a$ and $b$ is $\theta$. The area of such a parallelogram is $ab\sin\theta$. The height of the parallelepiped is $c\sin\phi$, so the volume is

$$V = \int_0^{c\sin\phi} ab\sin\theta\, dx = \boxed{abc\sin\phi\sin\theta}.$$

## AP® Practice Problems

1. Determine the point(s) of intersection of $f(x) = 3x$ and $g(x) = x^2$:

$$\begin{aligned} \text{Let } 3x &= x^2 \\ x^2 - 3x &= 0 \\ x(x-3) &= 0 \\ x &= 0 \text{ or } x = 3 \end{aligned}$$

Then

$$\begin{aligned} V &= \int_0^3 \left(3x - x^2\right)^2 dx \\ &= \int_0^3 \left(9x^2 - 6x^3 + x^4\right) dx \\ &= \left[3x^3 - \frac{3x^4}{2} + \frac{x^5}{5}\right]_0^3 \\ &= \left[3\,(3)^3 - \frac{3\,(3^4)}{2} + \frac{3^5}{5} - 0\right] \\ &= \boxed{\frac{81}{10}} \end{aligned}$$

CHOICE B

2. $$\begin{aligned} V &= \int_0^2 \frac{1}{2} \cdot \pi \left(\frac{e^2 - e^x}{2}\right)^2 dx \\ &= \boxed{\frac{1}{2}\int_0^2 \pi \left(\frac{e^2 - e^x}{2}\right)^2 dx} \end{aligned}$$

CHOICE C

3. $$\begin{aligned} V &= \int_0^4 \left(e^{-x}+1-0\right)^2\,dx \\ &= \int_0^4 \left(e^{-2x}+2e^{-x}+1\right)dx \\ &= \left[\frac{e^{-2x}}{-2}-2e^{-x}+x\right]_0^4 \\ &= \left(\frac{e^{-8}}{-2}-2e^{-4}+4\right)-\left(\frac{e^{0}}{-2}-2e^{0}-0\right) \\ &= \frac{1}{-2e^8}-\frac{2}{e^4}+4+\frac{1}{2}+2 \\ &= \frac{1}{2}\left(\frac{-1}{e^8}-\frac{4}{e^4}+13\right) \\ &= \boxed{\frac{1}{2}\left(13-e^{-8}-4e^{-4}\right)} \end{aligned}$$

CHOICE D

4. $$\begin{aligned} V &= \int_0^3 \left(-x^2+9\right)^2\,dx \\ &= \int_0^3 \left(x^4-18x^2+81\right)dx \\ &= \left[\frac{x^5}{5}-6x^3+81x\right]_0^3 \\ &= \frac{3^5}{5}-6\,(3)^3+81\,(3)-0 \\ &= \boxed{\frac{648}{5}} \end{aligned}$$

CHOICE C

5. $$\begin{aligned} V &= \int_0^3 \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\,(2x)^2\,dx \\ &= \sqrt{3}\int_0^3 x^2\,dx \\ &= \frac{\sqrt{3}}{3}\left[x^3\right]_0^3 \\ &= \frac{\sqrt{3}}{3}\left[27-0\right] \\ &= \boxed{9\sqrt{3}} \end{aligned}$$

CHOICE C

6. $$\begin{aligned} V &= \int_0^4 (x-0)^2\,dy \\ &= \int_0^4 x^2\,dy \\ &= \int_0^4 y\,dy \\ &= \left[\frac{y^2}{2}\right]_0^4 \\ &= \frac{16}{2} - 0 \\ &= \boxed{8} \end{aligned}$$

   CHOICE B

7. For $y = \ln x$, $x = e^y$.

   Then

$$\begin{aligned} V &= \int_0^1 \frac{1}{2}\cdot\pi\left(\frac{e-x}{2}\right)^2 dy \\ &= \frac{\pi}{8}\int_0^1 (e-e^y)^2\,dy \\ &= \frac{\pi}{8}\int_0^1 \left(e^2 - 2e\cdot e^y + e^{2y}\right) dy \\ &= \frac{\pi}{8}\int_0^1 \left(e^2 - 2e^{y+1} + e^{2y}\right) dy \\ &= \frac{\pi}{8}\left[e^2 y - 2e^{y+1} + \frac{1}{2}e^{2y}\right]_0^1 \\ &= \frac{\pi}{8}\left[\left(e^2 - 2e^2 + \frac{1}{2}e^2\right) - \left(-2e + \frac{1}{2}e^0\right)\right] \\ &= \frac{\pi}{8}\left[-\frac{e^2}{2} + 2e - \frac{1}{2}\right] \\ &= \boxed{\frac{\pi}{16}\left[-e^2 + 4e - 1\right]} \end{aligned}$$

   CHOICE B

8. **(a)** Determine the point of intersection of $y = 2\sqrt{x}$ and $y = (x-2)^2$, for the upper limit of the integral:

$$\begin{aligned} 2\sqrt{x} &= (x-2)^2 \\ \left(2\sqrt{x}\right)^2 &= (x-2)^4 \\ 4x &= x^4 - 8x^3 + 24x^2 - 32x + 16 \\ x^4 - 8x^3 + 24x^2 - 36x + 16 &= 0 \\ (x-4)\left(x^3 - 4x^2 + 8x - 4\right) &= 0 \\ x &= 4 \end{aligned}$$

Then

$$\begin{aligned} A &= \int_1^4 2\sqrt{x} - (x^2 - 4x + 4)\,dx \\ &= \int_1^4 (2\sqrt{x} - x^2 + 4x - 4)\,dx \\ &= \left[\frac{4x^{3/2}}{3} - \frac{x^3}{3} + 2x^2 - 4x\right]_1^4 \\ &= \frac{4}{3}(4)^{3/2} - \frac{4^3}{3} + 2\left(4^2\right) - 4(4) - \left(\frac{4}{3} - \frac{1}{3} + 2 - 4\right) \\ &= \boxed{\frac{19}{3}} \end{aligned}$$

**(b)** $V = \pi \int_1^4 \left[\left(2\sqrt{x}\right)^2 - \left((x-2)^2\right)^2\right] dx = \boxed{\pi \int_1^4 \left[4x - (x-2)^4\right] dx}$

**(c)** By the shell method:

$$\boxed{V = 2\pi \int_1^4 x\left[2\sqrt{x} - (x-2)^2\right] dx}$$

**(d)** $\boxed{V = \int_1^4 \left[2\sqrt{x} - (x-2)^2\right] dx}$

## 6.5 Arc Length; Surface Area of a Solid of Revolution

### Concepts and Vocabulary

1. False. The integrand should be $\sqrt{1+[f'(x)]^2}$, not $\sqrt{1+[f(x)]^2}$: $s = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$.
2. True. By regarding $f$ as a piecewise continuous function, we may be able to find the arc length of each continuous piece and add up all the pieces.

### Skill Building

3. With $f(x) = 3x - 1$, we have $f'(x) = 3$. Since $f'(x)$ is continuous everywhere, the Arc Length Formula applies, so that the arc length is

$$s = \int_1^3 \sqrt{1^2 + 3^2}\,dx = \int_1^3 \sqrt{10}\,dx = \boxed{2\sqrt{10}}.$$

Using the Distance Formula, we get $s = \sqrt{(3-1)^2 + (8-2)^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$, and the answers are the same.

4. With $f(x) = -4x + 1$, we have $f'(x) = -4$. Since $f'(x)$ is continuous everywhere, the Arc Length Formula applies, so that the arc length is

$$s = \int_{-1}^1 \sqrt{1^2 + (-4)^2}\,dx = \int_{-1}^1 \sqrt{17}\,dx = \boxed{2\sqrt{17}}.$$

Using the Distance Formula, we get $s = \sqrt{(1-(-1))^2 + (-3-5)^2 +} = \sqrt{2^2 + 8^2} = \sqrt{68} = 2\sqrt{17}$, and the answers are the same.

5. First solve for $y$, giving $y = \frac{2}{3}x + \frac{4}{3}$ and $y' = \frac{2}{3}$. Since $f'(x)$ is continuous everywhere, the Arc Length Formula applies, so that the arc length is

$$s = \int_1^4 \sqrt{1^2 + \left(\frac{2}{3}\right)^2}\, dx = \int_1^4 \frac{\sqrt{13}}{3}\, dx = \boxed{\sqrt{13}}.$$

Using the Distance Formula, we get $s = \sqrt{(4-1)^2 + (4-2)^2} = \sqrt{9+4} = \sqrt{13}$, and the answers are the same.

6. First solve for $y$, giving $y = -\frac{3}{4}x + 3$ and $y' = -\frac{3}{4}$. Since $f'(x)$ is continuous everywhere, the Arc Length Formula applies, so that the arc length is

$$s = \int_0^4 \sqrt{1^2 + \left(-\frac{3}{4}\right)^2}\, dx = \int_0^4 \frac{5}{4}\, dx = \boxed{5}.$$

Using the Distance Formula, we get $s = \sqrt{(4-0)^2 + (0-3)^2} = \sqrt{16+9} = 5$, and the answers are the same.

7. The derivative of $y = x^{2/3} + 1$ is $y = \frac{2}{3}x^{-1/3}$, so that $y$ has a continuous derivative on an interval containing 1 and 8. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_1^8 \sqrt{1^2 + \left(\frac{2}{3}x^{-1/3}\right)^2}\, dx = \frac{1}{3}\int_1^8 \sqrt{9 + 4x^{-2/3}}\, dx = \frac{1}{3}\int_1^8 x^{-1/3}\sqrt{9x^{2/3}+4}\, dx.$$

Now use the substitution $u = 9x^{2/3} + 4$, so that $du = 6x^{-1/3}\, dx$. Then $x = 1$ corresponds to $u = 13$ while $x = 8$ corresponds to $u = 40$, and we get

$$s = \frac{1}{3}\cdot\frac{1}{6}\int_{13}^{40} \sqrt{u}\, du = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40} = \frac{1}{27}\left(40^{3/2} - 13^{3/2}\right) = \boxed{\frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)}.$$

8. The derivative of $y = x^{2/3} + 6$ is $y = \frac{2}{3}x^{-1/3}$, so that $y$ has a continuous derivative on an interval containing 1 and 8. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_1^8 \sqrt{1^2 + \left(\frac{2}{3}x^{-1/3}\right)^2}\, dx = \frac{1}{3}\int_1^8 \sqrt{9 + 4x^{-2/3}}\, dx = \frac{1}{3}\int_1^8 x^{-1/3}\sqrt{9x^{2/3}+4}\, dx.$$

Now use the substitution $u = 9x^{2/3} + 4$, so that $du = 6x^{-1/3}\, dx$. Then $x = 1$ corresponds to $u = 13$ while $x = 8$ corresponds to $u = 40$, and we get

$$s = \frac{1}{3}\cdot\frac{1}{6}\int_{13}^{40} \sqrt{u}\, du = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40} = \frac{1}{27}\left(40^{3/2} - 13^{3/2}\right) = \boxed{\frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right)}.$$

9. The derivative of $y = x^{3/2}$ is $y = \frac{3}{2}x^{1/2}$, so that $y$ has a continuous derivative on an interval containing 0 and 4. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_0^4 \sqrt{1^2 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx = \frac{1}{2}\int_0^4 \sqrt{4+9x}\, dx.$$

Now substitute $u = 4 + 9x$; then $du = 9\, dx$. Further, $x = 0$ corresponds to $u = 4$ while $x = 4$ corresponds to $u = 40$, so we get

$$\frac{1}{2}\int_0^4 \sqrt{4+9x}\, dx = \frac{1}{2}\cdot\frac{1}{9}\int_4^{40} u^{1/2}\, du = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_4^{40} = \frac{1}{27}\left(40\sqrt{40} - 8\right) = \boxed{\frac{1}{27}(80\sqrt{10} - 8)}.$$

10. The derivative of $x^{3/2}+4$ is $\frac{3}{2}x^{1/2}$, so that $y$ has a continuous derivative on an interval containing 1 and 4. Therefore the Arc Length Formula applies, and the arc length is

$$s=\int_1^4\sqrt{1^2+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx=\frac{1}{2}\int_1^4\sqrt{4+9x}\,dx.$$

Now substitute $u=4+9x$; then $du=9\,dx$. Further, $x=1$ corresponds to $u=13$ while $x=4$ corresponds to $u=40$, so we get

$$\begin{aligned}\frac{1}{2}\int_1^4\sqrt{4+9x}\,dx=\frac{1}{2}\cdot\frac{1}{9}\int_{13}^{40}u^{1/2}\,du&=\frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40}\\&=\frac{1}{27}\left(40\sqrt{40}-13\sqrt{13}\right)=\boxed{\frac{1}{27}(80\sqrt{10}-13\sqrt{13})}.\end{aligned}$$

11. First solve for $y$, giving (since we are given $y\geq 0$) $y=\frac{2}{3}x^{3/2}$. Then $y'=x^{1/2}$ has a continuous derivative on an interval containing 0 and 1, so that the Arc Length Formula applies, and the arc length is

$$s=\int_0^1\sqrt{1^2+\left(x^{1/2}\right)^2}\,dx=\int_0^1\sqrt{1+x}\,dx=\left[\frac{2}{3}(1+x)^{3/2}\right]_0^1=\boxed{\frac{2}{3}(2\sqrt{2}-1)}.$$

12. We have $y'=\frac{x^2}{2}-\frac{1}{2x^2}$, which is continuous on an interval containing 1 and 3. Therefore the Arc Length Formula applies, and the arc length is

$$\begin{aligned}s&=\int_1^3\sqrt{1^2+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2}\,dx\\&=\int_1^3\sqrt{1+\frac{x^4}{4}-\frac{1}{2}+\frac{1}{4x^4}}\,dx\\&=\int_1^3\sqrt{\frac{x^4}{4}+\frac{1}{2}+\frac{1}{4x^4}}\,dx\\&=\int_1^3\sqrt{\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)^2}\,dx\\&=\int_1^3\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)\,dx\\&=\left[\frac{x^3}{6}-\frac{1}{2x}\right]_1^3\\&=\boxed{\frac{14}{3}}.\end{aligned}$$

13. We have $y' = (x^2+1)^{1/2} \cdot 2x = 2x\sqrt{x^2+1}$, which is continuous on an interval containing 1 and 4. Therefore the Arc Length Formula applies, and the arc length is

$$\begin{aligned} s &= \int_1^4 \sqrt{1^2 + \left(2x\sqrt{x^2+1}\right)^2}\,dx \\ &= \int_1^4 \sqrt{4x^4+4x^2+1}\,dx \\ &= \int_1^4 \sqrt{(2x^2+1)^2}\,dx \\ &= \int_1^4 (2x^2+1)\,dx \\ &= \left[\frac{2}{3}x^3 + x\right]_1^4 \\ &= \boxed{45}. \end{aligned}$$

14. We have $y' = \frac{1}{2}(x^2+2)^{1/2} \cdot 2x = x\sqrt{x^2+2}$, which is continuous on an interval containing 2 and 4. Therefore the Arc Length Formula applies, and the arc length is

$$\begin{aligned} s &= \int_2^4 \sqrt{1^2 + \left(x\sqrt{x^2+2}\right)^2}\,dx \\ &= \int_2^4 \sqrt{x^4+2x^2+1}\,dx \\ &= \int_2^4 \sqrt{(x^2+1)^2}\,dx \\ &= \int_2^4 (x^2+1)\,dx \\ &= \left[\frac{1}{3}x^3 + x\right]_2^4 \\ &= \boxed{\frac{62}{3}}. \end{aligned}$$

15. We have

$$y' = \frac{2}{9}\sqrt{3} \cdot \frac{3}{2}(3x^2+1)^{1/2} \cdot 6x = 2\sqrt{3}\,x\sqrt{3x^2+1}.$$

This is continuous on an interval containing $-1$ and 2, so the Arc Length Formula applies and the arc length is

$$\begin{aligned} s &= \int_{-1}^2 \sqrt{1^2 + \left(2\sqrt{3}\,x\sqrt{3x^2+1}\right)^2}\,dx \\ &= \int_{-1}^2 \sqrt{36x^4+12x^2+1}\,dx \\ &= \int_{-1}^2 \sqrt{(6x^2+1)^2}\,dx \\ &= \int_{-1}^2 (6x^2+1)\,dx \\ &= \left[2x^3 + x\right]_{-1}^2 \\ &= \boxed{21}. \end{aligned}$$

16. We have

$$y' = \frac{3}{2}\left(1 - x^{2/3}\right)^{1/2} \cdot \left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3}\sqrt{1 - x^{2/3}}.$$

This is continuous on an interval containing $\frac{1}{8}$ and 1, so that the Arc Length Formula applies and the arc length is

$$\begin{aligned} s &= \int_{1/8}^{1} \sqrt{1^2 + \left(-x^{-1/3}\sqrt{1 - x^{2/3}}\right)^2}\, dx \\ &= \int_{1/8}^{1} \sqrt{x^{-2/3}}\, dx \\ &= \int_{1/8}^{1} x^{-1/3}\, dx \\ &= \left[\frac{3}{2}x^{2/3}\right]_{1/8}^{1} \\ &= \boxed{\frac{9}{8}}. \end{aligned}$$

17. Solving for $y$ gives $y = \frac{x^4}{8} + \frac{1}{4x^2}$. Then $y' = \frac{x^3}{2} - \frac{1}{2x^3}$, which is continuous on an interval containing 1 and 2. Therefore the Arc Length Formula applies, and the arc length is

$$\begin{aligned} s &= \int_{1}^{2} \sqrt{1^2 + \left(\frac{x^3}{2} - \frac{1}{2x^3}\right)^2}\, dx \\ &= \int_{1}^{2} \sqrt{1 + \frac{x^6}{4} - \frac{1}{2} + \frac{1}{4x^6}}\, dx \\ &= \int_{1}^{2} \sqrt{\frac{x^6}{4} + \frac{1}{2} + \frac{1}{4x^6}}\, dx \\ &= \int_{1}^{2} \sqrt{\left(\frac{x^3}{2} + \frac{1}{2x^3}\right)^2}\, dx \\ &= \int_{1}^{2} \left(\frac{x^3}{2} + \frac{1}{2x^3}\right) dx \\ &= \left[\frac{x^4}{8} - \frac{1}{4x^2}\right]_{1}^{2} \\ &= \boxed{\frac{33}{16}}. \end{aligned}$$

18. Solving for $y$ gives $y^2 = \frac{4}{9}(1+x^2)^3$, so that $y = \frac{2}{3}(1+x^2)^{3/2}$ (since $y \geq 0$). Then $y' = (1+x^2)^{1/2} \cdot 2x = 2x\sqrt{1+x^2}$. This is continuous everywhere, so that the Arc Length Formula applies and the arc length is

$$\begin{aligned} s &= \int_0^{2\sqrt{2}} \sqrt{1^2 + \left(2x\sqrt{1+x^2}\right)^2}\,dx \\ &= \int_0^{2\sqrt{2}} \sqrt{4x^4 + 4x^2 + 1}\,dx \\ &= \int_0^{2\sqrt{2}} \sqrt{(2x^2+1)^2}\,dx \\ &= \int_0^{2\sqrt{2}} (2x^2+1)\,dx \\ &= \left[\frac{2}{3}x^3 + x\right]_0^{2\sqrt{2}} \\ &= \boxed{\frac{38}{3}\sqrt{2}}. \end{aligned}$$

19. We get $y' = \frac{1}{\sin x} \cdot (-\cos x) = -\cot x$. This is continuous on an interval containing $\frac{\pi}{6}$ and $\frac{\pi}{3}$, so the Arc Length Formula applies. Using the identity $1 + \cot^2 x = \csc^2 x$, the arc length is

$$\begin{aligned} s &= \int_{\pi/6}^{\pi/3} \sqrt{1^2 + (-\cot x)^2}\,dx \\ &= \int_{\pi/6}^{\pi/3} \sqrt{1 + \cot^2 x}\,dx \\ &= \int_{\pi/6}^{\pi/3} \csc x\,dx \\ &= \left[\ln\left|\csc x - \cot x\right|\right]_{\pi/6}^{\pi/3} \\ &= \boxed{\ln\left|\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right| - \ln\left|2 - \sqrt{3}\right|}. \end{aligned}$$

This expression can be simplified to $\boxed{-\ln(2\sqrt{3} - 3)}$.

20. We get $y' = \frac{1}{\cos x} \cdot \sin x = \tan x$. This is continuous on an interval containing $\frac{\pi}{6}$ and $\frac{\pi}{3}$, so the Arc Length Formula applies. Using the identity $1 + \tan^2 x = \sec^2 x$, the arc length is

$$\begin{aligned} s &= \int_{\pi/6}^{\pi/3} \sqrt{1^2 + (\tan x)^2}\, dx \\ &= \int_{\pi/6}^{\pi/3} \sqrt{1 + \tan^2 x}\, dx \\ &= \int_{\pi/6}^{\pi/3} \sec x\, dx \\ &= \left[\ln|\sec x + \tan x|\right]_{\pi/6}^{\pi/3} \\ &= \ln\left|2 + \sqrt{3}\right| - \ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| \\ &= \ln(2 + \sqrt{3}) - \ln(\sqrt{3}) \\ &= \ln\frac{2 + \sqrt{3}}{\sqrt{3}} = \boxed{\ln\frac{2\sqrt{3} + 3}{3}}. \end{aligned}$$

21. Since $y \geq 0$, solving for $y$ gives $y = \frac{1}{2}(x+1)^{3/2}$, so that $y' = \frac{3}{4}\sqrt{x+1}$, which is continuous on an interval containing $-1$ and $16$. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_{-1}^{16} \sqrt{1^2 + \left(\frac{3}{4}\sqrt{x+1}\right)^2}\, dx = \frac{1}{4}\int_{-1}^{16} \sqrt{9x + 25}\, dx.$$

Now use the substitution $u = 9x + 25$, so that $du = 9\, dx$. Then $x = -1$ corresponds to $u = 16$ and $x = 16$ corresponds to $u = 169$, so we get

$$\frac{1}{4}\int_{-1}^{16} \sqrt{9x+25}\, dx = \frac{1}{4} \cdot \frac{1}{9}\int_{16}^{169} u^{1/2}\, du = \frac{1}{36}\left[\frac{2}{3}u^{3/2}\right]_{16}^{169} = \frac{1}{54}(13^3 - 4^3) = \boxed{\frac{79}{2}}.$$

22. The derivative of $x^{3/2} + 8$ is $\frac{3}{2}x^{1/2}$, so that $y$ has a continuous derivative on an interval containing 0 and 4. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_0^4 \sqrt{1^2 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx = \frac{1}{2}\int_0^4 \sqrt{4 + 9x}\, dx = \frac{1}{2}\left[\frac{2}{27}(4+9x)^{3/2}\right]_0^4 = \boxed{\frac{1}{27}(80\sqrt{10} - 8)}.$$

23. Since $y' = \frac{2}{3}x^{-1/3}$ is not continuous (or even defined) at $x = 0$, we try partitioning the $y$-axis instead. Solving $y = x^{2/3}$ for $x$ gives $x = y^{3/2}$, so that $x' = \frac{3}{2}\sqrt{y}$. Note that this is the same curve as the one we are interested in, just expressed differently, so that it has the same arc length. Now, $x = 0$ corresponds to $y = 0$, and $x = 1$ to $y = 1$. Since $\frac{3}{2}\sqrt{y}$ is continuous on an interval containing 0 and 1, the Arc Length Formula for partitioning along the $y$-axis applies, and the arc length is

$$s = \int_0^1 \sqrt{1 + \left(\frac{3}{2}\sqrt{y}\right)^2}\, dy = \frac{1}{2}\int_0^1 \sqrt{4 + 9y}\, dy.$$

Now substitute $u = 4 + 9y$; then $du = 9\, dy$. Further, $y = 0$ corresponds to $u = 4$ while $y = 1$ corresponds to $u = 13$, so we get

$$\frac{1}{2}\int_0^1 \sqrt{4+9y}\, dy = \frac{1}{2} \cdot \frac{1}{9}\int_4^{13} u^{1/2}\, du = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_4^{13} = \boxed{\frac{1}{27}(13\sqrt{13} - 8)}.$$

24. Since $y = x^{2/3}$ is symmetric about the $y$-axis, its arc length from $x = -1$ to $x = 0$ is the same as its arc length from $x = 0$ to $x = 1$. But this arc length was just computed in Problem 23 to be $\boxed{\frac{1}{27}(13\sqrt{13} - 8)}$.

25. Since we wish to partition along the $y$-axis, we solve for $x$, giving $x = 2y^{3/2} - 1$. (Note that we took the positive square root of $4y^3$ since we have $x \geq -1$.) Then $x' = 3y^{1/2} = 3\sqrt{y}$, which is continuous on an interval containing 0 and 1. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_0^1 \sqrt{1^2 + (3\sqrt{y})^2}\,dy = \int_0^1 \sqrt{9y+1}\,dy.$$

Now make the substitution $u = 9y + 1$, so that $du = 9\,dy$. Then $y = 0$ corresponds to $u = 1$, and $y = 1$ corresponds to $u = 10$, so we get

$$\int_0^1 \sqrt{9y+1}\,dy = \frac{1}{9}\int_1^{10} u^{1/2}\,du = \frac{1}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \boxed{\frac{2}{27}(10\sqrt{10} - 1)}.$$

26. We get $x' = (y-5)^{1/2} = \sqrt{y-5}$, which is continuous on an interval containing 5 and 6, so the Arc Length Formula applies, and the arc length is

$$s = \int_5^6 \sqrt{1^2 + \left(\sqrt{y-5}\right)^2}\,dy = \int_5^6 \sqrt{y-4}\,dy = \left[\frac{2}{3}(y-4)^{3/2}\right]_5^6 = \boxed{\frac{2}{3}(2\sqrt{2} - 1)}.$$

27. **(a)** We have $y' = 2x$, which is continuous everywhere, so the Arc Length Formula applies and the arc length is

$$s = \int_0^2 \sqrt{1^2 + (2x)^2}\,dx = \int_0^2 \sqrt{4x^2+1}\,dx.$$

**(b)** Using technology, this evaluates to $s \boxed{\approx 4.64678}$.

28. We will determine arc length by subdividing along the $y$-axis.

**(a)** $x' = 2y$, which is continuous everywhere, so the Arc Length Formula applies and the arc length is

$$s = \int_1^3 \sqrt{1^2 + (2y)^2}\,dy = \int_1^3 \sqrt{4y^2+1}\,dy.$$

**(b)** Using technology, this evaluates to $s \boxed{\approx 8.26815}$.

29. **(a)** $y' = \frac{1}{2}(25 - x^2)^{-1/2} \cdot (-2x) = -x(25 - x^2)^{-1/2}$. This is continuous on an interval containing 0 and 4, so the Arc Length Formula applies, and the arc length is

$$s = \int_0^4 \sqrt{1^2 + \left(-x(25-x^2)^{-1/2}\right)^2}\,dx = \int_0^4 \sqrt{1 + \frac{x^2}{25-x^2}}\,dx = \int_0^4 \sqrt{\frac{25}{25-x^2}}\,dx.$$

**(b)** Using technology, this evaluates to $s \boxed{\approx 4.63648}$.

30. We will determine arc length by partitioning along the $y$-axis.

**(a)** $x' = \frac{1}{2}(4 - y^2)^{-1/2} \cdot (-2y) = -y(4 - y^2)^{-1/2}$. This is continuous on an interval containing 0 and 1, so the Arc Length Formula applies, and the arc length is

$$s = \int_0^1 \sqrt{1^2 + \left(-y(4-y^2)^{-1/2}\right)^2}\,dy = \int_0^1 \sqrt{1 + \frac{y^2}{4-y^2}}\,dy = \int_0^1 \sqrt{\frac{4}{4-y^2}}\,dy.$$

**(b)** Using technology, this evaluates to $s \boxed{\approx 1.0472}$.

31. **(a)** $y' = \cos x$, which is continuous everywhere, so the Arc Length Formula applies. The arc length is

$$s = \int_0^{\pi/2} \sqrt{1^2 + (\cos x)^2}\, dx = \int_0^{\pi/2} \sqrt{1 + \cos^2 x}\, dx.$$

**(b)** Using technology, this evaluates to $s \boxed{\approx 1.9101}$.

32. We will determine arc length by partitioning along the $y$-axis.

**(a)** $x' = 1 + \frac{1}{y}$, which is continuous on an interval containing 1 and 4, so the Arc Length Formula applies. The arc length is

$$s = \int_1^4 \sqrt{1^2 + \left(1 + \frac{1}{y}\right)^2}\, dy = \int_1^4 \sqrt{\frac{1}{y^2} + \frac{2}{y} + 2}\, dy = \int_1^4 \frac{\sqrt{2y^2 + 2y + 1}}{y}\, dy.$$

**(b)** Using technology, this evaluates to $s \boxed{\approx 5.32324}$.

33. If $f(x) = 3x + 5$, then $f'(x) = 3$. This is continuous everywhere, so the Surface Area Formula applies. Therefore the surface area is

$$S = 2\pi \int_0^2 (3x+5)\sqrt{1 + (3)^2}\, dx = 2\pi \int_0^2 (3x+5)\sqrt{10}\, dx = 2\pi\sqrt{10} \int_0^2 (3x+5)\, dx$$
$$= 2\pi\sqrt{10}\left[\frac{3}{2}x^2 + 5x\right]_0^2 = 2\pi\sqrt{10}\left[\left(\frac{3}{2}\cdot 2^2 + 5\cdot 2\right) - 0\right] = 2\pi\sqrt{10}\cdot 16 = \boxed{32\pi\sqrt{10}}.$$

34. Surface area $= 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx$

For $f(x) = -x + 5, \quad f'(x) = -1$

$$\begin{aligned} S &= 2\pi \int_1^5 (-x+5)\sqrt{1 + 1^2}\, dx \\ &= 2\pi \int_1^5 (-x+5)\sqrt{2}\, dx \\ &= 2\pi\sqrt{2}\left[\frac{-x^2}{2} + 5x\right]_1^5 \\ &= 2\pi\sqrt{2}\left[\left(\frac{-25}{2} + 25\right) - \left(\frac{-1}{2} + 5\right)\right] \\ &= 2\pi\sqrt{2}\left[\frac{25}{2} - \frac{9}{2}\right] \\ &= \boxed{16\pi\sqrt{2}} \end{aligned}$$

35. If $f(x) = \sqrt{3x} = (3x)^{1/2}$, then $f'(x) = \frac{1}{2}(3x)^{-1/2}\cdot 3 = \frac{3}{2\sqrt{3x}}$. This is continuous on an interval containing 1 and 2, so the Surface Area Formula applies. Therefore the surface area is

$$S = 2\pi \int_1^2 \sqrt{3x}\sqrt{1 + \left(\frac{3}{2\sqrt{3x}}\right)^2}\, dx = 2\pi \int_1^2 \sqrt{3x}\sqrt{1 + \frac{9}{12x}}\, dx = 2\pi \int_1^2 \sqrt{3x}\sqrt{1 + \frac{3}{4x}}\, dx$$
$$= 2\pi \int_1^2 \sqrt{3x}\cdot\frac{\sqrt{4x+3}}{2\sqrt{x}}\, dx = \pi\sqrt{3}\int_1^2 \sqrt{4x+3}\, dx.$$

Now let $u = 4x + 3$, so $du = 4\,dx$, so $dx = \frac{1}{4}\,du$. Also, when $x = 1$, then $u = 4(1) + 3 = 7$ and when $x = 2$, then $u = 4(2) + 3 = 11$. Therefore

$$S = \pi\sqrt{3}\int_7^{11} u^{1/2} \cdot \frac{1}{4}\,du = \frac{\pi\sqrt{3}}{4}\int_7^{11} u^{1/2}\,du = \frac{\pi\sqrt{3}}{4}\left[\frac{2}{3}u^{3/2}\right]_7^{11} = \frac{\pi\sqrt{3}}{6}\left(11^{3/2} - 7^{3/2}\right)$$
$$= \boxed{\frac{\sqrt{3}}{6}\pi\left(11\sqrt{11} - 7\sqrt{7}\right)}.$$

36. Surface area $= S = 2\pi \displaystyle\int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx$

For $f(x) = \sqrt{2x} = (2x)^{1/2}$, $\quad f'(x) = \frac{1}{2}(2x)^{-1/2}(2) = (2x)^{-1/2}$

$$\begin{aligned} S &= 2\pi\int_2^8 \sqrt{2x}\left(\sqrt{1 + \left((2x)^{-1/2}\right)^2}\right) dx \\ &= 2\pi\int_2^8 \sqrt{2x}\left(\sqrt{1 + \frac{1}{2x}}\right) dx \\ &= 2\pi\int_2^8 \sqrt{2x+1}\,dx \\ &= \left[\frac{2\pi}{3}(2x+1)^{3/2}\right]_2^8 \\ &= \boxed{\frac{2\left(17^{3/2} - 5^{3/2}\right)\pi}{3} = \frac{2}{3}\pi\left(17\sqrt{17} - 5\sqrt{5}\right)} \end{aligned}$$

37. If $f(x) = 2\sqrt{x} = 2x^{1/2}$, then $f'(x) = 2 \cdot \frac{1}{2}x^{-1/2} = \frac{1}{\sqrt{x}}$. This is continuous on an interval containing 1 and 4, so the Surface Area Formula applies. Therefore the surface area is

$$S = 2\pi\int_1^4 2\sqrt{x} \cdot \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}\,dx = 2\pi\int_1^4 2\sqrt{x} \cdot \sqrt{1 + \frac{1}{x}}\,dx = 2\pi\int_1^4 2\sqrt{x} \cdot \sqrt{\frac{x+1}{x}}\,dx$$
$$= 2\pi\int_1^4 2\sqrt{x} \cdot \frac{\sqrt{x+1}}{\sqrt{x}}\,dx = 4\pi\int_1^4 \sqrt{x+1}\,dx.$$

Now let $u = x + 1$, so $du = dx$. Also, when $x = 1$, then $u = (1) + 1 = 2$ and when $x = 4$, then $u = (4) + 1 = 5$. Therefore

$$S = 4\pi\int_2^5 u^{1/2}\,dx = 4\pi\left[\frac{2}{3}u^{3/2}\right]_2^5 = 4\pi \cdot \frac{2}{3}\left(5^{3/2} - 2^{3/2}\right) = \boxed{\frac{8}{3}\pi\left(5\sqrt{5} - 2\sqrt{2}\right)}.$$

38. Surface area $= S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$

For $f(x) = 4\sqrt{x}, \quad f'(x) = 2x^{-1/2}$

$$\begin{aligned} S &= 2\pi \int_4^9 4\sqrt{x}\sqrt{1+\left(2x^{-1/2}\right)^2}\,dx \\ &= 2\pi \int_4^9 4\sqrt{x}\sqrt{1+4x^{-1}}\,dx \\ &= 2\pi \int_4^9 4\sqrt{x+4}\,dx \\ &= \frac{16\pi}{3}\left[(x+4)^{3/2}\right]_4^9 \\ &= \boxed{\frac{16\pi\left(13^{3/2}-8^{3/2}\right)}{3} = \frac{16}{3}\pi\left(13\sqrt{13}-16\sqrt{2}\right)} \end{aligned}$$

39. If $f(x) = x^3$, then $f'(x) = 3x^2$. This is continuous everywhere, so the Surface Area Formula applies. Therefore the surface area is

$$S = 2\pi \int_0^2 x^3\sqrt{1+(3x^2)^2}\,dx = 2\pi \int_0^2 x^3\sqrt{1+9x^4}\,dx.$$

Now let $u = 1+9x^4$, so $du = 36x^3\,dx$, so $x^3\,dx = \frac{1}{36}\,du$. Also, when $x = 0$, then $u = 1+9(0)^4 = 1$, and when $x = 2$, then $u = 1+9(2)^4 = 145$. Therefore

$$S = 2\pi \int_1^{145} \sqrt{u}\cdot\frac{1}{36}\,du = \frac{\pi}{18}\int_1^{145} u^{1/2}\,du = \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{145} = \frac{\pi}{27}\left(145^{3/2}-1^{3/2}\right)$$

$$= \boxed{\frac{1}{27}\pi\left(145\sqrt{145}-1\right)}.$$

40. Surface area $= S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$

For $f(x) = \dfrac{x^3}{3}, \quad f'(x) = x^2$

$$\begin{aligned} S &= 2\pi \int_1^3 \left(\frac{x^3}{3}\right)\sqrt{1+(x^2)^2}\,dx \\ &= \frac{2\pi}{3}\int_1^3 x^3\sqrt{1+x^4}\,dx \end{aligned}$$

Let $u = 1+x^4, \quad du = 4x^3\,dx, \quad \dfrac{du}{4} = x^3\,dx$

$$\begin{aligned} S &= \frac{2\pi}{3}\int_2^{82}\sqrt{u}\left(\frac{du}{4}\right) \\ &= \frac{\pi}{6}\int_2^{82} u^{1/2}\,du \\ &= \frac{\pi}{6}\left(\frac{2}{3}\right)\left[u^{3/2}\right]_2^{82} \\ &= \frac{\pi}{9}\left(82^{3/2}-2^{3/2}\right) \\ &= \boxed{\frac{\left(82^{3/2}-2^{3/2}\right)\pi}{9} = \frac{1}{9}\pi\left(82\sqrt{82}-2\sqrt{2}\right)} \end{aligned}$$

41. If $f(x) = \sqrt{4-x^2} = \left(4-x^2\right)^{1/2}$, then $f'(x) = \frac{1}{2}\left(4-x^2\right)^{-1/2}(-2x) = -\frac{x}{\sqrt{4-x^2}}$. This is continuous on an interval containing 0 and 1, so the Surface Area Formula applies. Therefore the surface area is

$$S = 2\pi \int_0^1 \sqrt{4-x^2} \cdot \sqrt{1+\left(-\frac{x}{\sqrt{4-x^2}}\right)^2}\, dx = 2\pi \int_0^1 \sqrt{4-x^2} \cdot \sqrt{1+\frac{x^2}{4-x^2}}\, dx$$

$$= 2\pi \int_0^1 \sqrt{4-x^2} \cdot \frac{\sqrt{4-x^2+x^2}}{\sqrt{4-x^2}}\, dx = 2\pi \int_0^1 \sqrt{4}\, dx = 2\pi \cdot 2 \int_0^1 dx = 4\pi [x]_0^1 = 4\pi(1-0) = \boxed{4\pi}.$$

42. Surface area $= S = 2\pi \displaystyle\int_a^b f(x)\sqrt{1+(f'(x))^2}\, dx$

For $f(x) = \sqrt{9-4x^2} = \left(9-4x^2\right)^{1/2}$, $f'(x) = \dfrac{1}{2}\left(9-4x^2\right)^{-1/2}(-8x) = \dfrac{-4x}{\left(9-4x^2\right)^{1/2}}$

$$\begin{aligned}
S &= 2\pi \int_0^1 \sqrt{9-4x^2}\sqrt{1+\left(\frac{-4x}{\left(9-4x^2\right)^{1/2}}\right)^2}\, dx \\
&= 2\pi \int_0^1 \sqrt{9-4x^2}\sqrt{1+\frac{16x^2}{9-4x^2}}\, dx \\
&= 2\pi \int_0^1 \sqrt{9-4x^2+16x^2}\, dx \\
&= 2\pi \int_0^1 \sqrt{9+12x^2}\, dx \\
&= 2\pi\sqrt{3} \int_0^1 \sqrt{3+4x^2}\, dx
\end{aligned}$$

Let $x = \dfrac{\sqrt{3}\tan u_1}{2}$, $\quad dx = \dfrac{\sqrt{3}\sec^2 u_1}{2}\, du_1$, $\quad u_1 = \operatorname{Tan}^{-1}\left(\dfrac{2x}{\sqrt{3}}\right)$

For $x = 0$, $u_1 = \operatorname{Tan}^{-1}(0) = 0$. For $x = 1$, $u_1 = \operatorname{Tan}^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$

$$\begin{aligned}
S &= 2\pi\sqrt{3} \int_0^{\operatorname{Tan}^{-1}\left(2/\sqrt{3}\right)} \left[\sqrt{3+4\left(\frac{\sqrt{3}\tan u_1}{2}\right)^2}\right]\left(\frac{\sqrt{3}\sec^2 u_1}{2}\, du_1\right) \\
&= 2\pi\sqrt{3} \int_0^{\operatorname{Tan}^{-1}\left(2/\sqrt{3}\right)} \left[\sqrt{3+4\left(\frac{3\tan^2 u_1}{4}\right)}\right]\left(\frac{\sqrt{3}\sec^2 u_1}{2}\, du_1\right) \\
&= 2\pi\sqrt{3} \int_0^{\operatorname{Tan}^{-1}\left(2/\sqrt{3}\right)} \left[\sqrt{3+3\tan^2 u_1}\right]\left(\frac{\sqrt{3}\sec^2 u_1}{2}\, du_1\right) \\
&= 3\pi\sqrt{3} \int_0^{\operatorname{Tan}^{-1}\left(2/\sqrt{3}\right)} \left[\sqrt{1+\tan^2 u_1}\right]\sec^2 u_1 (du_1) \\
&= 3\pi\sqrt{3} \int_0^{\operatorname{Tan}^{-1}\left(2/\sqrt{3}\right)} [\sec u_1]\sec^2 u_1 (du_1) \\
&= 3\pi\sqrt{3} \int_0^{\operatorname{Tan}^{-1}\left(2/\sqrt{3}\right)} \sec^3 u_1 (du_1)
\end{aligned}$$

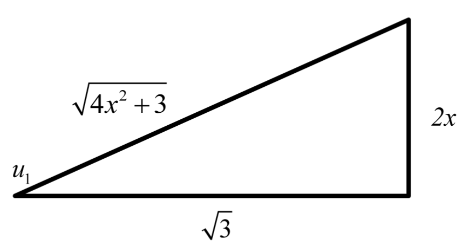
Using Integration by Parts

$$\begin{aligned} \text{Let } u_2 &= \sec u_1, & v &= \sec^2 u_1 \\ du_2 &= \sec u_1 \tan u_1 \, du_1, & dv &= \tan u_1 \end{aligned}$$

$$\int u_2 \, dv = u_2 v - \int v \, du$$

$$\begin{aligned} \int \sec^3 u_1 \, du_1 &= \sec u_1 \tan u_1 - \int \sec u_1 \tan u_1 (\tan u_1) \, du_1 \\ &= \sec u_1 \tan u_1 - \int \sec u_1 \tan^2 u_1 \, du_1 \\ &= \sec u_1 \tan u_1 - \int \sec u_1 \left(\sec^2 u_1 - 1\right) du_1 \\ &= \sec u_1 \tan u_1 - \int \left(\sec^3 u_1 - \sec u_1\right) du_1 \\ &= \sec u_1 \tan u_1 - \int \sec^3 u_1 du_1 + \int \sec u_1 \, du_1 \\ 2\int \sec^3 u_1 \, du_1 &= \sec u_1 \tan u_1 + \int \sec u_1 \, du_1 \\ &= \sec u_1 \tan u_1 + \int \sec u_1 \left(\frac{\sec u_1 + \tan u_1}{\sec u_1 + \tan u_1}\right) du_1 \\ &= \sec u_1 \tan u_1 + \int \left(\frac{\sec^2 u_1 + \sec u_1 \tan u_1}{\sec u_1 + \tan u_1}\right) du_1 \\ \text{Let } u_3 &= \sec u_1 + \tan u_1, \quad du_3 = \sec^2 u_1 + \sec u_1 \tan u_1 du_1 \\ &= \sec u_1 \tan u_1 + \int \frac{du_3}{u_3} \\ &= \sec u_1 \tan u_1 + \ln|u_3| \\ &= \sec u_1 \tan u_1 + \ln|\sec u_1 + \tan u_1| \\ \int \sec^3 u_1 \, du_1 &= \frac{1}{2}[\sec u_1 \tan u_1 + \ln|\sec u_1 + \tan u_1|] \\ S &= \frac{3\pi\sqrt{3}}{2}[\sec u_1 \tan u_1 + \ln|\sec u_1 + \tan u_1|]_0^{\mathrm{Tan}^{-1}\left(2/\sqrt{3}\right)} \end{aligned}$$

Reconsider the original substitution: $x = \dfrac{\sqrt{3}\tan u_1}{2}, \quad \tan u_1 = \dfrac{2x}{\sqrt{3}}$



$$\tan u_1 = \frac{2x}{\sqrt{3}}, \quad \sec u_1 = \frac{\sqrt{4x^2+3}}{\sqrt{3}}$$

$$S = \frac{3\pi\sqrt{3}}{2}\left[\sec u_1 \tan u_1 + \ln|\sec u_1 + \tan u_1|\right]_0^{\mathrm{Tan}^{-1}(2/\sqrt{3})}$$

$$S = \frac{3\pi\sqrt{3}}{2}\left[\left(\frac{\sqrt{4x^2+3}}{\sqrt{3}}\right)\left(\frac{2x}{\sqrt{3}}\right) + \ln\left|\frac{\sqrt{4x^2+3}}{\sqrt{3}} + \frac{2x}{\sqrt{3}}\right|\right]_0^1$$

$$= \frac{3\pi\sqrt{3}}{2}\left[\left(\frac{2\sqrt{7}}{3}\right) + \ln\left(\frac{2+\sqrt{7}}{\sqrt{3}}\right) - \left(\ln\left(\frac{\sqrt{3}}{\sqrt{3}}\right)\right)\right]$$

$$= \frac{3\pi\sqrt{3}}{2}\left[\frac{2\sqrt{7}}{3} + \ln\left(\frac{2+\sqrt{7}}{\sqrt{3}}\right) - \ln 1\right]$$

$$= \frac{3\pi\sqrt{3}}{2}\left[\frac{2\sqrt{7}}{3} + \ln\left(\frac{2+\sqrt{7}}{\sqrt{3}}\right)\right]$$

$$= \frac{3\pi\sqrt{3}}{2}\left[\frac{2\sqrt{7}}{3} + \ln\left(2+\sqrt{7}\right) - \ln\sqrt{3}\right]$$

$$= \frac{\pi\sqrt{3}}{2}\left[2\sqrt{7} + 3\ln\left(2+\sqrt{7}\right) - 3\ln\sqrt{3}\right]$$

$$= \boxed{\frac{\sqrt{3}}{2}\pi\left[3\ln\left(2+\sqrt{7}\right) + 2\sqrt{7} - 3\ln 3\right] \approx 7.146\pi}$$

43. **(a)** If $f(x) = x^2$, then $f'(x) = 2x$. This is continuous everywhere, so the Surface Area Formula applies. Therefore the surface area is

$$S = 2\pi\int_1^3 x^2\sqrt{1+(2x)^2}\,dx = \boxed{2\pi\int_1^3 x^2\sqrt{1+4x^2}\,dx}.$$

**(b)** Using technology, this evaluates to $S \approx \boxed{257.508}$.

44. **(a)** $\boxed{S = 2\pi\int_0^2 \left(4-x^2\right)\sqrt{1+(-2x)^2}\,dx}$

**(b)** $\boxed{S \approx 63.560}$

Integral evaluated by both the Texas Instruments TI-89 Titanium and www.integral-calculator.com

45. **(a)** If $f(x) = x^{2/3}$, then $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$. This is continuous on an interval containing 1 and 8, so the Surface Area Formula applies. Therefore the surface area is

$$S = 2\pi\int_1^8 x^{2/3}\sqrt{1+\left(\frac{2}{3x^{1/3}}\right)^2}\,dx\; S = \boxed{2\pi\int_1^8 x^{2/3}\sqrt{1+\frac{4}{9x^{2/3}}}\,dx}.$$

**(b)** Using technology, this evaluates to $S \approx \boxed{126.220}$.

46. **(a)** $\boxed{2\pi \int_0^4 x^{3/2}\sqrt{1+\frac{9x}{4}}\,dx}$

**(b)** 217.031

Integral evaluated by both the Texas Instruments TI-89 Titanium and www.integral-calculator.com

47. **(a)** If $f(x) = e^x$, then $f'(x) = e^x$. This is continuous everywhere, so the Surface Area Formula applies. Therefore the surface area is

$$S = 2\pi \int_0^3 e^x\sqrt{1+(e^x)^2}\,dx\; S = \boxed{2\pi \int_0^3 e^x\sqrt{1+e^{2x}}\,dx}.$$

**(b)** Using technology, this evaluates to $S \approx \boxed{1273.371}$.

48. **(a)** $2\pi \int_1^e \ln x\sqrt{1+\frac{1}{x^2}}\,dx$

**(b)** 7.055

Integral evaluated by both the Texas Instruments TI-89 Titanium and www.integral-calculator.com

49. **(a)** If $f(x) = \sin x$, then $f'(x) = \cos x$. This is continuous everywhere, so the Surface Area Formula applies. Therefore the surface area is

$$S = 2\pi \int_0^{\pi/2} \sin x\sqrt{1+(\cos x)^2}\,dx = \boxed{2\pi \int_0^{\pi/2} \sin x\sqrt{1+\cos^2 x}\,dx}.$$

**(b)** Using technology, this evaluates to $S \approx \boxed{7.212}$.

50. **(a)** $2\pi \int_0^{\pi/4} \tan x\sqrt{1+\sec^4 x}\,dx$

**(b)** 3.839

Integral evaluated by both the Texas Instruments TI-89 Titanium and www.integral-calculator.com

51. If $F(x) = \int_0^x \sqrt{16t^2-1}\,dt$, then $F'(x) = \sqrt{16x^2-1}$. This is continuous on an interval containing 0 and 2, so the Arc Length Formula applies. Therefore the arc length is

$$s = \int_0^2 \sqrt{1+\left(\sqrt{16x^2-1}\right)^2}\,dx = \int_0^2 \sqrt{1+(16x^2-1)}\,dx = \int_0^2 \sqrt{16x^2}\,dx = \int_0^2 4x\,dx$$

$$= 4\left[\frac{1}{2}x\right]_0^2 = 2\left(2^2-0^2\right) = \boxed{8}.$$

52. $$F(x) = \int_0^x \sqrt{4t-1}\,dt$$

$$\begin{aligned}
F'(x) &= \frac{d}{dx}\left(\int_0^x \sqrt{4t-1}\,dt\right) \\
&= \sqrt{4x-1} \\
L &= \int_0^2 \sqrt{1+(F'(x))^2}\,dx \\
&= \int_0^2 \sqrt{1+\left(\sqrt{4x-1}\right)^2}\,dx \\
&= \int_0^2 \sqrt{1+4x-1}\,dx \\
&= \int_0^2 \sqrt{4x}\,dx \\
&= 2\int_0^2 x^{1/2}\,dx \\
&= \left[2\left(\frac{2}{3}\right)x^{3/2}\right]_0^2 \\
&= \left[\left(\frac{4}{3}\right)x^{3/2}\right]_0^2 \\
&= \left(\frac{4}{3}\right)\left(2^{3/2}-0\right) \\
&= \boxed{\frac{8\sqrt{2}}{3}}
\end{aligned}$$

53. If $F(x) = \int_0^{3x} \sqrt{e^t - \frac{1}{9}}\,dt$, then $F'(x) = \sqrt{e^{(3x)} - \frac{1}{9}} \cdot \frac{d}{dx}(3x) = 3\sqrt{e^{3x} - \frac{1}{9}}$. This is continuous on an interval containing 0 and 4, so the Arc Length Formula applies. Therefore the arc length is

$$s = \int_0^4 \sqrt{1+\left(3\sqrt{e^{3x}-\frac{1}{9}}\right)^2}\,dx = \int_0^4 \sqrt{1+9\left(e^{3x}-\frac{1}{9}\right)}\,dx = \int_0^4 \sqrt{1+(9e^{3x}-1)}\,dx$$
$$= \int_0^4 \sqrt{9e^{3x}}\,dx = \int_0^4 3e^{3x/2}\,dx.$$

Now let $u = \frac{3x}{2}$, so $du = \frac{3}{2}\,dx$, so $dx = \frac{2}{3}\,du$. Also, when $x = 0$, then $u = \frac{3(0)}{2} = 0$, and when $x = 4$, then $u = \frac{3(4)}{2} = 6$. Therefore

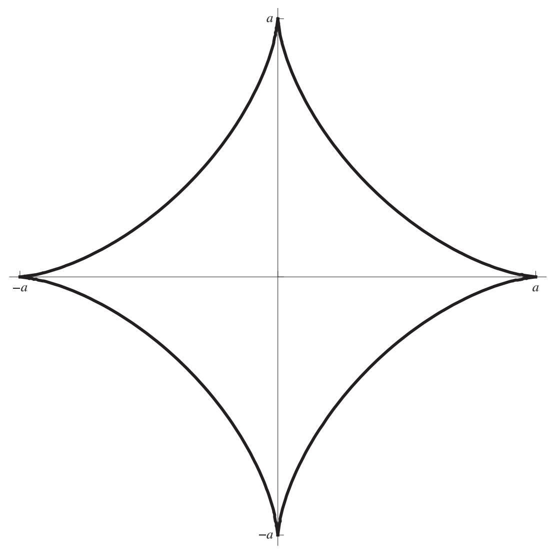
$$s = 3\int_0^6 e^u \cdot \frac{2}{3}\,du = 2[e^u]_0^6 = 2\left(e^6 - e^0\right) = \boxed{2\left(e^6-1\right)}$$

54. $F(x) = \int_0^{4x} \sqrt{t^4 - \frac{1}{4}}\, dt$

$$F'(x) = \frac{d}{dx}\left(\int_0^{4x} \sqrt{t^4 - \frac{1}{4}}\, dt\right)$$
$$= 4\left(\sqrt{(4x)^4 - \frac{1}{4}}\right)$$
$$= 4\left(\sqrt{\frac{1024x^4 - 1}{4}}\right)$$
$$= 2\sqrt{1024x^4 - 1}$$
$$L = \int_1^2 \sqrt{1 + (F'(x))^2}\, dx$$
$$= \int_1^2 \sqrt{1 + 4(1024x^4 - 1)}\, dx$$
$$= \int_1^2 \sqrt{4096x^4 - 3}\, dx$$
$$\approx \boxed{149.321}, \text{using technology}$$

**Applications and Extensions**

55. The hypocycloid is shown below:




All four arcs are the same (the graph is symmetric in both $x$ and $y$, since $(-x)^{2/3} = x^{2/3}$ and $(-y)^{2/3} = y^{2/3}$), and the arc in the first quadrant is symmetric about the line $x = y$, so the total arc length is the length of half of that arc, multiplied by 8. The arc in the first quadrant is divided in half at the point where the $x$-coordinate and the $y$-coordinate are equal, which is when $2x^{2/3} = a^{2/3}$, or $x = \frac{a}{2^{3/2}}$. So consider the arc from $x = \frac{a}{2^{3/2}}$ to $x = a$ in the first quadrant. Then $x, y \geq 0$, and solving for $y$ gives

$$y = \left(a^{2/3} - x^{2/3}\right)^{3/2},$$

so that

$$y' = \frac{3}{2}\left(a^{2/3} - x^{2/3}\right)^{1/2} \cdot \left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3}\left(x^{2/3} - a^{2/3}\right)^{1/2}.$$

Now, $y'$ is continuous on an interval containing $\frac{a}{2^{3/2}}$ and $a$, so the Arc Length Formula applies and the total length of the hypocycloid is therefore

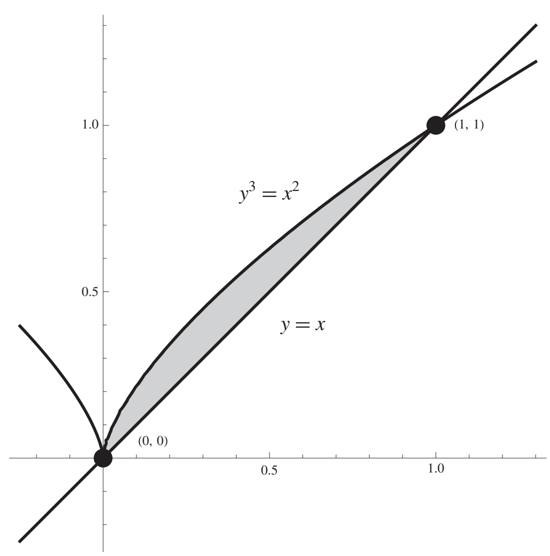
$$\begin{aligned}
s &= 8\int_{a/(2^{3/2})}^{a} \sqrt{1^2 + \left(-x^{-1/3}\left(a^{2/3}-x^{2/3}\right)^{1/2}\right)^2}\,dx \\
&= 8\int_{a/(2^{3/2})}^{a} \sqrt{1 + x^{-2/3}\left(a^{2/3}-x^{2/3}\right)}\,dx \\
&= 8\int_{a/(2^{3/2})}^{a} \sqrt{a^{2/3}x^{-2/3}}\,dx \\
&= 8\int_{a/(2^{3/2})}^{a} a^{1/3}x^{-1/3}\,dx \\
&= 8\left[\frac{3}{2}a^{1/3}x^{2/3}\right]_{a/(2^{3/2})}^{a} \\
&= 12\left(a^{1/3}a^{2/3} - a^{1/3}\cdot\frac{a^{2/3}}{(2^{3/2})^{2/3}}\right) \\
&= 12\left(a - \frac{a}{2}\right) = 6a.
\end{aligned}$$

The arc length of the hypocycloid is $\boxed{6a}$.

56. The curve whose arc length we wish to find is in the first quadrant; there, the curve is $y = \sqrt{x^3} = x^{3/2}$, so that $y' = \frac{3}{2}\sqrt{x}$, which is continuous on an interval containing 1 and 3. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_1^3 \sqrt{1^2 + \left(\frac{3}{2}\sqrt{x}\right)^2}\,dx = \frac{1}{2}\int_1^3 \sqrt{4+9x}\,dx = \frac{1}{2}\left[\frac{2}{27}(4+9x)^{3/2}\right]_1^3 = \boxed{\frac{1}{27}(31\sqrt{31} - 13\sqrt{13})}.$$

57. The region in question is shown below:



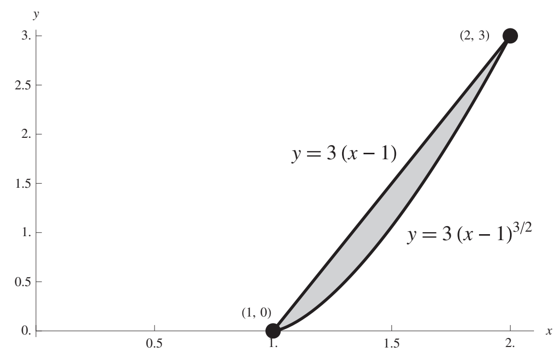
The perimeter of the region is composed of two pieces. The length of the first, along the line $y = x$, can be computed using the Distance Formula: $s = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$. For the second, we use the Arc Length Formula. If we try to partition along the $x$-axis, we get $y = x^{2/3}$, so that $y' = \frac{2}{3}x^{-1/3}$, which is not continuous at $x = 0$. So partition instead along the $y$-axis. Solving the equation for $x$ gives $x = y^{3/2}$, so that $x' = \frac{3}{2}\sqrt{y}$, which is continuous on an interval containing 0 and 1. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_0^1 \sqrt{1^2 + \left(\frac{3}{2}\sqrt{y}\right)^2}\,dy = \frac{1}{2}\int_1^3 \sqrt{4+9y}\,dy = \frac{1}{2}\left[\frac{2}{27}(4+9y)^{3/2}\right]_1^3 = \frac{1}{27}(31\sqrt{31} - 13\sqrt{13}).$$

So the total arc length is

$$\boxed{\sqrt{2} + \frac{1}{27}(31\sqrt{31} - 13\sqrt{13})}.$$

58. The region in question is shown below:




The perimeter of the region is composed of two pieces. The length of the first, along the line $y = 3(x-1)$, can be computed using the Distance Formula: $s = \sqrt{(2-1)^2 + (3-0)^2} = \sqrt{10}$. For the second, we use the Arc Length Formula. Partitioning along the $x$-axis, we have $y' = 3 \cdot \frac{3}{2}(x-1)^{1/2} = \frac{9}{2}\sqrt{x-1}$, which is continuous on an interval containing 1 and 3. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_1^3 \sqrt{1^2 + \left(\frac{9}{2}\sqrt{x-1}\right)^2}\,dx = \frac{1}{2}\int_1^3 \sqrt{81x - 77}\,dx = \frac{1}{2}\left[\frac{2}{243}(81x-77)^{3/2}\right]_1^3 = \frac{1}{243}(166\sqrt{166} - 8).$$

Therefore the total arc length is

$$\boxed{\sqrt{10} + \frac{1}{243}(166\sqrt{166} - 8)}.$$

59. Solving for $x$ gives $x = \frac{y^4+3}{6y} = \frac{y^3}{6} + \frac{1}{2y}$. Then $x' = \frac{y^2}{2} - \frac{1}{2y^2}$, which is differentiable on an interval containing 1 and 2. Therefore the Arc Length Formula applies, and the arc length is

$$\begin{aligned} s &= \int_1^2 \sqrt{1^2 + \left(\frac{y^2}{2} - \frac{1}{2y^2}\right)^2}\, dy \\ &= \int_1^2 \sqrt{1 + \frac{y^4}{4} - \frac{1}{2} + \frac{1}{4y^4}}\, dy \\ &= \int_1^2 \sqrt{\frac{y^4}{4} + \frac{1}{2} + \frac{1}{4y^4}}\, dy \\ &= \int_1^2 \sqrt{\left(\frac{y^2}{2} + \frac{1}{2y^2}\right)^2}\, dy \\ &= \int_1^2 \left(\frac{y^2}{2} + \frac{1}{2y^2}\right) dy \\ &= \left[\frac{y^3}{6} - \frac{1}{2y}\right]_1^2 \\ &= \boxed{\frac{17}{12}}. \end{aligned}$$

60. We have $y' = \sinh x$, which is continuous everywhere, so that the Arc Length Formula applies. Therefore the arc length is

$$s = \int_0^2 \sqrt{1^2 + (\sinh x)^2}\, dx = \int_0^2 \sqrt{1 + \sinh^2 x}\, dx.$$

Since $\cosh^2 x - \sinh^2 x = 1$, we see that $1 + \sinh^2 x = \cosh^2 x$. Since $\cosh x > 0$ everywhere, we get for the arc length

$$s = \int_0^2 \sqrt{\cosh^2 x}\, dx = \int_0^2 \cosh x\, dx = [\sinh x]_0^2 = \sinh 2 - \sinh 0 = \boxed{\sinh 2}.$$

61. We have $y' = \frac{1}{\csc x} \cdot (-\cot x \csc x) = -\cot x$, which is continuous on an interval containing $\frac{\pi}{4}$ and $\frac{\pi}{2}$. Therefore the Arc Length Formula applies, and the arc length is

$$s = \int_{\pi/4}^{\pi/2} \sqrt{1^2 + (-\cot x)^2}\, dx = \int_{\pi/4}^{\pi/2} \sqrt{1 + \cot^2 x}\, dx.$$

Now use the identity $1 + \cot^2 x = \csc^2 x$; since we are integrating for angles in the first quadrant, $\csc x \geq 0$, so we get for the arc length

$$\begin{aligned} x &= \int_{\pi/4}^{\pi/2} \sqrt{\csc^2 x}\, dx \\ &= \int_{\pi/4}^{\pi/2} \csc x\, dx = [\ln|\csc x - \cot x|]_{\pi/4}^{\pi/2} \\ &= \ln 1 - \ln(\sqrt{2} - 1) = \boxed{-\ln(\sqrt{2} - 1)}. \end{aligned}$$

62. Solving the equation for $y$ in the first quadrant (where $y \geq 0$) gives

$$y = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Then

$$y' = \frac{b}{a} \cdot \frac{1}{2}(a^2 - x^2)^{-1/2} \cdot (-2x) = -\frac{bx}{a\sqrt{a^2 - x^2}}.$$

This is continuous on an interval containing 0 and $\frac{a}{2}$, so the Arc Length Formula applies and we get for the arc length

$$\begin{aligned}
s &= \int_0^{a/2} \sqrt{1^2 + \left(-\frac{bx}{a\sqrt{a^2 - x^2}}\right)^2}\, dx \\
&= \int_0^{a/2} \sqrt{1 + \frac{b^2x^2}{a^2(a^2 - x^2)}}\, dx \\
&= \int_0^{a/2} \sqrt{\frac{b^2x^2 + a^4 - a^2x^2}{a^2(a^2 - x^2)}}\, dx \\
&= \int_0^{a/2} \sqrt{\frac{a^4 + x^2(b^2 - a^2)}{a^2(a^2 - x^2)}}\, dx \\
&= \int_0^{a/2} \sqrt{\frac{a^4 + a^4\left(\frac{x}{a}\right)^2\left(\left(\frac{b}{a}\right)^2 - 1\right)}{a^4\left(1 - \left(\frac{x}{a}\right)^2\right)}}\, dx \\
&= \int_0^{a/2} \sqrt{\frac{1 + \left(\frac{x}{a}\right)^2\left(\left(\frac{b}{a}\right)^2 - 1\right)}{1 - \left(\frac{x}{a}\right)^2}}\, dx.
\end{aligned}$$

63. **(a)** As mentioned in the comment to Example 3 on page 509, the derivative of $y$ as a function of $x$ will not be continuous at $x = 2$. Therefore solve for $x$ as a function of $y$ and integrate along $y$:

$$\begin{aligned}
x^2 + 4y^2 &= 4 \\
x^2 &= 4 - 4y^2 \\
x &= \sqrt{4 - 4y^2} = 2\sqrt{1 - y^2} \text{ (positive in the first quadrant)}
\end{aligned}$$

Next, to find one of the limits of integration, find the $y$-value of the intersection of this with $y = x$:

$$\begin{aligned}
y &= 2\sqrt{1 - y^2} \\
y^2 &= \left(2\sqrt{1 - y^2}\right)^2 = 4\left(1 - y^2\right) = 4 - 4y^2 \\
5y^2 &= 4 \\
y &= \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \text{ (positive in the first quadrant)}
\end{aligned}$$

Find the other limit of integration, when $x = 2$:

$$\begin{aligned}
(2)^2 + 4y^2 &= 4 \\
4 + y^2 &= 4 \\
y^2 &= 0 \\
y &= 0
\end{aligned}$$

If $x = 2\sqrt{1-y^2} = 2\left(1-y^2\right)^{1/2}$, then $x' = 2 \cdot \frac{1}{2}\left(1-y^2\right)^{-1/2} \cdot 2y = \frac{2y}{\sqrt{1-y^2}}$. This is continuous on an interval containing 0 and $\frac{2\sqrt{5}}{5}$, so the Arc Length Formula applies. Therefore the arc length is

$$s = \int_0^{2\sqrt{5}/5} \sqrt{1 + \left(\frac{2y}{\sqrt{1-y^2}}\right)^2}\, dy = \int_0^{2\sqrt{5}/5} \sqrt{1 + \frac{4y^2}{1-y^2}}\, dy$$

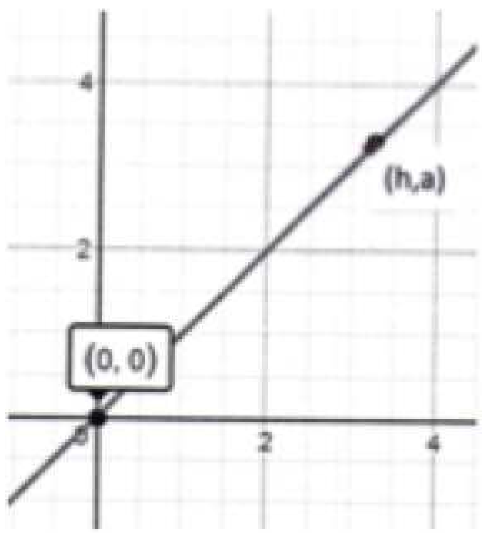
$$= \int_0^{2\sqrt{5}/5} \sqrt{\frac{(1-y^2)+4y^2}{1-y^2}}\, dy = \boxed{\int_0^{2\sqrt{5}/5} \sqrt{\frac{1+3y^2}{1-y^2}}\, dy}.$$

**(b)** Using technology, this evaluates to $s \approx \boxed{1.519}$.

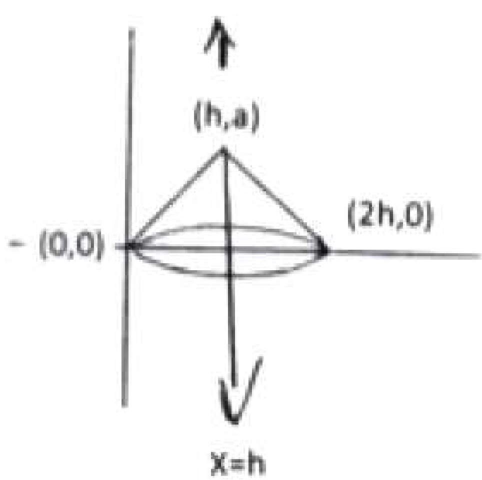
**(c)** From Example 3, the arc length from (0, 1) to $\left(\frac{2\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$ is $s \approx 0.903$. Adding that to the result from Part (b) above gives the arc length of one quarter of the ellipse, so the arc length of the entire ellipse is $s \approx 4(0.903 + 1.519) = \boxed{9.688}$.

64. **(a)** $y = \dfrac{ax}{h}$

**(b)**




**(c)**



**(d)** $$\begin{aligned} S &= 2\pi \int_0^h \frac{ax}{h}\sqrt{1+\frac{a^2}{h^2}}\,dx \\ &= \int_0^h \frac{ax}{h}\sqrt{\frac{h^2+a^2}{h^2}}\,dx \\ &= \frac{2\pi a\sqrt{a^2+h^2}}{h^2}\int_0^h x\,dx \\ &= \frac{2\pi a\sqrt{a^2+h^2}}{h^2}\left(\frac{x^2}{2}\right)\Bigg|_0^h \\ &= \frac{2\pi a\sqrt{a^2+h^2}}{h^2}\left(\frac{h^2}{2}\right) \\ &= \boxed{\pi a\sqrt{a^2+h^2}} \end{aligned}$$

65. Use the $x$-axis as the axis of the parabola and the axis of rotation, with the vertex of the parabola at the origin. Then find the equation of the parabola, using the fact that it contains the point $\left(\frac{1}{4},\ \frac{1}{2}\right)$:

$$\begin{aligned} x &= Ay^2 \\ \frac{1}{4} &= A\left(\frac{1}{2}\right)^2 = \frac{1}{4}A \\ A &= 4\cdot\frac{1}{4} = 1 \\ x &= 1\cdot y^2 = y^2 \end{aligned}$$

Solving for $y$ as a function of $x$ would give a function whose derivative is undefined at $x = 0$, so stay with $x$ as a function of $y$ and integrate along $y$.
So $x' = 2y$. This is continuous everywhere, so the Surface Area Formula applies. Therefore the surface area is

$$s = 2\pi\int_0^{1/2} y\sqrt{1+(2y)^2}\,dy = 2\pi\int_0^{1/2} y\sqrt{1+4y^2}\,dy$$

Now let $u = 1+4y^2$, so $du = 8y\,dy$, so $y\,dy = \frac{1}{8}\,du$. Also, when $y = 0$, then $u = 1+4(0)^2 = 1$, and when $y = \frac{1}{2}$, then $u = 1+4\left(\frac{1}{2}\right)^2 = 2$. Therefore

$$s = 2\pi\int_1^2 \sqrt{u}\cdot\frac{1}{8}\,du = \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_1^2 = \frac{\pi}{6}\left(2^{3/2}-1^{3/2}\right) = \boxed{\frac{\pi}{6}\left(2\sqrt{2}-1\right)\ \text{m}^2}.$$

66. **(a)** We want to find the arc length of the curve $y = Ax^{3/2}$ from $y = 0$ to $y = h$. Partitioning along the $x$-axis, then, we get $y' = \frac{3}{2}Ax^{1/2}$, which is continuous for $x \ge 0$, so the Arc Length Formula applies. Note that $y = h$ gives $h = Ax^{3/2}$, so that $x = \left(\frac{h}{A}\right)^{2/3}$, and this is the upper bound for integration. Then the arc length is

$$s = \int_0^{(h/A)^{2/3}} \sqrt{1^2+\left(\frac{3}{2}Ax^{1/2}\right)^2}\,dx = \boxed{\frac{1}{2}\int_0^{(h/A)^{2/3}} \sqrt{4+9A^2x}\,dx}.$$

**(b)** The given numbers mean that $h = 150$ and that $y = h$ corresponds to $x = 250$. Therefore $h = Ax^{3/2}$ means that $150 = A\cdot 250^{3/2}$, so that $A = 150\cdot 250^{-3/2}$ $\boxed{\approx 0.0379}$.

**(c)** Substituting into the formula in part (a), we get

$$s = \frac{1}{2}\int_0^{(h/A)^{2/3}} \sqrt{4+9A^2x}\,dx \approx \frac{1}{2}\int_0^{250} \sqrt{4+9\cdot 0.0379^2 x}\,dx$$
$$\approx \frac{1}{2}\int_0^{250} \sqrt{4+0.0129x}\,dx \approx \boxed{295.099 \text{ m}}.$$

**(d)** If the slope were a straight line, then using the Pythagorean Theorem, its length would be

$$\sqrt{250^2+150^2} = \sqrt{85000} \approx 291.55.$$

The length computed in part (c) is slightly longer than the straight-line distance, so it appears reasonable.

67. **(a)** Since $y' = a\sinh\frac{x}{a}\cdot\frac{1}{a} = \sinh\frac{x}{a}$, and $y'(10) = \frac{3}{4}$, we get $\frac{3}{4} = \sinh\frac{10}{a}$, so that $a = \frac{10}{\sinh^{-1}\frac{3}{4}}$. Now, compute the height of the endpoints above $b$, the lowest point; this is

$$y - b = a\cosh\frac{10}{a} - a = \frac{10}{\sinh^{-1}\frac{3}{4}}\left(\cosh\sinh^{-1}\frac{3}{4} - 1\right).$$

Since $\cosh^2 x = 1+\sinh^2 x$ for all $x$, we can simplify the above formula as follows:

$$\begin{aligned}\frac{10}{\sinh^{-1}\frac{3}{4}}\left(\cosh\sinh^{-1}\frac{3}{4} - 1\right) &= \frac{10}{\sinh^{-1}\frac{3}{4}}\left(\sqrt{1+\sinh^2\sinh^{-1}\frac{3}{4}} - 1\right)\\ &= \frac{10}{\sinh^{-1}\frac{3}{4}}\left(\sqrt{1+\frac{9}{16}} - 1\right)\\ &= \frac{10}{\sinh^{-1}\frac{3}{4}}\left(\sqrt{\frac{25}{16}} - 1\right)\\ &= \frac{10}{\sinh^{-1}\frac{3}{4}}\cdot\frac{1}{4} = \frac{5}{2\sinh^{-1}\frac{3}{4}}.\end{aligned}$$

Therefore the height of the supports is $\boxed{b + \frac{5}{2\sinh^{-1}\frac{3}{4}}}$.

**(b)** We determine the arc length by partitioning along the $x$-axis. From part (a), we know that $y' = \sinh\frac{x}{a}$, so that the arc length is (using the identity $\cosh^2 x = 1+\sinh^2 x$ again)

$$\begin{aligned} s &= \int_{-10}^{10}\sqrt{1^2+\left(\sinh\frac{x}{a}\right)^2}\,dx\\ &= \int_{-10}^{10}\sqrt{1+\sinh^2\frac{x}{a}}\,dx\\ &= \int_{-10}^{10}\sqrt{\cosh^2\frac{x}{a}}\,dx\\ &= \int_{-10}^{10}\cosh\frac{x}{a}\,dx\\ &= \left[a\sinh\frac{x}{a}\right]_{-10}^{10}\\ &= 2a\sinh\frac{10}{a}.\end{aligned}$$

Now substitute the value of $a$ from part (a), giving

$$s = 2 \cdot \frac{10}{\sinh^{-1} \frac{3}{4}} \sinh \sinh^{-1} \frac{3}{4} = \frac{60}{4 \sinh^{-1} \frac{3}{4}} = \boxed{\frac{15}{\sinh^{-1} \frac{3}{4}} \approx 21.6404 \text{ m}}.$$

68. The length of the rope is the arc length. We have $y' = a \sinh \frac{x}{a} \cdot \frac{1}{a} = \sinh \frac{x}{a}$, so that the arc length is (using the identity $\cosh^2 x = 1 + \sinh^2 x$)

$$\begin{aligned} L &= \int_{-c}^{c} \sqrt{1^2 + \left(\sinh \frac{x}{a}\right)^2}\, dx \\ &= \int_{-c}^{c} \sqrt{1 + \sinh^2 \frac{x}{a}}\, dx \\ &= \int_{-c}^{c} \sqrt{\cosh^2 \frac{x}{a}}\, dx \\ &= \int_{-c}^{c} \cosh \frac{x}{a}\, dx \\ &= \left[a \sinh \frac{x}{a}\right]_{-c}^{c} \\ &= 2a \sinh \frac{c}{a}. \end{aligned}$$

Then since $d$ is the height of the rope when $x = c$, we get

$$(d-b+a)^2 - a^2 = \left(a \cosh \frac{c}{a} + b - a - b + a\right)^2 - a^2 = a^2\left(\cosh^2 \frac{c}{a} - 1\right) = a^2 \sinh^2 \frac{c}{a} = \frac{L^2}{4}.$$

Expanding the left-hand side and simplifying gives

$$(d-b)^2 + 2a(d-b) = \frac{L^2}{4}, \text{ so that } a = \frac{\frac{L^2}{4} - (d-b)^2}{2(d-b)} = \frac{L^2 - 4(d-b)^2}{8(d-b)}.$$

(Note that $d \neq b$, so this expression makes sense, since $d$ is the height of the ends of the catenary, while $b$ is its lowest height.)

69. **(a)** Let the parabola be $h(x) = ax^2 + bx + c$. Since $h(0) = 0$, we must have $c = 0$, so that the equation of motion is $h(x) = ax^2 + bx = x(ax + b)$. Since $h(0) = h(150) = 0$, we see that 150 must be a zero of $ax + b$, so that $h(x) = ax(x - 150) = ax^2 - 150ax$. It follows that $b = -150a$. Finally, $h'(x) = 2ax - 150a$, so that $h'(x) = 0$ when $x = 75$. We are given that $h(75) = 46$, so that

$$46 = h(75) = a \cdot 75 \cdot (75 - 150) = -75^2 a, \text{ so that } a = -\frac{46}{75^2}.$$

Therefore the equation of motion is

$$h(x) = -\frac{46}{75^2} x(x - 150) = \frac{46}{75^2}\left(150x - x^2\right).$$

**(b)** The total arc length is twice the length of the arc from $x = 0$ to the top of the arc, at $x = -\frac{b}{a}$. Using a CAS to integrate, we get

$$\begin{aligned} s = 2\int_0^{-b/a} \sqrt{1^2 + (2ax + b)^2}\, dx &= 2\int_0^{150} \sqrt{1 + \left(-\frac{92}{75^2} x + \frac{92}{75}\right)^2}\, dx \\ &= \boxed{2\sqrt{14089} + \frac{5625}{46} \sinh^{-1} \frac{92}{75} \approx 363.704 \text{ m}}. \end{aligned}$$

**Challenge Problems**

70. This function is not differentiable at $x = 0$. To see this, use the definition of the derivative:

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h \sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} \sin\frac{1}{h}.$$

But as $h \to 0$, $\frac{1}{h} \to \infty$, so that $\sin\frac{1}{h}$ oscillates infinitely often between $-1$ and $1$, so that this limit does not exist. Since it is not differentiable on an interval containing $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we cannot apply the Arc Length Formula.

71. **(a)** The total angle around point $O$ is $360°$, and the polygon divides it into $2^n$ equal angles, each of which must be $\frac{360}{2^n} = \frac{180}{2^{n-1}} = \frac{90}{2^{n-2}}$ degrees. But the dotted ray emanating from $O$ is the perpendicular bisector of the opposite side of an isosceles triangle, so it also bisects its angle. Therefore $\alpha = \frac{1}{2}\cdot\frac{90}{2^{n-2}} = \frac{45°}{2^{n-2}}$.

**(b)** The right triangle of which $\alpha$ is one angle has hypotenuse 1. The side opposite $\alpha$ therefore has length $\sin\alpha$. Since there are $2^n$ sides in the polygon, there are $2 \cdot 2^n = 2^{n+1}$ segments of length $\sin\alpha$, so that the sum of all of these segments has length

$$2^{n+1}\sin\alpha = 2^{n+1}\sin\frac{45°}{2^{n-2}}.$$

**(c)** A table showing the value of this expression for several values of $n$ is below. The true circumference of the circle is $2\pi r = 2\pi \approx 6.283185307$:

| 5 | 10 | 50 | 100 | 500 | 1000 |
|---|---|---|---|---|---|
| 6.273096981 | 6.283175451 | 6.283185307 | 6.283185307 | 6.283185307 | 6.283185307 |

72. Note that $a_0 = \frac{\sqrt{2}}{2} = \sin 45°$. Then use the half-angle formula

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\sqrt{1-\sin^2\theta}}{2}}.$$

Since we start with $\theta = 45°$, all angles will be in the first quadrant, so we can always take positive square roots. Then

$$a_1 = \sin\frac{45°}{2}, \quad a_2 = \sin\frac{45°}{4}, \quad \ldots, \quad a_n = \sin\frac{45°}{2^n}.$$

Then by Problem 47, we know that $2^{n+3}\sin\frac{45°}{2^n} = 2^{n+3}a_n$ is a good approximation to the circumference of the circle, which is $2\pi$. Therefore $2^{n+2}a_n$ is a good approximation to $\frac{1}{2}\cdot 2\pi = \pi$. A table showing the value of $2^{n+2}a_n$ for several values of $n$ is below. The true value is $\pi \approx 3.141592654$:

| 5 | 10 | 50 | 100 | 500 | 1000 |
|---|---|---|---|---|---|
| 3.141277251 | 3.141592346 | 3.141592654 | 3.141592654 | 3.141592654 | 3.141592654 |

The expression approaches $\pi$ rather rapidly.

73. Since $s$ is as given, with $y = f(x)$ we must have

$$s = e^x - f(x) = \int_0^x \sqrt{1 + (f'(t))^2}\,dt.$$

Taking the derivative of both sides and using the Fundamental Theorem gives

$e^x - f'(x) = \sqrt{1 + (f'(x))^2}$; now square both sides to get $e^{2x} - 2e^x f'(x) + (f'(x))^2 = 1 + (f'(x))^2$.

Cancel the quadratic terms and solve for $f'(x)$, giving

$$f'(x) = \frac{e^{2x}-1}{2e^x} = \frac{1}{2}(e^x - e^{-x}).$$

Now integrate to get

$$f(x) = \frac{1}{2}(e^x + e^{-x}) + C = \cosh x + C.$$

Since $f(0) = 1$, we get $C = 0$, so that $f(x) = \cosh x$. As a check, compute the arc length of $f$ from 0 to $x$ (use $t$ as a dummy variable):

$$s = \int_0^x \sqrt{1^2 + \sinh^2 t}\,dt = \int_0^x \sqrt{\cosh^2 t}\,dt = \int_0^x \cosh t\,dt = \sinh x - \sinh 0 = \sinh x.$$

But

$$e^x - f(x) = e^x - \cosh x = e^x - \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(e^x + e^{-x}) = \sinh x,$$

and the two are equal.

74. The key point here is that partitioning along either axis works except where the circle meets the axes. Where it meets the $x$-axis, we must partition along the $y$-axis since the curve has a vertical tangent line, and where it meets the $y$-axis, we must partition along the $x$-axis since the curve has a horizontal tangent line. Solving $x^2 + y^2 = 1$ for $y$ gives $y = \pm\sqrt{1-x^2}$, so that

$$y' = \pm\frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = \pm x(1-x^2)^{-1/2}.$$

Similarly, if we solve for $x$ and differentiate, we get $x' = \pm y(1-y^2)^{-1/2}$. So the arc length integrands when partitioning along each axis are

$$x\text{-axis}: \quad \sqrt{1^2 + \left(\pm x(1-x^2)^{-1/2}\right)^2} = \sqrt{1 + \frac{x^2}{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$y\text{-axis}: \quad \sqrt{1^2 + \left(\pm y(1-y^2)^{-1/2}\right)^2} = \sqrt{1 + \frac{y^2}{1-y^2}} = \frac{1}{\sqrt{1-y^2}}$$

Then:

**(a)** Since $P_1$ is in quadrant I and $P_2$ in quadrant II, the only axis crossing occurs at $(0,1)$, so we can partition along the $x$-axis for the whole integration, giving

$$s = \int_{x_2}^{x_1} \frac{1}{\sqrt{1-x^2}}\,dx.$$

**(b)** Since $P_1$ and $P_2$ are both in the third quadrant, there are no axis crossings, so we can partition along either axis. Using the $y$-axis, since $y_1 < y_2$, we get

$$s = \int_{y_1}^{y_2} \frac{1}{\sqrt{1-y^2}}\,dy.$$

**(c)** Since $P_1$ is in quadrant II and $P_2$ is in quadrant IV, and we are considering the counterclockwise arc from $P_1$ to $P_2$, there are two axis crossings, at $(-1,0)$ and at $(0,-1)$. From $P_1$ to $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ (for example; any other point in the third quadrant would work as well), we can partition along the $y$-axis, and then from $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ to $P_2$ we partition along the $x$-axis. Therefore the arc length is

$$s = \int_{-\sqrt{2}/2}^{y_1} \frac{1}{\sqrt{1-y^2}}\,dy + \int_{-\sqrt{2}/2}^{x_2} \frac{1}{\sqrt{1-x^2}}\,dx.$$

## AP® Practice Problems

1. $f(x) = -x^4 + 2$

   $f'(x) = -4x^3$

   $$L = \int_0^{10} \sqrt{1 + (-4x^3)^2}\, dx = \boxed{\int_0^{10} \sqrt{1 + 16x^6}\, dx}$$

   CHOICE D

2. $y = \tan x$

   $y' = \sec^2 x$

   $$L = \int_{-\pi/2}^{\pi/2} \sqrt{1 + (\sec^2 x)^2}\, dx = \boxed{\int_{-\pi/2}^{\pi/2} \sqrt{1 + \sec^4 x}\, dx}$$

   CHOICE B

3. $f(x) = \ln(\cos x)$

   $f'(x) = \dfrac{-\sin x}{\cos x} = -\tan x$

   $$\begin{aligned} L &= \int_0^{\pi/3} \sqrt{1 + (-\tan x)^2}\, dx \\ &= \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx \\ &= \int_0^{\pi/3} \sqrt{\sec^2 x}\, dx \\ &= \int_0^{\pi/3} \sec x\, dx \\ &= [\ln|\sec x + \tan x|]_0^{\pi/3} \\ &= \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \ln|\sec 0 + \tan 0| \\ &= \ln\left|2 + \sqrt{3}\right| - \ln|1 + 0| \\ &= \boxed{\ln(2 + \sqrt{3})} \end{aligned}$$

   CHOICE B

4. **(a)** $f(x) = -x^3 + 4x^2$

   $f'(x) = -3x^2 + 8x$

   $$\boxed{L = \int_0^4 \sqrt{1 + (-3x^2 + 8x)^2}\, dx}$$

   **(b)** $$\boxed{V = \pi \int_0^4 \left(-x^3 + 4x^2\right)^2 dx}$$

   **(c)** By the Shell Method

   $$\boxed{V = 2\pi \int_0^4 x\left(-x^3 + 4x^2\right) dx}$$

## 6.6 Work

### Concepts and Vocabulary

1. True. See the first sentence of this section.
2. $W = Fx$. See the second sentence of this section.
3. A unit of work is called a *newton-meter*, or *joule*, in SI units and a *foot-pound* in the customary U.S. system of units.
4. The work $W$ done by a continuously varying force $F = F(x)$ acting on an object, which moves the object along a straight line in the direction of $F$ from $x = a$ to $x = b$, is given by the definite integral $\int_a^b F(x)\,dx$. See subsection 1.
5. A spring is said to be in *equilibrium* when it is neither extended nor compressed. See the discussion in subsection 2 of the text.
6. True. See the discussion of springs in subsection 2.
7. True. See the note in Example 4.
8. True. See the discussion at the end of this section in the text.

### Skill Building

9. The work is the integral of the force through the distance, so

$$W = \int_5^{20} (40 - x)\,dx = \left[40x - \frac{1}{2}x^2\right]_5^{20} = \boxed{\frac{825}{2}\text{ J}}.$$

10. The work is the integral of the force through the distance, so

$$W = \int_1^2 \frac{1}{x}\,dx = [\ln x]_1^2 = \boxed{\ln 2\text{ J}}.$$

11. Following the method of example 1, since the density of the chain is 3 kg/m, and the portion of chain $x$ m below the bridge must be lifted $40 - x$ m, we get (since $g = 9.8$ m/s$^2$)

$$W = 9.8\int_0^{40} 3(40 - x)\,dx = 9.8\left[120x - \frac{3}{2}x^2\right]_0^{40} = \boxed{23520\text{ J}}.$$

12. Following the method of Example 1, since the density of the chain is 2 lb/ft, and the portion of chain $x$ ft below the roof must be lifted $120 - x$ feet, we get

$$W = \int_0^{120} 2(120 - x)\,dx = \left[240x - x^2\right]_0^{120} = \boxed{14400\text{ ft-lbs}}.$$

13. Since the restoring force is $-3$ N, we have

$$-3 = -k \cdot \frac{1}{4}, \quad \text{so that} \quad k = \boxed{12\text{ N/m}}.$$

14. Since the restoring force is 6 ft-lb, we have

$$6 = -k \cdot \left(-\frac{1}{2}\right), \quad \text{so that} \quad k = \boxed{12\text{ lb/ft}}.$$

15. We are stretching the spring 0.6 m beyond equilibrium, so the work done by the spring force is

$$\int_0^{0.6} (-5x)\,dx = \left[-\frac{5}{2}x^2\right]_0^{0.6} = \boxed{-0.9 \text{ J}}.$$

16. We are compressing the spring 0.2 m from equilibrium, so the work done by the spring force is

$$\int_0^{-0.2} (-0.3x)\,dx = \left[-0.15x^2\right]_0^{-0.2} = \boxed{-0.006 \text{ J}}.$$

17. **(a)** Water that is $x$ feet above the bottom of the pool must be lifted $4 - x$ feet. The cross-sectional area of the pool at any height is the area of a circle with radius 12 ft, which is $144\pi$ ft$^2$. The water at the bottom must be lifted 4 feet, while the water at the top is lifted 0 feet, so the limits of integration are from 0 to 4, and the work performed is

$$\begin{aligned}\int_0^4 [62.42 \cdot 144\pi \cdot (4-x)]\,dx &= 8988.48\pi \int_0^4 (4-x)\,dx \\ &= 8988.48\pi \left[4x - \frac{1}{2}x^2\right]_0^4 \approx \boxed{2.259 \times 10^5 \text{ ft lb}}.\end{aligned}$$

**(b)** The same as in Part (a), except that now the water must be lifted $9 - x$ ft. So

$$\begin{aligned}W &= 8988.48\pi \int_0^4 (9-x)\,dx = 8988.45\pi \left[9x - \frac{1}{2}x^2\right]_0^4 \\ &= 8988.45\pi \left[\left(9 \cdot 4 - \frac{1}{2} \cdot 4^2\right) - 0\right]_0^4 = 8988.45\pi \cdot 28 = 251{,}677\pi \approx \boxed{7.907 \times 10^5 \text{ ft lb}}.\end{aligned}$$

18. **(a)** Gasoline that is $x$ meters above the bottom of the tank must be lifted $10 - x$ m. The cross-sectional area of the tank at any height is the area of a circle with radius 8, which is $64\pi$ m$^2$. The gasoline at the bottom must be lifted 10 m, while that at the top is lifted 0 m, so the limits of integration are from 0 to 10, and the work performed is

$$\begin{aligned}W &= \rho g\,(64\pi) \int_0^{10} (10-x)\,dx \\ &= (720)(9.8)(64\pi)\left[10x - \frac{x^2}{2}\right]_0^{10} \\ &= (451{,}584)\pi\,[100 - 50 - 0] \\ &= 22{,}579{,}200\pi \approx \boxed{7.093 \times 10^7 \text{ J}}\end{aligned}$$

**(b)**

$$\begin{aligned}W &= \rho g\,(64\pi) \int_0^{10} (15-x)\,dx \\ &= (720)(9.8)(64\pi)\left[15x - \frac{x^2}{2}\right]_0^{10} \\ &= (451{,}584)\pi\,[150 - 50 - 0] \\ &= 45{,}158{,}400\pi \approx \boxed{1.418 \times 10^8 \text{ J}}\end{aligned}$$

19. **(a)** Each horizontal cross section of the pyramid is a square; the side lengths of this square decrease linearly from 2 at the top to 0 at the bottom, so at a point $x$ meters below the top of the container, the side length is $2 - \frac{2}{5}x$ ft. Therefore the cross-sectional area of the slurry at height $x$ is $\left(2 - \frac{2}{5}x\right)^2$ ft$^2$, and this must be lifted $x$ m. So the work performed (since the container is only filled 4 meters deep) is

$$9.9 \int_1^5 \left[17.9 \cdot \left(2 - \frac{2}{5}x\right)^2 \cdot x\right] dx = 175.42 \int_1^5 \left(4x - \frac{8}{5}x^2 + \frac{4}{25}x^3\right) dx$$
$$= 175.42 \left[2x^2 - \frac{8}{15}x^3 + \frac{1}{25}x^4\right]_1^5 = \boxed{1197.534 \text{ J}}.$$

**(b)** The size of the square cross sections are only proportional to the depth while the water is still inside the container, so this will be easier to solve if we now let $x$ represent the distance from the *bottom* of the container.

If $s$ represents the side of each square cross section, then, by similar triangles, $\frac{s}{x} = \frac{2 \text{ m}}{5 \text{ m}}$, or $s = \frac{2}{5}x$, so $A = s^2 = \left(\frac{2}{5}x\right)^2 = \frac{4}{25}x^2$. Also, each cross section now has to be lifted $8 - x$ m. Therefore, the work done is

$$W = 175.42 \int_0^4 (8 - x) \cdot \frac{4}{25}x^2\, dx = \frac{175.42}{25} \int_0^4 \left(32x^2 - 4x^3\right) dx$$
$$= \frac{175.42}{25}\left[\frac{32}{3}x^3 - x^4\right]_0^4 = \frac{175.42}{25}\left[\left(\frac{32}{3} \cdot 4^3 - 4^4\right) - 0\right] = \frac{175.42}{25} \cdot \frac{1280}{3} \approx \boxed{2993.835 \text{ J}}$$

20. Each horizontal cross section of the vat is a rectangle; its sides decrease linearly from 2 m $\times$ 0.5 m at the top to 0 m $\times$ 0 m at the bottom, so at a point $x$ meters below the top of the vat, the dimensions of the cross section are $\left(2 - \frac{x}{2}\right)$ m $\times$ $\left(\frac{1}{2} - \frac{x}{8}\right)$ m. Therefore the area of this cross section is

$$\left(2 - \frac{x}{2}\right)\left(\frac{1}{2} - \frac{x}{8}\right) \text{m}^2 = 1 - \frac{x}{2} + \frac{x^2}{16} \text{ m}^2$$

The density of olive oil is 0.9 g/cm$^3$ $\times$ 0.001 kg/g $\times$ $100^3$ cm$^3$/m$^3$ = 900 kg/m$^3$.

**(a)** Each cross section must be lifted $x$ meters, $2 \le x \le 4$.

$$\begin{aligned}
W &= 9.8(900) \int_2^4 x\left(1 - \frac{x}{2} + \frac{x^2}{16}\right) dx \\
&= 8820 \int_2^4 x\left(\frac{16 - 8x + x^2}{16}\right) dx \\
&= \frac{8820}{16} \int_2^4 \left(16x - 8x^2 + x^3\right) dx \\
&= \frac{2205}{4}\left[8x^2 - \frac{8x^3}{3} + \frac{x^4}{4}\right]_2^4 \\
&= \frac{2205}{4}\left[\left(128 - \frac{512}{3} + 64\right) - \left(32 - \frac{64}{3} + 4\right)\right] \\
&= \frac{2205}{4} \cdot \frac{20}{3} \\
&= \boxed{3675 \text{ J}}.
\end{aligned}$$

**(b)** Now each cross section must be lifted $2 + x$ meters, $2 \le x \le 4$.

$$\begin{aligned}
W &= 9.8(900)\int_2^4 (2+x)\left(1-\frac{x}{2}+\frac{x^2}{16}\right)dx \\
&= 8820\int_2^4 (2+x)\left(\frac{16-8x+x^2}{16}\right)dx \\
&= \frac{8820}{16}\int_2^4 \left(32-16x+2x^2+16x-8x^2+x^3\right)dx \\
&= \frac{2205}{4}\int_2^4 \left(x^3-6x^2+32\right)dx \\
&= \frac{2205}{4}\left[\frac{x^4}{4}-2x^3+32x\right]_2^4 \\
&= \frac{2205}{4}\left[(64-128+128)-(4-16+64)\right] \\
&= \frac{2205}{4}\cdot 12 \\
&= \boxed{6615 \text{ J}}.
\end{aligned}$$

**Applications and Extensions**

21. The elevator itself is lifted 400 ft, so this takes $10{,}000 \times 400 = 4 \times 10^6$ ft-lbs of work. The remaining work is lifting the cables themselves. For each cable, the portion of the cable $x$ feet from the top must be lifted $x$ feet. The cable weighs 0.36 lbs/in, which is $0.36 \times 12 = 4.32$ lbs/ft, so the work involved in lifting all six cables is

$$6\int_0^{400} 4.32x\,dx = 6\left[2.16x^2\right]_0^{400} = 2.0736 \times 10^6 \text{ ft-lbs}.$$

The total work is the sum of these two, or $\boxed{6.0736 \times 10^6 \text{ ft-lbs}}$.

22. The 15 m of cable must be lowered another 60 m; since the cable's density is 9 kg/m, this takes $15 \times 9 \times 60 \times 9.8 = 79380$ J. The remainder of the work expended is to lower the additional cable. A bit of cable on the spool that ends up $x$ m below the spool must be lowered $x$ m, so the work expended is

$$9.8\int_0^{60} 9x\,dx = 9.8\left[\frac{9}{2}x^2\right]_0^{60} = 158760 \text{ J}.$$

So the total work expended by gravity is $79380 + 158760 = \boxed{238140 \text{ J}}$.

23. Pulling the bucket itself to the top requires lifting a mass of 75 kg through a distance of 10 m, which takes $75 \times 10 \times 9.8 = 7350$ J. The remainder of the work expended is for the chain itself. The mass density of the chain is $\frac{20}{10} = 2$ kg/m, and the portion of the chain $x$ m below the roof must be lifted $x$ m, so the work expended is

$$9.8\int_0^{10} 2x\,dx = 9.8\left[x^2\right]_0^{10} = 980 \text{ J}.$$

Therefore the total work is $7350 + 980 = \boxed{8330 \text{ J}}$.

24. Pulling the bucket itself to the top requires lifting a mass of 75 kg through a distance of 10 m, which takes $75 \times 10 \times 9.8 = 7350$ J. The remainder of the work expended is for the chain itself. The mass density of the chain is $\frac{15}{10} = 1.5$ kg/m, and the portion of the chain $x$ m below the roof must be lifted $x$ m, so the work expended is

$$9.8 \int_0^{10} 1.5x\,dx = 9.8\left[0.75x^2\right]_0^{10} = 735 \text{ J}.$$

Therefore the total work is $7350 + 735 = \boxed{8085 \text{ J}}$.

25. The force required to extend the spring from 1 m to 3 m, which is 2 m beyond its equilibrium point, is $-3$ N, so the spring constant is given by $-3 = -k \cdot 2$, and therefore $k = \frac{3}{2}$. Then the work expended to extend it 1 m from equilibrium (to a length of 2 m) is

$$\int_0^1 \left(-\frac{3}{2}\right) x\,dx = -\left[\frac{3}{4}x^2\right]_0^1 = \boxed{-\frac{3}{4} \text{ J}}.$$

26. The force required to compress the spring by $\frac{3}{2}$ m is 10 N, so the spring constant is given by the equation $10 = -k \cdot \left(-\frac{3}{2}\right)$, and therefore $k = \frac{20}{3}$. Then the work expended to compress the spring to a length of 1 m is

$$\int_0^1 \left(-\frac{20}{3}x\right) dx = \left[\frac{10}{3}x^2\right]_0^1 = \boxed{-\frac{10}{3} \text{ J}}.$$

27. The force required to extend the spring by 4 ft is 2 lb, so the spring constant is given by $-2 = -k \cdot 4$, and then $k = \frac{1}{2}$. So

$$-9 \text{ ft-lb} = \int_0^x \left(-\frac{1}{2}t\right) dt = -\left[\frac{1}{4}t^2\right]_0^x = -\frac{1}{4}x^2 \text{ ft-lb}.$$

Simplifying gives $x^2 = 36$, so that $x = 6$ (since we are extending the spring, we take the positive solution). Therefore the total length of the spring is $6 + 4 = \boxed{10 \text{ ft}}$.

28. The spring constant is $\frac{1}{2}$, from Problem 27, so using the integral from Problem 27, we get

$$-8 \text{ ft-lb} = -\frac{1}{4}x^2 \text{ ft-lb},$$

so that $x = \sqrt{32} = 4\sqrt{2}$. So the spring is extended $4\sqrt{2}$ ft, for a total length of $\boxed{4 + 4\sqrt{2} \text{ ft}}$.

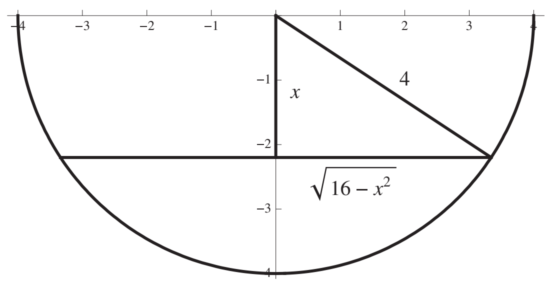
29. The radius of the cone, which is 4 m at the top and 0 m at the bottom, decreases linearly with height, so at a distance of $x$ m from the top, its radius is $4 - x$ m. Therefore the cross sectional area of the cone $x$ m from the top is $\pi(4-x)^2$ m$^2$. That cross section must be lifted $x$ m, so that the total work required is

$$\begin{aligned} 9.8 \int_0^4 1000 \cdot x \cdot \pi(4-x)^2\,dx &= 9800\pi \int_0^4 x(4-x)^2\,dx = 9800\pi \int_0^4 \left(16x - 8x^2 + x^3\right) dx \\ &= 9800\pi\left[8x^2 - \frac{8}{3}x^3 + \frac{1}{4}x^4\right]_0^4 = \boxed{\frac{627200}{3}\pi \text{ J}}. \end{aligned}$$

30. All the calculations from Problem 29 are still valid, except that the integration goes from 2 to 4 (since the integration variable $x$ is the distance below the top of the tank). Therefore we get for the work

$$9.8 \cdot 1000\pi\left[8x^2 - \frac{8}{3}x^3 + \frac{1}{4}x^4\right]_2^4 = \boxed{\frac{196000}{3}\pi \text{ J}}.$$

31. To determine the radius of the cross section $x$ m below the top of the bowl, consider the following diagram, which shows a vertical cross section through the center of the bowl:




At a height $x$ m below the top, the radius of the cross section is $\sqrt{16-x^2}$, so that the area of the cross section at that height is $(16-x^2)\pi$. That cross section must be lifted $x$ m, so the work performed is

$$9.8\int_2^4 1000x(16-x^2)\pi\,dx = 9800\pi\int_2^4 (16x-x^3)\,dx = 9800\pi\left[8x^2-\frac{1}{4}x^4\right]_2^4 = \boxed{352800\pi \text{ J}}.$$

32. The area of each cross section is the same as in Problem 31. Now, that cross section must be lifted $x+2$ m, since we want to lift it 2 m above the top of the tank. The integration bounds are now from 0 to 4, since the tank is full. Then the work performed is

$$\begin{aligned}9.8\int_0^4 1000(x+2)(16-x^2)\pi\,dx &= 9800\pi\int_0^4 \left(-x^3-2x^2+16x+32\right)\,dx\\ &= 9800\pi\left[-\frac{1}{4}x^4-\frac{2}{3}x^3+8x^2+32x\right]_0^4 = \boxed{\frac{4390400}{3}\pi \text{ J}}.\end{aligned}$$

33. The area of each cross section is $4\pi$ m$^2$, since the tank has a diameter of 4 m.

**(a)** The cross section $x$ m below the top must be lifted $x$ m. We wish to lift only the top half of the water, so the integration bounds are from 0 to 3 and the total work is

$$9.8\int_0^3 1000x\cdot 4\pi\,dx = 9.8\int_0^3 4000\pi x\,dx = 9.8\left[2000\pi x^2\right]_0^3 \approx \boxed{5.542\times 10^5 \text{ J}}.$$

**(b)** Now each cross section must be lifted $x+3$ m. We wish to lift the entire tank, so the integration bounds are 0 to 6. Therefore the work done is

$$\begin{aligned}W &= 9.8\cdot 4000\pi\int_0^3 (x+3)\,dx = 39{,}200\pi\left[\frac{1}{2}x^2+3x\right]_0^3\\ &= 39{,}200\pi\left[\left(\frac{1}{2}\cdot 3^2+3\cdot 3\right)-0\right] = 39{,}200\pi\cdot\frac{27}{2} = 529{,}200\pi \approx \boxed{1.663\times 10^6 \text{ J}}.\end{aligned}$$

34. The area of each horizontal cross section is $30\times 20 = 600$ ft$^2$, and the cross section $x$ ft below the top must be raised $x$ ft. The integration bounds are from 1 to 6 since there is water in the bottom 5 feet of the pool. Therefore the total work is

$$\int_1^6 62.42x\cdot 600\,dx = \left[18726x^2\right]_1^6 = \boxed{655410 \text{ ft-lbs}}.$$

35. The area of each horizontal cross section is $25 \times 15 = 375$ ft$^2$, and the cross section $x$ ft below the top must be raised $x$ ft. The integration bounds are from 1 to 5 since there is water in the bottom 4 feet of the pool. Therefore the total work is

$$\int_1^5 62.42x \cdot 375\, dx = \left[11703.8x^2\right]_1^5 = 280890 \text{ ft-lbs.}$$

Since the motor can do 550 ft-lbs of work per second, the time required to empty the pool is

$$\frac{280890}{550} \text{ s} \approx 511 \text{ s} = \frac{511}{60} \text{ min} \approx \boxed{8.5 \text{ min}}.$$

It takes between 8 and 9 minutes to empty the pool.

36. From the discussion in the text, the work required to move a mass $m$ kg from the surface of the Earth to a distance $d$ meters above the center of the Earth is

$$gRm\left(1 - \frac{R}{R+d}\right) \text{ J},$$

where $g \approx 9.8\text{m/s}^2$ is the acceleration due to Earth's gravity and $R \approx 6.37 \times 10^6$ m is the radius of the Earth. So to move a mass of 30 kg a distance of 30 km = 30000 m above the Earth's surface requires

$$9.8 \cdot 6.37 \times 10^6 \cdot 30\left(1 - \frac{6370000}{6370000 + 30000}\right) \approx \boxed{8.77866 \times 10^6 \text{ J}}.$$

37. From the discussion in the text, the work required to move a mass $m$ kg from the surface of the Earth to a distance $d$ meters above the surface of the Earth is

$$gRm\left(1 - \frac{R}{R+d}\right) \text{ J},$$

where $g \approx 9.8\text{m/s}^2$ is the acceleration due to Earth's gravity and $R \approx 6.37 \times 10^6$ m is the radius of the Earth. So to move a mass of 1000 kg a distance of 800 km = 800000 m above the Earth's surface requires

$$9.8 \cdot 6.37 \times 10^6 \cdot 1000\left(1 - \frac{6370000}{6370000 + 800000}\right) \approx \boxed{6.96524 \times 10^9 \text{ J}}.$$

38. To move the charge from $x = 2a$ to $x = a$, we must apply a force equal to the negative of the electrical force, or $-\frac{m}{x^2}$. Here we are dealing with a unit charge, so $m = 1$, and then the work is

$$\int_{2a}^{a}\left(-\frac{1}{x^2}\right) dx = \left[\frac{1}{x}\right]_{2a}^{a} = \boxed{\frac{1}{2a}}.$$

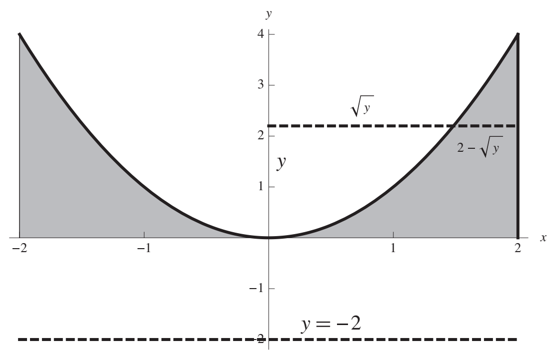
39. Since the water weighs 20 lb at the bottom, and loses a fourth of its weight, it weighs 15 lb at the top. So at a depth $x$ from the top of the well, the weight of the water in the bucket is $15 + 5\frac{x}{25} = 15 + \frac{x}{5}$ lb. Since the bucket weighs 1.5 lb, the total weight being lifted at height $x$ is $16.5 + \frac{x}{5}$ lb. To compute the work, note that there is no need to multiply by $x$ in this integration, since we are lifting this weight "infinitesimally higher", so the expression above is the complete integrand. Another way of looking at this is that in the previous problems, we lifted the cross section all the way to the top, so we needed to multiply by $x$, the distance to the top, in order to figure out how far to lift it. Here we are lifting it only $dx$.

$$\int_0^{25}\left(16.5 + \frac{x}{5}\right) dx = \left[16.5x + \frac{1}{10}x^2\right]_0^{25} = \boxed{475 \text{ ft-lbs}}.$$

40. The force is $F = -300000x$, so the work done is

$$\int_0^{0.1} (-300000x)\, dx = -\left[150000x^2\right]_0^{0.1} = \boxed{-1500 \text{ J}}.$$

41. A vertical cross section of the container, with the level of the water source shown, is below:




At height $y$, a horizontal cross section of the container is a washer with inner radius $\sqrt{y}$ and outer radius 2 (see the figure), so its area is $\pi\left(2^2 - \left(\sqrt{y}\right)^2\right) = (4-y)\pi$. To lift that water to height $y$, it must be lifted $y+2$ units since the source is 2 units below the $x$-axis. Finally, the container is bounded by $x = \pm 2$, so by $y = 2^2 = 4$. Therefore the total work performed is

$$\int_0^4 (y+2)(4-y)\pi\,dy = \pi\int_0^4 (-y^2+2y+8)\,dy = \pi\left[-\frac{1}{3}y^3 + y^2 + 8y\right]_0^4 = \boxed{\frac{80}{3}\pi}.$$

### Challenge Problems

42. **(a)** Since the volume is given in cubic feet, and we want cubic inches, we multiply by $12^3 = 1728$ to get 3456 in$^3$ for the current volume and 6912 in$^3$ for the desired volume. Since $PV^{1.4} = c$, and $P = 100$ when $V = 3456$ in$^3$, we get $100 \cdot 3456^{1.4} = c$, or $c \approx 8.995 \times 10^6$. Then since we are doubling the volume from 3456 to 6912, the work done is

$$W = \int_{3456}^{6912} P\,dV \approx \int_{3456}^{6912} \frac{8.995\times 10^6}{V^{1.4}}\,dV = 8.995\times 10^6 \int_{3456}^{6912} V^{-1.4}\,dV$$
$$= 8.995\times 10^6\left[-\frac{1}{0.4}V^{-0.4}\right]_{3456}^{6912} \boxed{\approx 209209 \text{ in-lb}}.$$

**(b)** The work is

$$W = \int_{2.4}^{4.6} P\,dV = \int_{2.4}^{4.6} \frac{120}{V^{1.2}}\,dV = 120\left[-\frac{1}{0.2}V^{-0.2}\right]_{2.4}^{4.6} = 600\left(\frac{1}{2.4^{0.2}} - \frac{1}{4.6^{0.2}}\right)\boxed{\approx 61.45 \text{ in-lb}}.$$

## 6.7 Hydrostatic Pressure and Force

### Concepts and Vocabulary

1. Pressure is defined as the *force* exerted per unit area.
2. True. The pressure is $\rho g h$, where $h$ is the depth of the plate.
3. (b), Pascal, is correct. See Table 2.
4. True. See the discussion following Table 2.

## Skill Building

5. At each height ($y$-coordinate), the depth of the water above that height is $5-y$, so the force on a slice at that height is $3\rho g(5-y)\Delta y$. Therefore the total force is the integral over the possible slices, or

$$\int_0^5 3\rho g(5-y)\,dy = 3\rho g\left[5y-\frac{1}{2}y^2\right]_0^5 = \frac{75}{2}\rho g.$$

Using SI units, we have $\rho g = 9800$, so we get $\frac{75}{2}\cdot 9800 = \boxed{367500 \text{ N}}$.

6. At each height ($y$-coordinate), the depth of the water above that height is $-y$. At that height, the coordinates of the two edges of the end of the container are, using the Pythagorean Theorem, $\pm\sqrt{4-y^2}$. So the force on a slice at that height is $2\sqrt{4-y^2}\cdot\rho g(-y)\Delta y$. Therefore the total force is

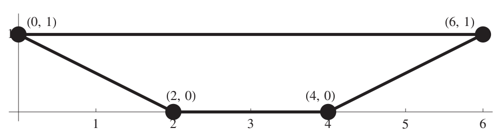
$$\int_{-2}^0 2\sqrt{4-y^2}\cdot\rho g(-y)\,dy = -2\rho g\int_{-2}^0 y\sqrt{4-y^2}\,dy.$$

Make the substitution $u=4-y^2$, so that $du=-2y\,dy$. Then $y=-2$ corresponds to $u=0$, and $y=0$ to $u=4$, so we get

$$-2\rho g\int_{-2}^0 y\sqrt{4-y^2}\,dy = -2\rho g\cdot\left(-\frac{1}{2}\right)\int_0^4 u^{1/2}\,du = \rho g\left[\frac{2}{3}u^{3/2}\right]_0^4 = \frac{16}{3}\rho g.$$

Using SI units, we have $\rho g = 9800$, so we get $\frac{16}{3}\cdot 9800 = \boxed{\frac{156800}{3}\approx 52266.7 \text{ N}}$.

7. Assuming that the diagonal lines make the same angle with the $x$-axis, we get the following diagram of the end of the container:




The diagonal line on the left passes through $(2,0)$ and $(0,1)$, so it has equation

$$y-1=\frac{1-0}{0-2}(x-0)=-\frac{1}{2}x, \qquad \text{or} \qquad y=1-\frac{1}{2}x, \qquad \text{or} \qquad x=2-2y.$$

The diagonal line on the right passes through $(4,0)$ and $(6,1)$, so it has equation

$$y=\frac{1-0}{6-4}(x-4), \qquad \text{or} \qquad y=\frac{1}{2}x-2, \qquad \text{or} \qquad x=4+2y.$$

At a height $y$, the distance below the surface of the water is $1-y$, so the force on a slice at that height is

$$\rho g(1-y)((4+2y)-(2-2y)) = \rho g(1-y)(2+4y) = \rho g(-4y^2+2y+2).$$

Therefore the total force on the plate is

$$\int_0^1 \rho g(-4y^2+2y+2) = \rho g\left[-\frac{4}{3}y^3+y^2+2y\right]_0^1 = \frac{5}{3}\rho g = \boxed{\frac{49000}{3} \approx 16333.3\text{ N}}.$$

8. The diagonal side of the container passes through the points $(3,0)$ and $(0,4)$, so its equation is given by $4x+3y=12$, or $x = 3-\frac{3}{4}y$. At a given $y$-coordinate, a cross section is $4-y$ feet below the surface of the water. Solving $4x+3y=12$ for $x$, we see that the width of that cross section is $x = 3-\frac{3}{4}y$. Then the total force on the end of the container is

$$\begin{aligned}\int_0^4 \rho g(4-y)\left(3-\frac{3}{4}y\right)\,dy &= \rho g\int_0^4\left(12-6y+\frac{3}{4}y^2\right)\,dy \\ &= \rho g\left[12y-3y^2+\frac{1}{4}y^3\right]_0^4 = 16\rho g = \boxed{1000\text{ lbs}}.\end{aligned}$$

9. At any depth, the width of the plate is 2 m, so the force on a cross section at height $y$ is $2\rho g y\,\Delta y$. Therefore the total force on the plate is

$$\int_0^6 2\rho g y\,dy = \left[\rho g y^2\right]_0^6 = 36\rho g = \boxed{352800\text{ N}}.$$

10. The analysis in the solution to Problem 9 is still valid; the only difference is that the integral goes from $y=1$ to $y=7$, since the plate is suspended 1 m below the water surface. Therefore the total force on the plate is

$$\int_1^7 2\rho g y\,dy = \left[\rho g y^2\right]_1^7 = 48\rho g = \boxed{470400\text{ N}}.$$

## Applications and Extensions

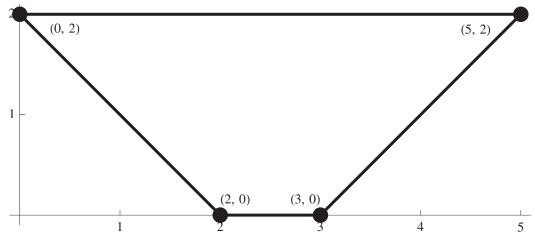
11. At any depth, the width of the pool is 20 ft. Therefore the force on the short side of the pool is (since the pool is 6 ft deep)

$$\int_0^6 \rho g 20y\,dy = 62.5\left[10y^2\right]_0^6 = \boxed{22500\text{ lbs}}.$$

12. The long side of the pool is 30 ft long, so, since the pool is again 6 ft deep, the total force is

$$\int_0^6 \rho g 30y\,dy = 62.5\left[15y^2\right]_0^6 = \boxed{33750\text{ lbs}}.$$

13. Place the trapezoid on a Cartesian grid:



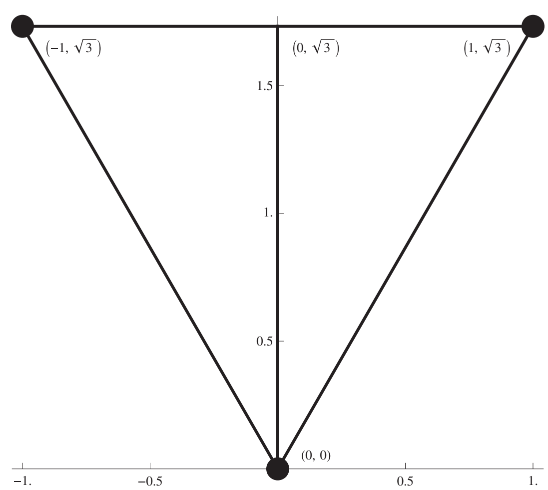
Then the equations of the two diagonal lines are:

$$y - 0 = \frac{0-2}{2-0}(x-2) \qquad \text{or} \qquad y = 2 - x \qquad \text{or} \qquad x = 2 - y$$
$$y - 0 = \frac{0-2}{3-5}(x-3) \qquad \text{or} \qquad y = x - 3 \qquad \text{or} \qquad x = y + 3.$$

So at any height $y$, the height of the water above a slice at that point is $1 - y$ (since the trough is filled only to a depth of 1 m), and the width of the slice is $(y+3)-(2-y) = 2y+1$, so the force on the end of the trough is

$$\int_0^1 \rho g(1-y)(2y+1)\,dy = \rho g \int_0^1 (-2y^2 + y + 1)\,dy = \rho g \left[ -\frac{2}{3}y^3 + \frac{1}{2}y^2 + y \right]_0^1$$
$$= 9800 \cdot \frac{5}{6} = \boxed{\frac{24500}{3} \approx 8166.67 \text{ N}}.$$

14. Placing the bottom vertex of the equilateral triangle at the origin, we get the following picture:




Each of the diagonal lines makes an angle of 60° with the $x$-axis, so their equations are $\frac{y}{x} = \tan 60°$, or $x = \frac{1}{\sqrt{3}}y$ and $\frac{y}{x} = \tan 120°$, or $x = -\frac{1}{\sqrt{3}}y$. So at a height of $y$ above the bottom the width of the trough is $\frac{1}{\sqrt{3}}y - \left(-\frac{1}{\sqrt{3}}y\right) = \frac{2}{\sqrt{3}}y$. Since the equilateral triangle is 2 m on the side, its height is $\sqrt{3}$, so that a cross section at $y$ is $\sqrt{3} - y$ m below the surface. Therefore the force is

$$\int_0^{\sqrt{3}} \rho g(\sqrt{3} - y)\left(\frac{2}{\sqrt{3}}y\right)\,dy = \rho g \int_0^{\sqrt{3}} \left(-\frac{2}{\sqrt{3}}y^2 + 2y\right)\,dy$$
$$= \rho g \left[ -\frac{2}{3\sqrt{3}}y^3 + y^2 \right]_0^{\sqrt{3}} = 1\rho g = \boxed{9800 \text{ N}}.$$

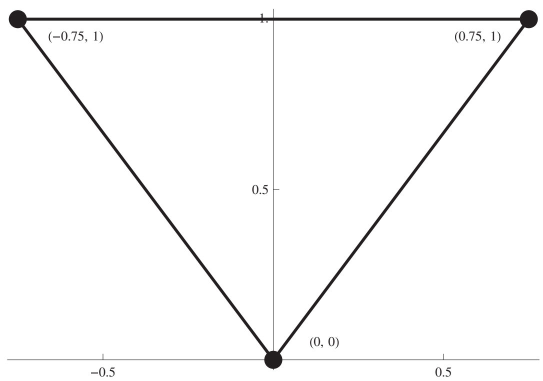
15. With the origin at the dot in the diagram, water at a given value of $y$ has width $2\sqrt{4-y^2}$, since the left edge of the trough will be at $x = -\sqrt{4-y^2}$ and the right edge will be at $\sqrt{4-y^2}$. Since the trough is filled, water at a given value of $y$ is at a depth of $-y$ m. So the total force is

$$\int_{-2}^{0} \rho g(-y)2\sqrt{4-y^2}\,dy = \rho g \int_{-2}^{0} \left(-2y\sqrt{4-y^2}\right)\,dy.$$

Now make the substitution $u = 4 - y^2$, so that $du = -2y\,dy$. Then $y = -2$ corresponds to $u = 0$, and $y = 0$ to $u = 4$, so we get

$$\rho g \int_{-2}^{0} \left(-2y\sqrt{4 - y^2}\right)\,dy = \rho g \int_{0}^{4} u^{1/2}\,du = \rho g \left[\frac{2}{3}u^{3/2}\right]_0^4 = \frac{16}{3}\rho g \boxed{\approx 52266 \text{ N}}.$$

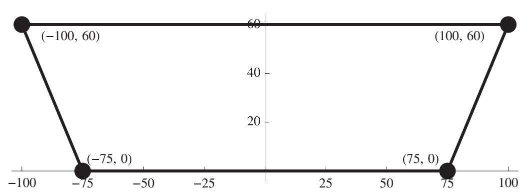
16. The face of the floodgate is shown below:




The diagonal lines have slopes $-\frac{1}{0.75} = -\frac{4}{3}$ and $\frac{1}{0.75} = \frac{4}{3}$, so their equations are $x = -\frac{3}{4}y$ and $x = \frac{3}{4}y$. Therefore for a given value of $y$, the width of a cross section of the floodgate is $\frac{3}{4}y - \left(-\frac{3}{4}y\right) = \frac{3}{2}y$, and this cross section is $1 - y$ m below the surface. Then the total force is

$$\int_0^1 \rho g(1-y)\left(\frac{3}{2}y\right)\,dy = \rho g \int_0^1 \left(\frac{3}{2}y - \frac{3}{2}y^2\right)\,dy = \rho g \left[\frac{3}{4}y^2 - \frac{1}{2}y^3\right]_0^1 = \boxed{2450 \text{ lbs}}.$$

17. Placing the origin at the center of the bottom of the dam, the face of the dam is shown below:




The two diagonal lines have equations

$$y - 0 = \frac{60 - 0}{-100 - (-75)}(x - (-75)) \quad \text{or} \quad y = -\frac{12}{5}(x + 75) \quad \text{or} \quad y = -\frac{12}{5}x - 180$$

$$y - 0 = \frac{60 - 0}{100 - 75}(x - 75) \quad \text{or} \quad y = \frac{12}{5}(x - 75) \quad \text{or} \quad y = \frac{12}{5}x - 180.$$

Solving for $x$ gives

$$x = -\frac{5}{12}y - 75 \quad \text{or} \quad x = \frac{5}{12}y + 75.$$

Therefore the width of a cross section at height $y$ is

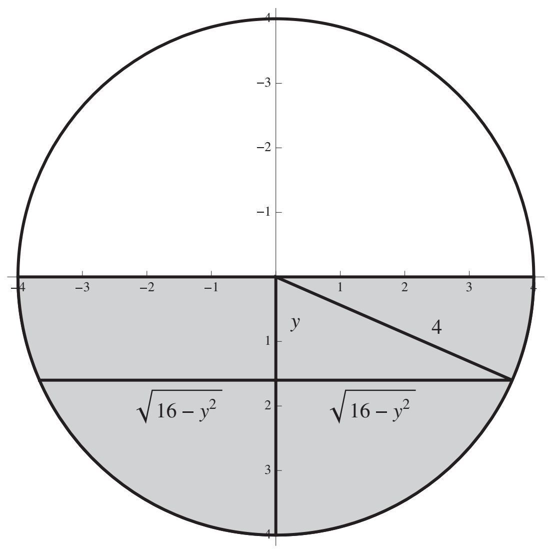
$$\left(\frac{5}{12}y + 75\right) - \left(-\frac{5}{12}y - 75\right) = \frac{5}{6}y + 150,$$

and such a cross section lies $60 - y$ m below the surface. Then the total force is

$$\int_0^{60} \rho g(60-y)\left(\frac{5}{6}y+150\right)\,dy = \rho g\int_0^{60}\left(9000-100y-\frac{5}{6}y^2\right)\,dy$$

$$= \rho g\left[9000y - 50y^2 - \frac{5}{18}y^3\right]_0^{60} = \boxed{2.94\times 10^9\ \text{N}}.$$

18. Placing the origin at the center of one end of the tank with the positive $y$-axis pointing down, the face of the tank is shown below:



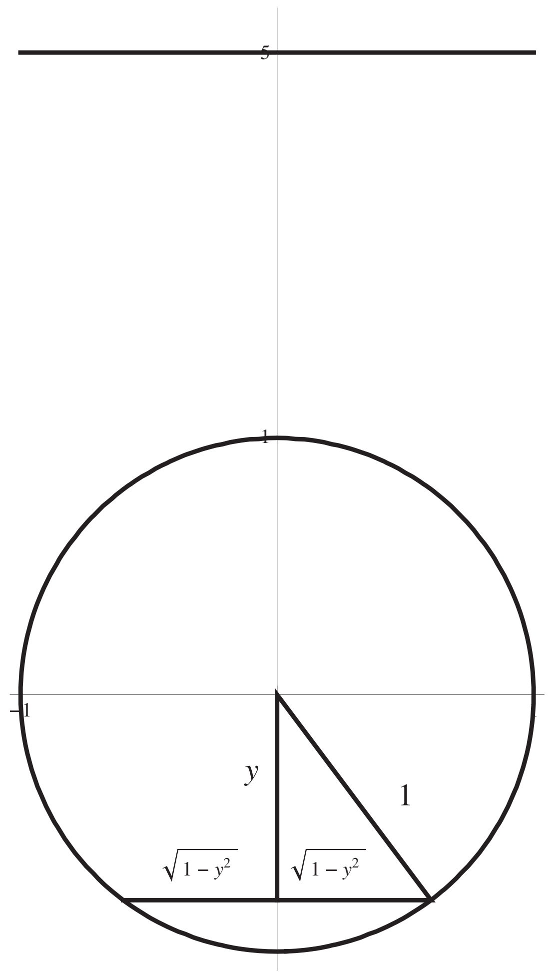

At a given depth $y$, the distance across that end of the tank is $2\sqrt{16-y^2}$, and that cross section lies $y$ ft below the surface. Therefore the total force is

$$\int_0^4 \rho g y\cdot 2\sqrt{16-y^2}\,dy = 2\rho g\int_0^4 y\sqrt{16-y^2}\,dy.$$

Now use the substitution $u = 16 - y^2$, so that $du = -2y\,dy$. Then $y = 0$ corresponds to $u = 16$, while $y = 4$ corresponds to $u = 0$, so we get

$$2\rho g\int_0^4 y\sqrt{16-y^2}\,dy = 2\rho g\cdot\left(-\frac{1}{2}\right)\int_{16}^0 u^{1/2}\,dy = -\rho g\left[\frac{2}{3}u^{3/2}\right]_{16}^0 = \frac{128}{3}\rho g = 60\cdot\frac{128}{3} = \boxed{2560\ \text{lbs}}.$$

19. Place the origin at the center of the viewing plate; then the situation is shown below:

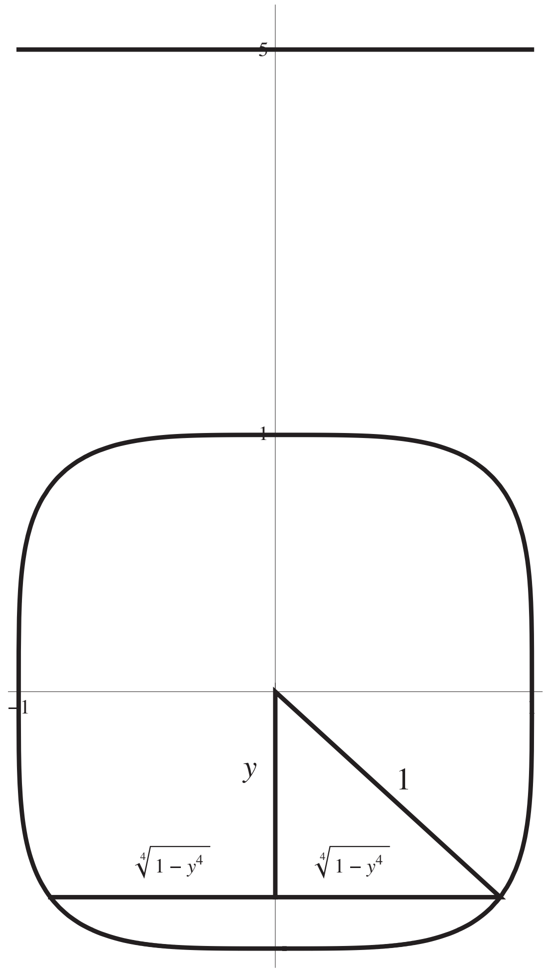



The equation of the viewing plate is $x^2 + y^2 = 1$, so at any $y$-coordinate, the width of a cross section of the plate is $2\sqrt{1-y^2}$ (see the figure), and that cross section is $5 - y$ m below the water surface. (Note that if $y < 0$, as drawn, then $5 - y > 5$, which makes sense since this part of the plate is *more* than 5 m below the surface.) Therefore the total force is

$$\int_{-1}^{1} \rho g(5-y) \cdot 2\sqrt{1-y^2}\, dy$$

Evaluating numerically, with $\rho g = 1025 \cdot 9.8$, gives $5\rho g\pi$ $\boxed{\approx 157{,}786 \text{ N}}$.

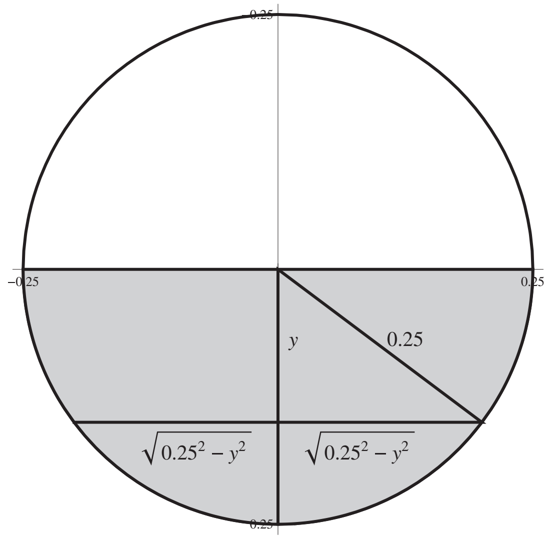
20. Place the origin at the center of the viewing plate; then the situation is shown below:




The equation of the viewing plate is $x^4 + y^4 = 1$, so at any $y$-coordinate, the width of a cross section of the plate is $2\sqrt[4]{1-y^4}$ (see the figure), and that cross section is $5 - y$ m below the water surface. (Note that if $y < 0$, as drawn, then $5 - y > 5$, which makes sense since this part of the plate is *more* than 5 m below the surface.) Therefore the total force is (evaluating numerically, with $\rho g = 1025 \cdot 9.8$)

$$\int_{-1}^{1} \rho g(5-y) \cdot 2\sqrt[4]{1-y^4}\,dy \approx 18.5407\rho g \boxed{\approx 186{,}241 \text{ N}}.$$

21. Placing the origin at the center of one end of the tank with the positive $y$-axis pointing down, the face of the tank is shown below:




Since the tank's radius is 0.25 m, it is half-full. At a given $y$-coordinate (depth), the width of the cross section of that end is $2\sqrt{0.25^2 - y^2}$, and the cross section lies $y$ m below the surface. So the total force is

$$\int_0^{0.25} \rho g y \cdot 2\sqrt{0.25^2 - y^2}\, dy$$

Now use the substitution $u = 0.25^2 - y^2$, so that $du = -2y\, dy$. Then $y = 0$ corresponds to $u = 0.25^2$, while $y = 0.25$ corresponds to $y = 0$, and we get

$$\int_0^{0.25} \rho g y \cdot 2\sqrt{0.25^2 - y^2}\, dy = -\rho g \int_{0.25^2}^{0} u^{1/2}\, du = -\rho g \left[\frac{2}{3} u^{3/2}\right]_{0.25^2}^{0} = \frac{2}{3}\rho g \cdot 0.25^3$$

$$= \frac{2}{3} \cdot 690 \cdot 9.8 \cdot 0.015625 = \boxed{70.4375 \text{ N}}.$$

22.

$$\begin{aligned} F &= \int_c^d \rho g (H - y)\,[f(y) - h(y)]\, dy \\ &= \int_{-2}^{0} 9800(0 - y)\left[\sqrt{16 - y^2} - \left(-\sqrt{16 - y^2}\right)\right] dy \\ &= 9800 \int_{-2}^{0} (-y)\left[2\sqrt{16 - y^2}\right] dy \\ &= 9800 \int_{-2}^{0} -2y\sqrt{16 - y^2}\, dy \end{aligned}$$

Let $u = 16 - y^2$. Then $du = -2y\, dy$, $u(-2) = 12$, and $u(0) = 16$.

$$\begin{aligned} &= 9800 \int_{12}^{16} u^{1/2}\, du \\ &= 9800 \left[\frac{2u^{3/2}}{3}\right]_{12}^{16} = 9800 \left[\frac{2\left(64 - 24\sqrt{3}\right)}{3}\right] = 9800 \left[\frac{128 - 48\sqrt{3}}{3}\right] \\ &\approx 146{,}547.766 \text{ N} \approx \boxed{1.466 \times 10^5 \text{ N}} \end{aligned}$$

## 6.8 Center of Mass; Centroid; the Pappus Theorem

### Concepts and Vocabulary

1. (c), center of mass, is correct. See the first few sentences of this chapter.
2. (c), $md$, is correct. See the discussion in subsection 1.
3. True. See the discussion in subsection 2.
4. (a) is correct; see Example 4.
5. False. See Example 5 for one instance where it does not.
6. (a) is correct. See the discussion in subsection 2.

### Skill Building

7. The center of mass is

$$\bar{x} = \frac{\sum_{i=1}^{2} m_i x_i}{\sum_{i=1}^{2} m_i} = \frac{20 \cdot 4 + 50 \cdot 10}{20 + 50} = \boxed{\frac{58}{7}}.$$

8. The center of mass is

$$\bar{x} = \frac{\sum_{i=1}^{2} m_i x_i}{\sum_{i=1}^{2} m_i} = \frac{10 \cdot (-2) + 3 \cdot 3}{10 + 3} = \boxed{-\frac{11}{13}}.$$

9. The center of mass is

$$\bar{x} = \frac{\sum_{i=1}^{4} m_i x_i}{\sum_{i=1}^{4} m_i} = \frac{4 \cdot (-1) + 3 \cdot 2 + 3 \cdot 4 + 5 \cdot 3}{4 + 3 + 3 + 5} = \boxed{\frac{29}{15}}.$$

10. The center of mass is

$$\bar{x} = \frac{\sum_{i=1}^{4} m_i x_i}{\sum_{i=1}^{4} m_i} = \frac{7 \cdot 6 + 3 \cdot (-2) + 2 \cdot (-4) + 4 \cdot (-1)}{7 + 3 + 2 + 4} = \frac{24}{16} = \boxed{\frac{3}{2}}.$$

11. The moments are

$$M_y = \sum_{i=1}^{3} m_i x_i = 4 \cdot 0 + 8 \cdot 2 + 1 \cdot 4 = 20$$
$$M_x = \sum_{i=1}^{3} m_i y_i = 4 \cdot 2 + 8 \cdot 1 + 1 \cdot 8 = 24$$

and the mass is $M = 4 + 8 + 1 = 13$, so that

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{20}{13}, \frac{24}{13}\right)}.$$

12. The moments are

$$M_y = \sum_{i=1}^{3} m_i x_i = 6 \cdot (-1) + 2 \cdot 12 + 10 \cdot (-1) = 8$$
$$M_x = \sum_{i=1}^{3} m_i y_i = 6 \cdot (-1) + 2 \cdot 6 + 10 \cdot (-2) = -14$$

and the mass is $M = 6 + 2 + 10 = 18$, so that

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{8}{18}, -\frac{14}{18}\right) = \boxed{\left(\frac{4}{9}, -\frac{7}{9}\right)}.$$

13. The moments are

$$M_y = \sum_{i=1}^{4} m_i x_i = 4 \cdot (-1) + 3 \cdot 2 + 3 \cdot 4 + 5 \cdot 3 = 29$$
$$M_x = \sum_{i=1}^{4} m_i y_i = 4 \cdot 2 + 3 \cdot 3 + 3 \cdot 5 + 5 \cdot 6 = 62$$

and the mass is $M = 4 + 3 + 3 + 5 = 15$, so that

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{29}{15}, \frac{62}{15}\right)}.$$

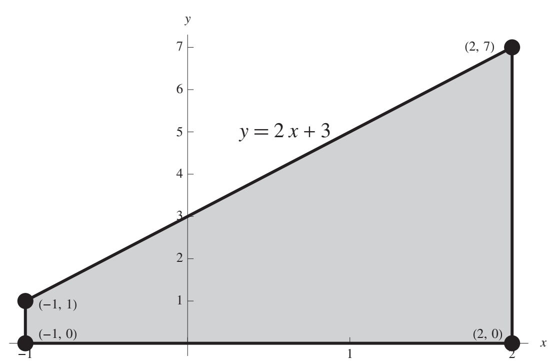
14. The moments are

$$M_y = \sum_{i=1}^{4} m_i x_i = 8 \cdot (-4) + 6 \cdot 0 + 3 \cdot 6 + 5 \cdot (-3) = -29$$
$$M_x = \sum_{i=1}^{4} m_i y_i = 8 \cdot 4 + 6 \cdot 5 + 3 \cdot 4 + 5 \cdot (-5) = 49$$

and the mass is $M = 8 + 6 + 3 + 5 = 22$, so that

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(-\frac{29}{22}, \frac{49}{22}\right)}.$$

15. The lamina is shown below:




Let the mass density of the lamina be $\rho$. Then the mass of the lamina is

$$M = \int_{-1}^{2} \rho(2x + 3)\, dx = \rho \left[x^2 + 3x\right]_{-1}^{2} = 12\rho,$$

and the moments are

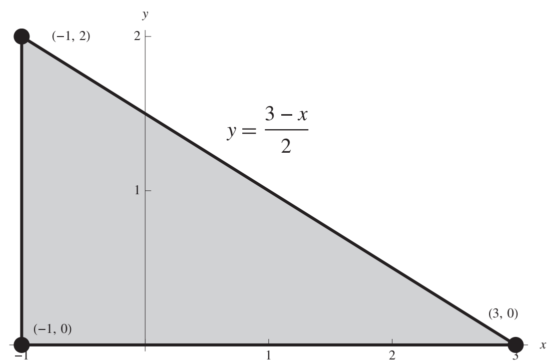
$$M_y = \rho \int_{-1}^{2} x(2x+3)\,dx = \rho \int_{-1}^{2} (2x^2+3x)\,dx = \rho\left[\frac{2}{3}x^3 + \frac{3}{2}x^2\right]_{-1}^{2} = \frac{21}{2}\rho$$

$$M_x = \frac{1}{2}\rho \int_{-1}^{2} (2x+3)^2\,dx = \frac{1}{2}\rho\left[\frac{1}{6}(2x+3)^3\right]_{-1}^{2} = \frac{342}{12}\rho = \frac{57}{2}\rho.$$

Therefore the centroid is

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{(21/2)\rho}{12\rho}, \frac{(57/2)\rho}{12\rho}\right) = \left(\frac{21}{2\cdot 12}, \frac{57}{2\cdot 12}\right) = \boxed{\left(\frac{7}{8}, \frac{19}{8}\right)}.$$

16. The lamina is shown below:




Let the mass density of the lamina be $\rho$. Then the mass of the lamina is

$$M = \rho \int_{-1}^{3} \frac{3-x}{2}\,dx = \rho \int_{-1}^{3} \left(\frac{3}{2} - \frac{1}{2}x\right)dx = \rho\left[\frac{3}{2}x - \frac{1}{4}x^2\right]_{-1}^{3} = 4\rho$$

and the moments are

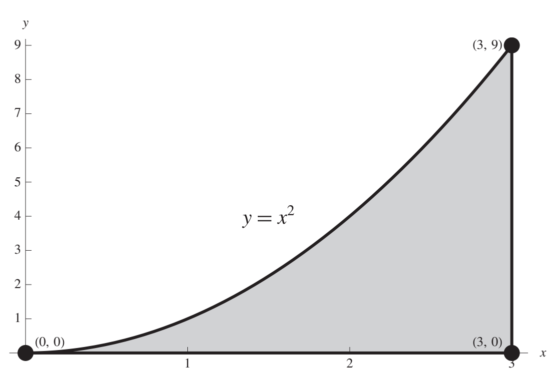
$$M_y = \rho \int_{-1}^{3} x\cdot\frac{3-x}{2}\,dx = \rho \int_{-1}^{3} \left(\frac{3}{2}x - \frac{1}{2}x^2\right)dx = \rho\left[\frac{3}{4}x^2 - \frac{1}{6}x^3\right]_{-1}^{3} = \frac{4}{3}\rho$$

$$M_x = \frac{1}{2}\rho \int_{-1}^{3} \left(\frac{3-x}{2}\right)^2 dx = \frac{1}{8}\rho \int_{-1}^{3} (3-x)^2\,dx = \frac{1}{8}\rho\left[-\frac{1}{3}(3-x)^3\right]_{-1}^{3} = \frac{8}{3}\rho.$$

Therefore the centroid is

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{(4/3)\rho}{4\rho}, \frac{(8/3)\rho}{4\rho}\right) = \left(\frac{4}{3\cdot 4}, \frac{8}{3\cdot 4}\right) = \boxed{\left(\frac{1}{3}, \frac{2}{3}\right)}.$$

17. The lamina is shown below:



Let the mass density of the lamina be $\rho$. Then the mass of the lamina is

$$M = \rho \int_0^3 x^2\,dx = \rho\left[\frac{1}{3}x^3\right]_0^3 = 9\rho$$

and the moments are

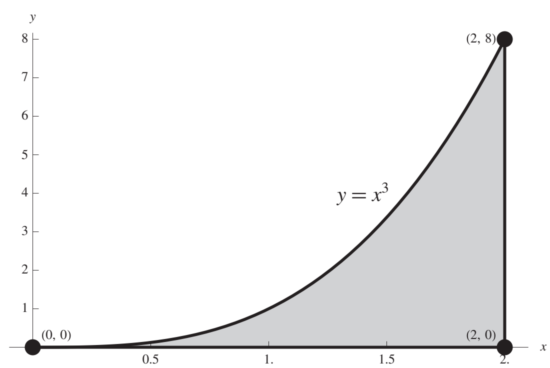
$$M_y = \rho \int_0^3 x \cdot x^2\,dx = \rho \int_0^3 x^3\,dx = \rho\left[\frac{1}{4}x^4\right]_0^3 = \frac{81}{4}\rho$$

$$M_x = \frac{1}{2}\rho \int_0^3 \left(x^2\right)^2\,dx = \frac{1}{2}\rho \int_0^3 x^4\,dx = \frac{1}{2}\rho\left[\frac{1}{5}x^5\right]_0^3 = \frac{243}{10}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{(81/4)\rho}{9\rho}, \frac{(243/10)\rho}{9\rho}\right) = \left(\frac{81}{4\cdot 9}, \frac{243}{10\cdot 9}\right) = \boxed{\left(\frac{9}{4}, \frac{27}{10}\right)}.$$

18. The lamina is shown below:




Let the mass density of the lamina be $\rho$. Then the mass of the lamina is

$$M = \rho \int_0^2 x^3\,dx = \rho\left[\frac{1}{4}x^4\right]_0^2 = 4\rho$$

and the moments are

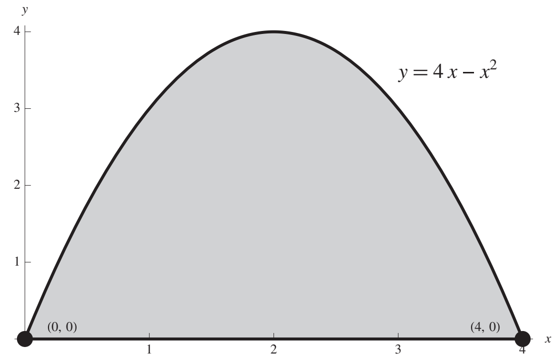
$$M_y = \rho \int_0^2 x \cdot x^3\,dx = \rho \int_0^2 x^4\,dx = \rho\left[\frac{1}{5}x^5\right]_0^2 = \frac{32}{5}\rho$$

$$M_x = \frac{1}{2}\rho \int_0^2 \left(x^3\right)^2\,dx = \frac{1}{2}\rho \int_0^2 x^6\,dx = \frac{1}{2}\rho\left[\frac{1}{7}x^7\right]_0^2 = \frac{64}{7}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{(32/5)\rho}{4\rho}, \frac{(64/7)\rho}{4\rho}\right) = \left(\frac{32}{5\cdot 4}, \frac{64}{7\cdot 4}\right) = \boxed{\left(\frac{8}{5}, \frac{16}{7}\right)}.$$

19. The lamina is shown below:




Let the mass density of the lamina be $\rho$. Then the mass of the lamina is

$$M = \rho \int_0^4 (4x - x^2)\, dx = \rho \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \frac{32}{3}\rho$$

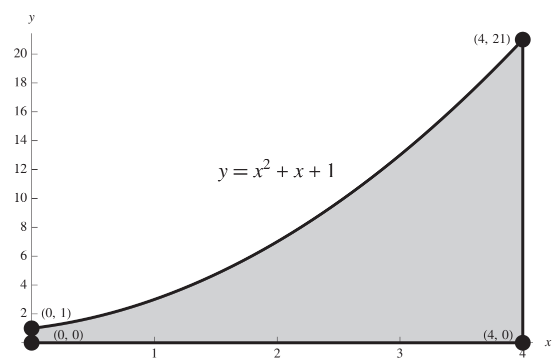
and the moments are

$$M_y = \rho \int_0^4 x \cdot (4x - x^2)\, dx = \rho \int_0^4 (4x^2 - x^3)\, dx = \rho \left[\frac{4}{3}x^3 - \frac{1}{4}x^4\right]_0^4 = \frac{64}{3}\rho$$
$$M_x = \frac{1}{2}\rho \int_0^4 (4x - x^2)^2\, dx = \frac{1}{2}\rho \int_0^4 \left(16x^2 - 8x^3 + x^4\right)\, dx = \frac{1}{2}\rho \left[\frac{16}{3}x^3 - 2x^4 + \frac{1}{5}x^5\right]_0^4 = \frac{256}{15}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{(64/3)\rho}{(32/3)\rho}, \frac{(256/15)\rho}{(32/3)\rho}\right) = \left(2, \frac{256 \cdot 3}{32 \cdot 15}\right) = \boxed{\left(2, \frac{8}{5}\right)}.$$

20. The lamina is shown below:




Let the mass density of the lamina be $\rho$. Then the mass of the lamina is

$$M = \rho \int_0^4 (x^2 + x + 1)\, dx = \rho \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + x\right]_0^4 = \frac{100}{3}$$

and the moments are

$$M_y = \rho \int_0^4 x \cdot (x^2 + x + 1)\, dx = \rho \int_0^4 (x^3 + x^2 + x)\, dx = \rho \left[ \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 \right]_0^4 = \frac{280}{3}\rho$$

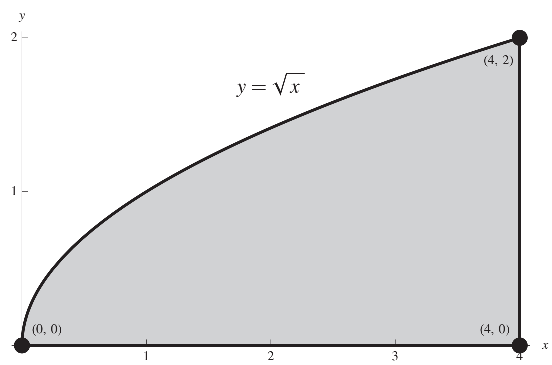
$$M_x = \frac{1}{2}\rho \int_0^4 (x^2 + x + 1)^2\, dx = \frac{1}{2}\rho \int_0^4 \left(x^4 + 2x^3 + 3x^2 + 2x + 1\right)\, dx$$

$$= \frac{1}{2}\rho \left[ \frac{1}{5}x^5 + \frac{1}{2}x^4 + x^3 + x^2 + x \right]_0^4 = \frac{1042}{5}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left( \frac{M_y}{M}, \frac{M_x}{M} \right) = \left( \frac{(280/3)\rho}{(100/3)\rho}, \frac{(1042/5)\rho}{(100/3)\rho} \right) = \left( \frac{280}{100}, \frac{1042 \cdot 3}{100 \cdot 5} \right) = \boxed{\left( \frac{14}{5}, \frac{1563}{250} \right)}.$$

21. The lamina is




Let the mass density of the lamina be $\rho$. Then the mass of the lamina is

$$M = \rho \int_0^4 x^{1/2}\, dx = \rho \left[ \frac{2}{3}x^{3/2} \right]_0^4 = \frac{16}{3}\rho$$

and the moments are

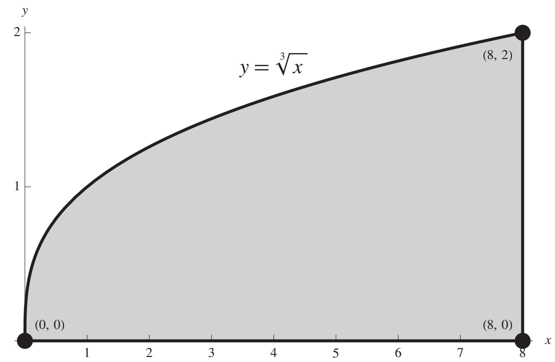
$$M_y = \rho \int_0^4 x \cdot x^{1/2}\, dx = \rho \int_0^4 x^{3/2}\, dx = \rho \left[ \frac{2}{5}x^{5/2} \right]_0^4 = \frac{64}{5}\rho$$

$$M_x = \frac{1}{2}\rho \int_0^4 \left(x^{1/2}\right)^2\, dx = \frac{1}{2}\rho \int_0^4 x\, dx = \frac{1}{2}\rho \left[ \frac{1}{2}x^2 \right]_0^4 = 4\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left( \frac{M_y}{M}, \frac{M_x}{M} \right) = \left( \frac{(64/5)\rho}{(16/3)\rho}, \frac{4\rho}{(16/3)\rho} \right) = \left( \frac{64 \cdot 3}{16 \cdot 5}, \frac{4 \cdot 3}{16} \right) = \boxed{\left( \frac{12}{5}, \frac{3}{4} \right)}.$$

22. The lamina is shown below:




Let the mass density of the lamina be $\rho$. Then the mass of the lamina is

$$M = \rho \int_0^8 x^{1/3}\,dx = \rho \left[\frac{3}{4}x^{4/3}\right]_0^8 = 12\rho$$

and the moments are

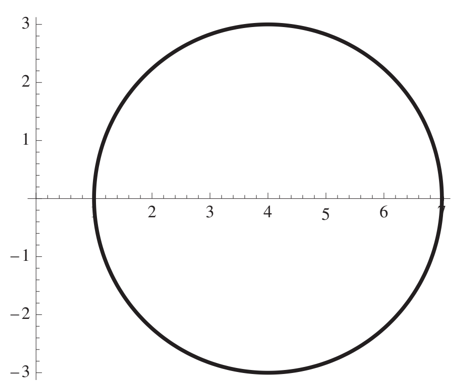
$$M_y = \rho \int_0^8 x \cdot x^{1/3}\,dx = \rho \int_0^8 x^{4/3}\,dx = \rho \left[\frac{3}{7}x^{7/3}\right]_0^8 = \frac{384}{7}\rho$$

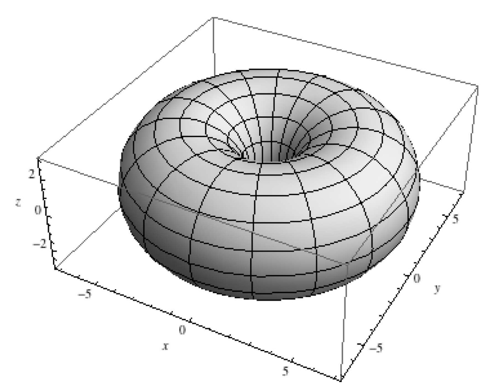
$$M_x = \frac{1}{2}\rho \int_0^8 \left(x^{1/3}\right)^2\,dx = \frac{1}{2}\rho \int_0^8 x^{2/3}\,dx = \frac{1}{2}\rho \left[\frac{3}{5}x^{5/3}\right]_0^8 = \frac{48}{5}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{(384/7)\rho}{12\rho}, \frac{(48/5)\rho}{12\rho}\right) = \left(\frac{384}{7 \cdot 12}, \frac{48}{5 \cdot 12}\right) = \boxed{\left(\frac{32}{7}, \frac{4}{5}\right)}.$$

23. The circle and the surface are shown below:

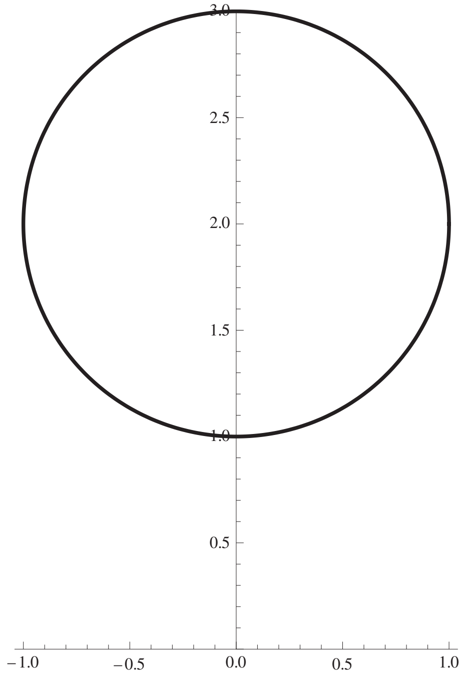


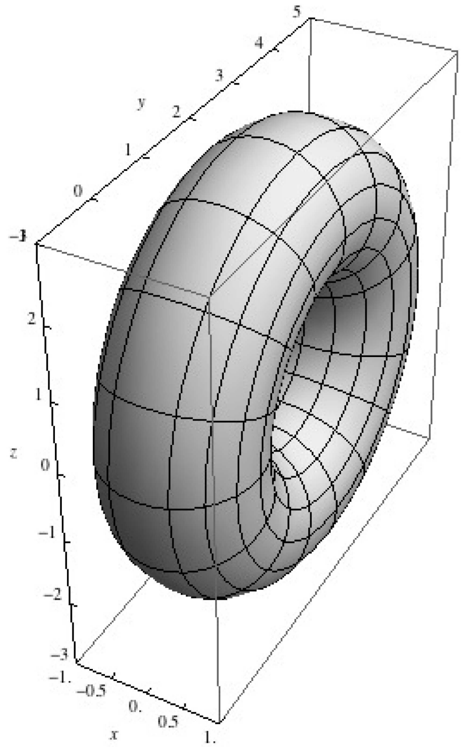




The circle has a center of $(4, 0)$ and a radius of 3. It is symmetric about the point $(4, 0)$, so its centroid is $(\bar{x}, \bar{y}) = (4, 0)$. The distance from $(4, 0)$ to the axis of revolution (the $y$-axis) is 4, and the area of the circle is $\pi \cdot 3^2 = 9\pi$. The circle does not intersect the $y$-axis, so by Pappus' Theorem, the volume of the solid of revolution is

$$2\pi A d = 2\pi \cdot 9\pi \cdot 4 = \boxed{72\pi^2}.$$

24. The circle and the surface are shown below:




The circle has a center of $(0, 2)$ and a radius of 1. It is symmetric about the point $(0, 2)$, so its centroid is $(\bar{x}, \bar{y}) = (0, 2)$. The distance from $(0, 2)$ to the axis of revolution (the $x$-axis) is 2, and the area of the circle is $\pi \cdot 1^2 = \pi$. The circle does not intersect the $x$-axis, so by Pappus's Theorem, the volume of the solid of revolution is

$$2\pi A d = 2\pi \cdot \pi \cdot 2 = \boxed{4\pi^2}.$$

## Applications and Extensions

25. If the mass density is $\rho$, then the mass is the mass of the circle of radius 1 plus the mass of the square of side 2, so it is $(4+\pi)\rho$. By symmetry, since the figure is symmetric around the line $x = 1$, we have $\bar{x} = 1$. To compute $\bar{y}$, we will need the moment $M_x$. This figure is the union of a non-overlapping square and circle, so its moment is the sum of the individual moments. The center of the square, at $(1, 1)$, is its centroid, and its mass is $4\rho$, so we have

$$1 = \frac{M_1}{4\rho}, \text{ so that } M_1 = 4\rho.$$

The center of the circle, at $(1, 3)$, is its centroid, and its mass is $\pi\rho$, so we have

$$3 = \frac{M_2}{\pi\rho}, \text{ so that } M_2 = 3\pi\rho.$$

So for the figure as a whole, we have $M_x = (4 + 3\pi)\rho$, and then

$$(\bar{x}, \bar{y}) = \left(1, \frac{(4+3\pi)\rho}{(4+\pi)\rho}\right) = \boxed{\left(1, \frac{4+3\pi}{4+\pi}\right)}.$$

26. This triangle has a base of 4 and a height of 4, so its area is $\frac{1}{2} \cdot 4 \cdot 4 = 8$ and then its mass is $8\rho$. By symmetry, $\bar{x} = 0$. To compute $\bar{y}$, note that the triangle is the sum of two non-overlapping triangles, one for $x \le 0$ and one for $x \ge 0$. For $x \le 0$, the diagonal side of the triangle is the line $y = 4 + 2x$, while for $x \ge 0$, the diagonal side is the line $y = 4 - 2x$. So the moment $M_x$ is the sum of these two moments, or

$$\begin{aligned} M_x &= \frac{1}{2}\rho \int_{-2}^{0} (4+2x)^2\,dx + \frac{1}{2}\rho \int_{0}^{2} (4-2x)^2\,dx \\ &= \frac{1}{2}\rho \left( \left[ \frac{1}{6}(4+2x)^3 \right]_{-2}^{0} + \left[ -\frac{1}{6}(4-2x)^3 \right]_{0}^{2} \right) \\ &= \frac{1}{2}\rho \left( \frac{32}{3} + \frac{32}{3} \right) = \frac{32}{3}\rho. \end{aligned}$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left( 0, \frac{(32/3)\rho}{8\rho} \right) = \boxed{\left( 0, \frac{4}{3} \right)}.$$

27. This triangle has a base of $2c$ and a height of $b$, so its area is $bc$ and then its mass is $bc\rho$, where $\rho$ is the mass density of the lamina. The triangle is composed of two nonoverlapping triangles, one of which has vertices $(-c, 0)$, $(a, 0)$ and $(a, b)$ and the other of which has vertices $(a, 0)$, $(c, 0)$ and $(a, b)$. The moments of the given region about either axis are the sum of the moments of these two triangles about that axis. For the first triangle, the equation of the diagonal edge is

$$y - 0 = \frac{b-0}{a-(-c)}(x - (-c)), \text{ or } y = \frac{b}{a+c}(x+c).$$

Therefore its moments are

$$\begin{aligned} M_y &= \rho \int_{-c}^{a} x \left( \frac{b}{a+c}(x+c) \right) dx \\ &= \rho \cdot \frac{b}{a+c} \int_{-c}^{a} (x^2 + cx)\,dx \\ &= \rho \cdot \frac{b}{a+c} \left[ \frac{1}{3}x^3 + \frac{1}{2}cx^2 \right]_{-c}^{a} \\ &= \rho \cdot \frac{b}{a+c} \left( \frac{1}{6}(2a^3 + 3a^2c - c^3) \right) \\ &= \rho \cdot \frac{b}{a+c} \left( \frac{1}{6}(2a-c)(a+c)^2 \right) \\ &= \frac{1}{6}b(2a-c)(a+c)\rho \end{aligned}$$

$$\begin{aligned} M_x &= \frac{1}{2}\rho \int_{-c}^{a} \left( \frac{b}{a+c}(x+c) \right)^2 dx \\ &= \frac{1}{2}\rho \left( \frac{b}{a+c} \right)^2 \int_{-c}^{a} (x+c)^2\,dx \\ &= \frac{1}{2}\rho \left( \frac{b}{a+c} \right)^2 \left[ \frac{1}{3}(x+c)^3 \right]_{-c}^{a} \\ &= \frac{1}{2}\rho \frac{b^2}{(a+c)^2} \cdot \frac{1}{3}(a+c)^3 \\ &= \frac{1}{6}b^2(a+c)\rho. \end{aligned}$$

For the second triangle, the equation of the diagonal edge is

$$y - 0 = \frac{b-0}{a-c}(x-c), \text{ or } y = \frac{b}{a-c}(x-c).$$

Therefore its moments are

$$\begin{aligned}
M_y &= \rho \int_a^c x\left(\frac{b}{a-c}(x-c)\right) dx \\
&= \rho \cdot \frac{b}{a-c}\int_a^c (x^2 - cx)\, dx \\
&= \rho \cdot \frac{b}{a-c}\left[\frac{1}{3}x^3 - \frac{1}{2}cx^2\right]_a^c \\
&= \rho \cdot \frac{b}{a-c}\left(-\frac{1}{6}(2a+c)(a-c)^2\right) \\
&= -\frac{1}{6}b(2a+c)(a-c)\rho
\end{aligned}$$

$$\begin{aligned}
M_x &= \frac{1}{2}\rho \int_a^c \left(\frac{b}{a-c}(x-c)\right)^2 dx \\
&= \frac{1}{2}\rho\left(\frac{b}{a-c}\right)^2 \int_a^c (x-c)^2\, dx \\
&= \frac{1}{2}\rho\left(\frac{b}{a-c}\right)^2 \left[\frac{1}{3}(x-c)^3\right]_a^c \\
&= \frac{1}{2}\rho\frac{b^2}{(a-c)^2}\cdot\left(-\frac{1}{3}(a-c)^3\right) \\
&= -\frac{1}{6}b^2(a-c)\rho.
\end{aligned}$$

So the moments for the figure as a whole are

$$\begin{aligned}
M_y &= \frac{1}{6}b(2a-c)(a+c)\rho - \frac{1}{6}b(2a+c)(a-c)\rho = \frac{1}{3}abc\rho \\
M_x &= \frac{1}{6}b^2(a+c)\rho - \frac{1}{6}b^2(a-c)\rho = \frac{1}{3}b^2c\rho.
\end{aligned}$$

Then the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{(abc/3)\rho}{bc\rho}, \frac{(b^2c/3)\rho}{bc\rho}\right) = \boxed{\left(\frac{a}{3}, \frac{b}{3}\right)}.$$

28. If the mass density is $\rho$, then the mass is the mass of the semicircle of radius 1 plus the mass of the $2 \times 2$ square, so it is $\left(4 + \frac{1}{2}\pi\right)\rho$. By symmetry, since the figure is symmetric around the line $y$-axis, we have $\bar{x} = 0$. To compute $\bar{y}$, we will need the moment $M_x$. This figure is the union of a non-overlapping square and circle, so its moment is the sum of the individual moments. The center of the square, at $(0, -1)$, is its centroid, and its mass is $4\rho$, so we have

$$-1 = \frac{M_1}{4\rho}, \text{ so that } M_1 = -4\rho.$$

For the semicircle, we compute

$$M_2 = \frac{1}{2}\rho\int_{-1}^1 \left(\sqrt{1-x^2}\right)^2 dx = \rho\int_{-1}^1 (1-x^2)\, dx = \frac{1}{2}\rho\left[x - \frac{1}{3}x^3\right]_{-1}^1 = \frac{2}{3}\rho.$$

So for the entire lamina,

$$M_x = -4\rho + \frac{2}{3}\rho = -\frac{10}{3}\rho,$$

and the centroid is

$$(\bar{x}, \bar{y}) = \left(0, \frac{M_x}{M}\right) = \left(0, \frac{-(10/3)\rho}{(4 + (1/2)\pi)\rho}\right) = \boxed{\left(0, -\frac{20}{24 + 3\pi}\right)}.$$

29. The mass of the lamina is

$$M = \rho \int_0^a \frac{h}{a^2} x^2\, dx = \rho \cdot \frac{h}{a^2} \int_0^a x^2\, dx = \rho \cdot \frac{h}{a^2} \left[\frac{1}{3}x^3\right]_0^a = \rho \cdot \frac{h}{a^2} \cdot \frac{1}{3}a^3 = \frac{1}{3}ah\rho.$$

The moments about the axes are

$$M_y = \rho \int_0^a x\left(\frac{h}{a^2}x^2\right) dx = \rho \cdot \frac{h}{a^2} \int_0^a x^3\, dx = \rho \cdot \frac{h}{a^2} \left[\frac{1}{4}x^4\right]_0^a = \frac{1}{4}a^2 h\rho$$

$$M_x = \frac{1}{2}\rho \int_0^a \left(\frac{h}{a^2}x^2\right)^2 dx = \frac{1}{2}\rho \cdot \frac{h^2}{a^4} \int_0^a x^4\, dx = \frac{1}{2}\rho \cdot \frac{h^2}{a^4} \left[\frac{1}{5}x^5\right]_0^a = \frac{1}{10}ah^2.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{(a^2h/4)\rho}{(ah/3)\rho}, \frac{(ah^2/10)\rho}{(ah/3)\rho}\right) = \boxed{\left(\frac{3a}{4}, \frac{3h}{10}\right)}.$$

30. By symmetry, $\bar{x} = 0$. Now, this figure is the union of two identical figures, one on each side of the $y$-axis; each of these figures has a moment about the $x$-axis that was determined in Problem 29 to be $\frac{1}{10}ah^2\rho$, so the total moment about the $x$-axis if $\frac{1}{5}ah^2\rho$. The mass of the lamina is twice the mass of the lamina in Problem 29, or $\frac{2}{3}ah\rho$. Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(0, \frac{M_x}{M}\right) = \left(0, \frac{(ah^2/5)\rho}{(2ah/3)\rho}\right) = \boxed{\left(0, \frac{3h}{10}\right)}.$$

31. **(a)** The mass of the bat is

$$M = \int_0^L \lambda\, dx = \int_0^L kx\, dx = \left[\frac{1}{2}kx^2\right]_0^L = \boxed{\frac{1}{2}kL^2}.$$

**(b)** Solving the result of part (a) for $k$ gives

$$k = \boxed{\frac{2M}{L^2}}.$$

**(c)** The center of mass of the bat is

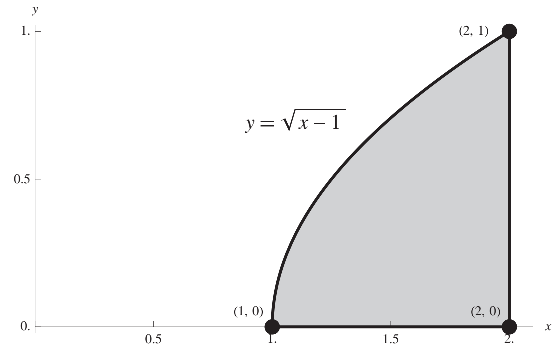
$$\bar{x} = \frac{\int_0^L x\lambda\, dx}{M} = \frac{2}{kL^2} \int_0^L kx^2\, dx = \frac{2}{L^2}\left[\frac{1}{3}x^3\right]_0^L = \boxed{\frac{2}{3}L}.$$

The center of mass is $\frac{2}{3}$ of the way from the handle to the end.

**(d)** Answers will vary. The "sweet spot" is at the center of mass, since at that point the bat is neither underbalanced nor overbalanced, and you will be more likely to make solid contact.

**(e)** Answers will vary. A small slice through the bat at any $x$ gives a slice whose density is approximately $\lambda = kx$ and whose width is $\Delta x$, so the total mass is the limit of $\sum kx\Delta x$ as $\Delta x \to 0$; this is a Riemann sum whose limit is $\int_0^L kx\, dx = \int_0^L \lambda\, dx$.

32. The region is shown below:




The density is $\rho = 1$, so the mass (area) is

$$\int_1^2 \sqrt{x-1}\,dx = \left[\frac{2}{3}(x-1)^{3/2}\right]_1^2 = \frac{2}{3}.$$

Since we are revolving about the $y$-axis, we need to know the distance of the centroid from the $y$-axis to apply Pappus' Theorem; this distance is $\bar{x}$. So we compute $M_y$:

$$\begin{aligned} M_y &= \int_1^2 x\sqrt{x-1}\,dx \\ &= \int_1^2 \left((x-1)\sqrt{x-1} + \sqrt{x-1}\right)\,dx \\ &= \int_1^2 \left((x-1)^{3/2}\,dx + (x-1)^{1/2}\right)\,dx \\ &= \left[\frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2}\right]_1^2 = \frac{16}{15}. \end{aligned}$$

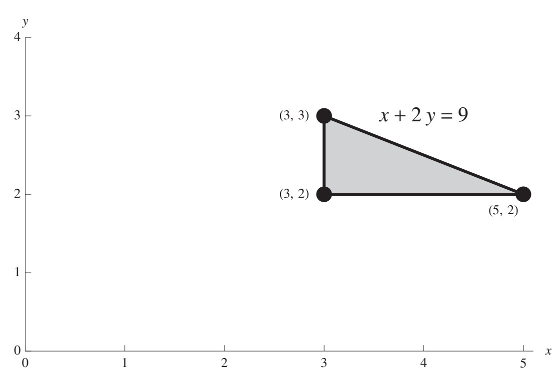
Then

$$\bar{x} = \frac{M_y}{M} = \frac{16/15}{2/3} = \frac{8}{5},$$

which is also the distance from the centroid to the axis of revolution. Therefore by Pappus's Theorem, the volume of the solid of revolution is

$$2\pi A d = 2\pi \cdot \frac{2}{3} \cdot \frac{8}{5} = \boxed{\frac{32\pi}{15}}.$$

33. The region is shown below:



This triangle has base $5 - 3 = 2$ and height $3 - 2 = 1$, so its area is 1. We are revolving about the $y$-axis, so to apply Pappus' Theorem we will need to know the distance from the centroid of the region to the $y$-axis; this distance is $\bar{x}$, so we must compute $M_y$. This is

$$M_y = \int_3^5 x\left(\frac{9-x}{2} - 2\right)\,dx = \int_3^5 \left(\frac{5}{2}x - \frac{1}{2}x^2\right)\,dx = \left[\frac{5}{4}x^2 - \frac{1}{6}x^3\right]_3^5 = \frac{11}{3}.$$

Therefore $\bar{x} = \frac{11}{3}$ (since $M = 1$), so that by Pappus' Theorem the volume of the solid of revolution is

$$2\pi A d = 2\pi \cdot 1 \cdot \frac{11}{3} = \boxed{\frac{22}{3}\pi}.$$

34. This cone is the solid of revolution about the $y$-axis of a triangle whose vertices are $(0,0)$, $(R,0)$, and $(0,H)$. The equation of the diagonal edge of this triangle is $Hx + Ry = RH$, or $y = H - \frac{H}{R}x$. The area of the triangle is $\frac{1}{2}RH$. We will want the distance of the centroid from the $y$-axis, so we compute $M_y$:

$$M_y = \int_0^R x\left(H - \frac{H}{R}x\right)\,dx = \int_0^R \left(Hx - \frac{H}{R}x^2\right)\,dx = \left[\frac{H}{2}x^2 - \frac{H}{3R}x^3\right]_0^R = \frac{1}{6}HR^2.$$

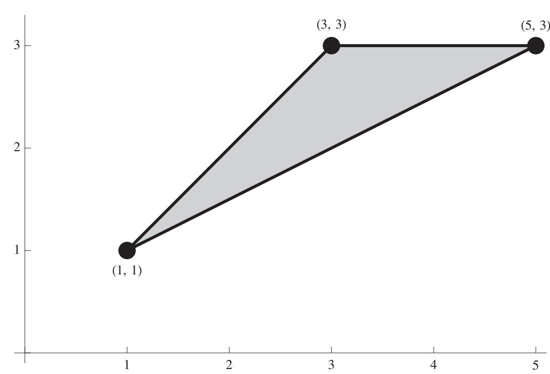
Then

$$\bar{x} = \frac{(1/6)HR^2}{(1/2)RH} = \frac{1}{3}R,$$

so by Pappus' Theorem, the volume of the cone is

$$2\pi A d = 2\pi A\bar{x} = 2\pi \cdot \frac{1}{2}RH \cdot \frac{1}{3}R = \boxed{\frac{1}{3}\pi R^2 H}.$$

35. The region is shown below:




We are revolving this about the $x$-axis, so using Pappus' Theorem, we will want the $y$-coordinate of the centroid, $\bar{y}$. We must split this up into two integrals, one from $x = 1$ to $x = 3$ and the other from $x = 3$ to $x = 5$. The equation of the diagonal line from $(1,1)$

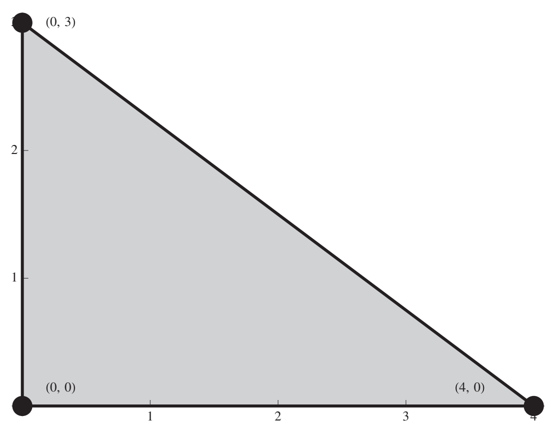
to $(3,3)$ is $y = x$; the equation of the other diagonal line is $y = \frac{1}{2}(x+1)$. Then computing $M_x$ gives

$$\begin{aligned}
M_x &= \frac{1}{2}\int_1^3 \left(x^2 - \left(\frac{1}{2}(x+1)\right)^2\right)\,dx + \frac{1}{2}\int_3^5 \left(3^2 - \left(\frac{1}{2}(x+1)\right)^2\right)\,dx \\
&= \frac{1}{2}\int_1^3 \left(\frac{3}{4}x^2 - \frac{1}{2}x - \frac{1}{4}\right)\,dx + \frac{1}{2}\int_3^5 \left(-\frac{1}{4}x^2 - \frac{1}{2}x + \frac{35}{4}\right)\,dx \\
&= \frac{1}{2}\left[\frac{1}{4}x^3 - \frac{1}{4}x^2 - \frac{1}{4}x\right]_1^3 + \frac{1}{2}\left[-\frac{1}{12}x^3 - \frac{1}{4}x^2 + \frac{35}{4}x\right]_3^5 \\
&= \frac{1}{2}\left(\frac{15}{4} - \left(-\frac{1}{4}\right)\right) + \frac{1}{2}\left(\frac{325}{12} - \frac{87}{4}\right) \\
&= \frac{14}{3}.
\end{aligned}$$

Therefore $\bar{y} = \frac{M_x}{M} = \frac{7}{3}$, so by Pappus' Theorem, the volume of the solid is

$$2\pi A d = 2\pi \cdot 2 \cdot \frac{7}{3} = \boxed{\frac{28}{3}\pi}.$$

36. The region is shown below:




The triangle has a base of 3 and a height of 4, so its area is $\frac{1}{2}\cdot 4 \cdot 3 = 6$. The equation of the diagonal edge of this triangle is $4x + 3y = 12$, or $y = 4 - \frac{4}{3}x$. Therefore the moments are

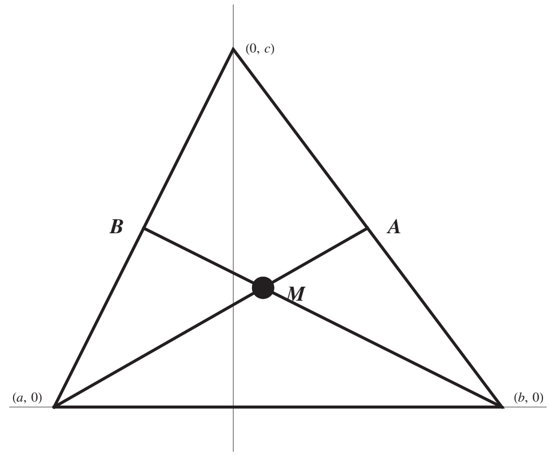
$$\begin{aligned}
M_y &= \int_0^3 x\left(4 - \frac{4}{3}x\right)\,dx = \int_0^3 \left(4x - \frac{4}{3}x^2\right)\,dx = \left[2x^2 - \frac{4}{9}x^3\right]_0^3 = 6 \\
M_x &= \frac{1}{2}\int_0^3 \left(4 - \frac{4}{3}x\right)^2\,dx = \frac{1}{2}\left[-\frac{1}{4}\left(4 - \frac{4}{3}x\right)^3\right]_0^3 = -\frac{1}{8}\left[\left(4 - \frac{4}{3}x\right)^3\right]_0^3 = 8,
\end{aligned}$$

so that

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(1, \frac{4}{3}\right)}.$$

## Challenge Problems

37. The triangle with two of the medians drawn is shown below:




Here $A$ is the midpoint of the side opposite $(a, 0)$, $B$ is the midpoint of the side opposite $(b, 0)$, and $M$ is the point of intersection of those two medians. So we must show that $M$ is the centroid of the triangle.

Since $A$ is the midpoint of the segment from $(0, c)$ to $(b, 0)$, its coordinates are $\frac{1}{2}((0,c) + (b,0)) = \left(\frac{b}{2}, \frac{c}{2}\right)$. Similarly the coordinates of $B$ are $\left(\frac{a}{2}, \frac{c}{2}\right)$. So the equations of the two medians are

$$y - 0 = \frac{\frac{c}{2} - 0}{\frac{b}{2} - a}(x - a), \text{ or } y = \frac{c}{b - 2a}(x - a)$$
$$y - 0 = \frac{\frac{c}{2} - 0}{\frac{a}{2} - b}(x - b), \text{ or } y = \frac{c}{a - 2b}(x - b).$$

(Note that $b - 2a > 0$ since $b > 0$ and $a < 0$, so that the denominator in the first expression is nonzero. Similarly, $a - 2b < 0$. Therefore the denominators in these equations cannot be zero.) These two lines intersect when

$$\frac{c}{b - 2a}(x - a) = \frac{c}{a - 2b}(x - b).$$

Clearing fractions gives

$$\begin{aligned}
c(a - 2b)(x - a) &= c(b - 2a)(x - b)\\
acx - a^2c - 2bcx + 2abc &= bcx - b^2c - 2acx + 2abc\\
(ac - 2bc)x - a^2c &= (bc - 2ac)x - b^2c\\
(ac - 2bc - bc + 2ac)x &= (b^2 - a^2)c\\
3c(a - b)x &= (b - a)(b + a)c\\
-3x &= -(a + b)\\
x &= \frac{a + b}{3}.
\end{aligned}$$

Again note that dividing through by $(a - b)c$ is justified since $a - b < 0$ and $c > 0$, so the product cannot be zero. Then

$$y = \frac{c}{b - 2a}(x - a) = \frac{c}{b - 2a} \cdot \left(\frac{a + b}{3} - a\right) = \frac{c}{b - 2a} \cdot \frac{b - 2a}{3} = \frac{c}{3}.$$

So the point $M$ has coordinates $\left(\frac{1}{3}(a+b),\ \frac{1}{3}c\right)$. Now consider the third median, from $(0,c)$ to the $x$-axis. If $a=-b$, then the midpoint of the lower side of the triangle is the origin, and the median is the line $x=0$ from $(0,c)$ to the origin. In this case, the point $M$ has coordinates $\left(0,\ \frac{1}{3}c\right)$ since $a+b=0$, and so the median from $(0,c)$ passes through $M$ as well. If $a\neq -b$, then the midpoint of the lower side of the triangle is $\frac{a+b}{2}$, and the median has equation

$$y-c=\frac{0-c}{\frac{a+b}{2}-0}(x-0),\ \text{or}\ y=c-\frac{2c}{a+b}x.$$

Substituting the $x$-coordinate of $M$ for $x$ in this equation gives

$$y=c-\frac{2c}{a+b}\cdot\frac{a+b}{3}=c-\frac{2}{3}c=\frac{1}{3}c,$$

so that again the median passes through $M$. Therefore $M$ is the point of intersection of all three medians.

It remains to show that $M$ is the centroid of the triangle. The triangle has base $b-a$ and height $c$, so its area is $\frac{1}{2}c(b-a)$. This triangle is composed of two smaller triangles, one in the first quadrant and one in the second quadrant. The first quadrant triangle is bounded above by the line $y=-\frac{c}{b}x+c$ (note that $b\neq 0$), so its moments are

$$M_y=\int_0^b x\left(-\frac{c}{b}x+c\right)\,dx=\int_0^b\left(cx-\frac{c}{b}x^2\right)\,dx=\left[\frac{1}{2}cx^2-\frac{c}{3b}x^3\right]_0^b=\frac{1}{6}cb^2$$

$$M_x=\frac{1}{2}\int_0^b\left(-\frac{c}{b}x+c\right)^2\,dx=\frac{1}{2}\left[-\frac{b}{3c}\left(-\frac{c}{b}x+c\right)^3\right]_0^b=\frac{1}{6}bc^2.$$

The second quadrant triangle is bounded above by the line $y=-\frac{c}{a}x+c$ (note that $a\neq 0$), so its moments are

$$M_y=\int_a^0 x\left(-\frac{c}{a}x+c\right)\,dx=\int_a^0\left(cx-\frac{c}{a}x^2\right)\,dx=\left[\frac{1}{2}cx^2-\frac{c}{3a}x^3\right]_a^0=-\frac{1}{6}ca^2$$

$$M_x=\frac{1}{2}\int_a^0\left(-\frac{c}{a}x+c\right)^2\,dx=\frac{1}{2}\left[-\frac{a}{3c}\left(-\frac{c}{a}x+c\right)^3\right]_a^0=-\frac{1}{6}ac^2.$$

The moments of the entire figure are the sum of the moments of each figure, so the centroid of this triangle is

$$\begin{aligned}(\bar{x},\bar{y})&=\left(\frac{M_y}{M},\ \frac{M_x}{M}\right)\\&=\left(\frac{(1/6)cb^2-(1/6)ca^2}{(1/2)c(b-a)},\ \frac{(1/6)bc^2-(1/6)ac^2}{(1/2)c(b-a)}\right)\\&=\left(\frac{b^2-a^2}{3(b-a)},\ \frac{bc-ac}{3(b-a)}\right)\\&=\left(\frac{1}{3}(a+b),\ \frac{1}{3}c\right).\end{aligned}$$

These are the coordinates of the point $M$ computed above, so that the centroid of a triangle is the intersection of the medians.

38. Suppose the region is defined as shown in the diagram. As in the text, partition the interval $[a,b]$ into $n$ subintervals

$$[a,x_1],\quad [x_1,x_2],\ldots,[x_{n-1},b]\quad a=x_0,\ b=x_n.$$

Each of these has width $\Delta x = \frac{b-a}{n}$. Let $u_i$ be the midpoint of the $i^{\text{th}}$ interval for $i = 1, 2, \ldots, n$. That is,

$$u_i = \frac{x_{i-1} + x_i}{2}.$$

So we have partitioned the lamina into $n$ nonoverlapping regions $R_i$ for $i = 1, \ldots, n$, each of which is roughly rectangular. The centroid of $R_i$ is the point $\left(u_i, \frac{f(u_i)+g(u_i)}{2}\right)$, since that is the center of the rectangle. Further, the height of each rectangle is $f(u_i) - g(u_i)$, and the width is $\Delta x$, so the mass $m_i$ of $R_i$ is

$$m_i = \rho A_i = \rho(f(u_i) - g(u_i))\Delta x.$$

The moment of $R_i$ about the $y$-axis is

$$M_y(R_i) = m_i u_i = \rho u_i (f(u_i) - g(u_i))\Delta x,$$

since the distance of the centroid from the $y$-axis is $u_i$. Adding up these moments over all rectangles gives

$$M_y \approx \rho \sum_{i=1}^{n} u_i (f(u_i) - g(u_i))\Delta x.$$

This is a Riemann sum whose limit as $n \to \infty$ is

$$M_y = \rho \int_a^b x(f(x) - g(x))\, dx.$$

Similarly, the moment of $R_i$ about the $x$-axis is the product of its mass and the distance of its centroid from the $x$-axis:

$$M_x(R_i) = m_i \cdot \frac{1}{2}(f(u_i)+g(u_i)) = \rho(f(u_i)-g(u_i))\Delta x \cdot \frac{1}{2}(f(u_i)+g(u_i)) = \frac{1}{2}\rho(f(u_i)^2 - g(u_i)^2)\Delta x.$$

Adding up these moments over all rectangles gives

$$M_x \approx \frac{1}{2}\rho \sum_{i=1}^{n} (f(u_i)^2 - g(u_i)^2)\Delta x.$$

This is a Riemann sum whose limit as $n \to \infty$ is

$$M_x = \frac{1}{2}\rho \int_a^b (f(x)^2 - g(x)^2)\, dx.$$

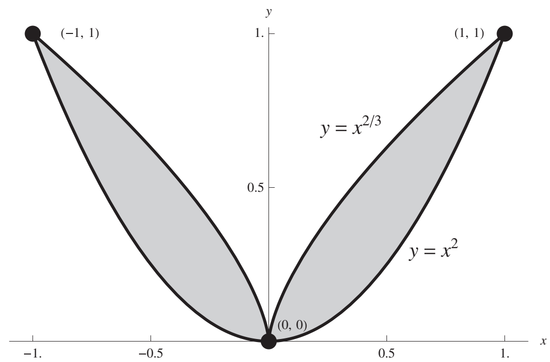
Finally, the mass of the lamina is $\rho$ times the area between the curves, which is

$$M = \rho \int_a^b (f(x) - g(x))\, dx.$$

Therefore the centroid of the lamina is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{\rho \int_a^b x(f(x) - g(x))\, dx}{\rho \int_a^b (f(x) - g(x))\, dx}, \frac{\frac{1}{2}\rho \int_a^b (f(x)^2 - g(x)^2)\, dx}{\rho \int_a^b (f(x) - g(x))\, dx}\right)$$
$$= \left(\frac{\int_a^b x(f(x) - g(x))\, dx}{\int_a^b (f(x) - g(x))\, dx}, \frac{\frac{1}{2}\int_a^b (f(x)^2 - g(x)^2)\, dx}{\int_a^b (f(x) - g(x))\, dx}\right).$$

39. The region is shown below:




The mass of the lamina is

$$M = \rho \int_{-1}^{1} \left(x^{2/3} - x^2\right)\, dx = \rho \left[\frac{3}{5}x^{5/3} - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{8}{15}\rho.$$

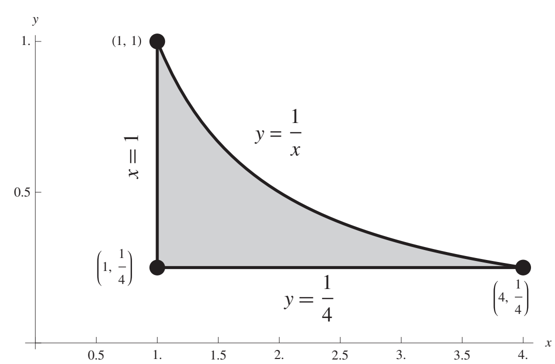
By symmetry, $\bar{x} = 0$, so we need only compute $M_x$ to be able to find $\bar{y}$. From Problem 38, $M_x$ is

$$M_x = \frac{1}{2}\int_{-1}^{1} \left(\left(x^{2/3}\right)^2 - \left(x^2\right)^2\right) \rho\, dx = \frac{1}{2}\rho \int_{-1}^{1} \left(x^{4/3} - x^4\right)\, dx = \frac{1}{2}\rho \left[\frac{3}{7}x^{7/3} - \frac{1}{5}x^5\right]_{-1}^{1} = \frac{8}{35}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(0, \frac{M_x}{M}\right) = \boxed{\left(0, \frac{3}{7}\right)}.$$

40. The region is shown below:




The mass of the lamina is

$$M = \int_1^4 \left(\frac{1}{x} - \frac{1}{4}\right) \rho\, dx = \rho \left[\ln x - \frac{1}{4}x\right]_1^4 = \left(\ln 4 - \frac{3}{4}\right)\rho.$$

From Problem 38, the moments about the two axes are

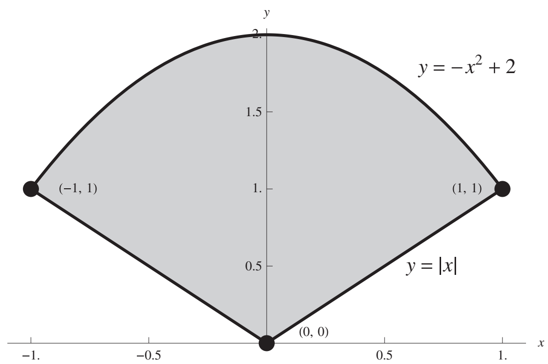
$$M_y = \int_1^4 x\left(\frac{1}{x} - \frac{1}{4}\right)\rho\, dx = \rho \int_1^4 \left(1 - \frac{1}{4}x\right)\, dx = \rho \left[x - \frac{1}{8}x^2\right]_1^4 = \frac{9}{8}\rho$$

$$M_x = \frac{1}{2}\int_1^4 \left(\left(\frac{1}{x}\right)^2 - \left(\frac{1}{4}\right)^2\right)\rho\, dx = \frac{1}{2}\rho \int_1^4 \left(\frac{1}{x^2} - \frac{1}{16}\right)\, dx = \frac{1}{2}\rho \left[-\frac{1}{x} - \frac{1}{16}x\right]_1^4 = \frac{9}{32}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{9}{8\ln 4 - 6}, \frac{9}{32\ln 4 - 24}\right)}.$$

41. The region is shown below:




By symmetry, the area of the region is twice the area of that portion of the region lying in the first quadrant. In the first quadrant, $y = |x|$ is the line $y = x$, so we get for the total mass

$$M = 2\int_0^1 \left((-x^2+2) - x\right)\,dx = 2\int_0^1 \left(-x^2 - x + 2\right)\rho\,dx = 2\rho\left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_0^1 = \frac{7}{3}\rho.$$

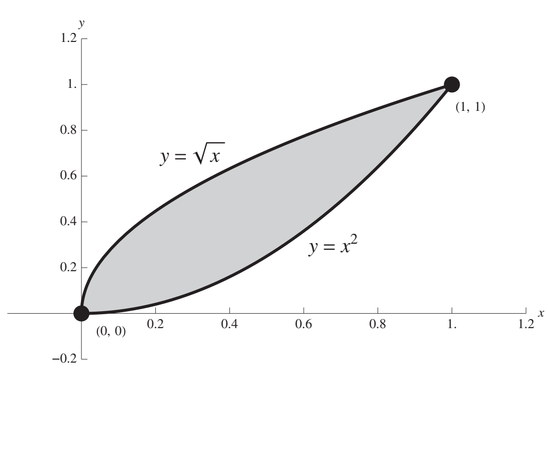
Again by symmetry, $\bar{x} = 0$, so we need only compute $\bar{y}$. Using symmetry yet again, the moment of the first quadrant region and the moment of the second quadrant region about the $x$-axis are equal, so to compute $M_x$, we compute it for the first quadrant region and double it (using the hint preceding Problems 25–30). Therefore the moment of the entire region about the $x$-axis is

$$M_x = 2\cdot\frac{1}{2}\int_0^1 \left((-x^2+2)^2 - x^2\right)\rho\,dx = \rho\int_0^1 \left(x^4 - 5x^2 + 4\right)\,dx = \rho\left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x\right]_0^1 = \frac{38}{15}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(0, \frac{M_x}{M}\right) = \boxed{\left(0, \frac{38}{35}\right)}.$$

42. The region is shown below:



The mass of the region is

$$M = \int_0^1 \left(\sqrt{x} - x^2\right) \rho\, dx = \rho \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1 = \frac{1}{3}\rho.$$

From Problem 38, the moments are

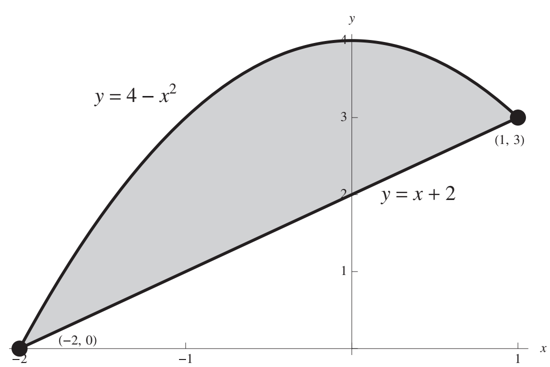
$$M_y = \int_0^1 x\left(\sqrt{x} - x^2\right) \rho\, dx = \int_0^1 \left(x^{3/2} - x^3\right) \rho\, dx = \rho \left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^1 = \frac{3}{20}\rho$$

$$M_x = \frac{1}{2}\int_0^1 \left(\left(\sqrt{x}\right)^2 - \left(x^2\right)^2\right) \rho\, dx = \frac{1}{2}\rho \int_0^1 \left(x - x^4\right)\, dx = \frac{1}{2}\rho \left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1 = \frac{3}{20}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{9}{20}, \frac{9}{20}\right)}.$$

43. The region is shown below:




The mass of the region is

$$M = \int_{-2}^1 \left((4 - x^2) - (x + 2)\right) \rho\, dx = \rho \int_{-2}^1 \left(-x^2 - x + 2\right)\, dx = \rho \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^1 = \frac{9}{2}\rho.$$

By Problem 38, the moments are

$$M_y = \int_{-2}^1 x\left((4 - x^2) - (x + 2)\right) \rho\, dx = \rho \int_{-2}^1 \left(-x^3 - x^2 + 2x\right)\, dx = \rho \left[-\frac{1}{4}x^4 - \frac{1}{3}x^3 + x^2\right]_{-2}^1 = -\frac{9}{4}\rho$$

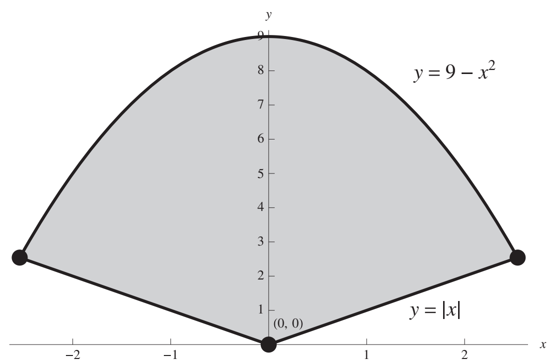
$$M_x = \frac{1}{2}\int_{-2}^1 \left((4 - x^2)^2 - (x + 2)^2\right) \rho\, dx = \frac{1}{2}\rho \int_{-2}^1 \left(x^4 - 9x^2 - 4x + 12\right)\, dx$$

$$= \frac{1}{2}\rho \left[\frac{1}{5}x^5 - 3x^3 - 2x^2 + 12x\right]_{-2}^1 = \frac{54}{5}\rho.$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(-\frac{1}{2}, \frac{12}{5}\right)}.$$

44. The region is shown below:




The point of intersection in the first quadrant is the point where $9 - x^2 = x$, so that $x^2 + x - 9 = 0$. Using the quadratic formula, this gives

$$x = \frac{1}{2}(-1 + \sqrt{37}), \text{ so that the other intersection point is } x = \frac{1}{2}(1 - \sqrt{37}).$$

The mass of the region is twice the mass of the region in the first quadrant (where $|x| = x$), so it is

$$M = 2\int_0^{\frac{1}{2}(-1+\sqrt{37})} (9 - x^2 - x)\rho\,dx = 2\rho\left[9x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right]_0^{\frac{1}{2}(-1+\sqrt{37})} = \frac{1}{6}(37\sqrt{37} - 55)\rho.$$

By symmetry, $\bar{x} = 0$, so we need only compute $\bar{y}$. Using symmetry yet again, the moment of the first quadrant region and the moment of the second quadrant region about the $x$-axis are equal, so to compute $M_x$, we compute it for the first quadrant region and double it (using the hint preceding Problems 25–30). Therefore the moment of the entire region about the $x$-axis is

$$\begin{aligned} M_x = 2 \cdot \frac{1}{2}\int_0^{\frac{1}{2}(-1+\sqrt{37})} \left((9 - x^2)^2 - x^2\right)\rho\,dx &= \rho\int_0^{\frac{1}{2}(-1+\sqrt{37})} \left(x^4 - 19x^2 + 81\right)\,dx \\ &= \rho\left[\frac{1}{5}x^5 - \frac{19}{3}x^3 + 81x\right]_0^{\frac{1}{2}(-1+\sqrt{37})} = \frac{2}{15}(148\sqrt{37} + 23)\rho. \end{aligned}$$

Now,

$$\begin{aligned} \frac{M_x}{M} = \frac{4(148\sqrt{37} + 23)}{5(37\sqrt{37} - 55)} &= \frac{4(148\sqrt{37} + 23)(37\sqrt{37} + 55)}{5(37\sqrt{37} - 55)(37\sqrt{37} + 55)} \\ &= \frac{815508 + 35964\sqrt{37}}{238140} = \frac{1}{245}(37\sqrt{37} + 839). \end{aligned}$$

Therefore the centroid is

$$(\bar{x}, \bar{y}) = \boxed{\left(0, \frac{1}{245}(37\sqrt{37} + 839)\right)}.$$

45. Suppose the region is defined as shown in Figures 69 and 70 in the text. As in the text, partition the interval $[a, b]$ into $n$ subintervals

$$[a, x_1], \quad [x_1, x_2], \ldots, [x_{n-1}, b] \quad a = x_0,\ b = x_n.$$

Each of these has width $\Delta x = \frac{b-a}{n}$. Let $u_i$ be the midpoint of the $i^{\text{th}}$ interval for $i = 1, 2, \ldots, n$. That is,

$$u_i = \frac{x_{i-1} + x_i}{2}.$$

So we have partitioned the lamina into $n$ nonoverlapping regions $R_i$ for $i = 1, \ldots, n$, each of which is roughly rectangular. Within $R_i$, the density of the lamina is roughly constant at $\rho(u_i)$, so the centroid of $R_i$ is the point $\left(u_i, \frac{1}{2}f(u_i)\right)$, since that is the center of $R_i$. Further, the height of each rectangle is $f(u_i)$, and the width is $\Delta x$, so the mass $m_i$ of $R_i$ is

$$m_i = \rho(u_i)A_i = \rho(u_i)f(u_i)\Delta x.$$

The mass of the lamina is therefore approximated by

$$M \approx \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} \rho(u_i)f(u_i)\Delta x.$$

This is a Riemann sum whose limit as $n \to \infty$ is

$$M = \int_a^b \rho(x)f(x)\,dx.$$

The moment of $R_i$ about the $y$-axis is

$$M_y(R_i) = m_i u_i = \rho(u_i)u_i f(u_i)\Delta x,$$

since the distance of the centroid from the $y$-axis is $u_i$. Adding up these moments over all rectangles gives

$$M_y \approx \sum_{i=1}^{n} \rho(u_i)u_i f(u_i)\Delta x.$$

This is a Riemann sum whose limit as $n \to \infty$ is

$$M_y = \int_a^b \rho(x)xf(x)\,dx.$$

Similarly, the moment of $R_i$ about the $x$-axis is the product of its mass and the distance of its centroid from the $x$-axis:

$$M_x(R_i) = m_i \cdot \frac{1}{2}f(u_i) = \rho(u_i)f(u_i)\Delta x \cdot \frac{1}{2}f(u_i) = \frac{1}{2}\rho(u_i)f(u_i)^2\Delta x.$$

Adding up these moments over all rectangles gives

$$M_x \approx \frac{1}{2}\sum_{i=1}^{n} \rho(u_i)f(u_i)^2\Delta x.$$

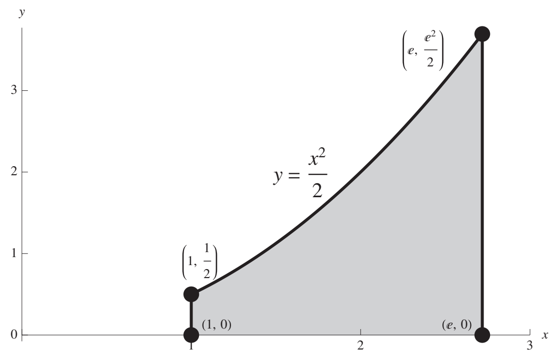
This is a Riemann sum whose limit as $n \to \infty$ is

$$M_x = \frac{1}{2}\int_a^b \rho(x)f(x)^2\,dx.$$

Therefore the centroid of the lamina is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{\int_a^b \rho(x)xf(x)\,dx}{\int_a^b \rho(x)f(x)\,dx}, \frac{\frac{1}{2}\int_a^b \rho(x)f(x)^2\,dx}{\int_a^b \rho(x)f(x)\,dx}\right).$$

46. The region is shown below:




From Problem 45, the mass of the lamina is

$$M = \int_a^b \rho(x) f(x)\,dx = \int_1^e \frac{1}{x^3} \cdot \frac{1}{2}x^2\,dx = \frac{1}{2}\int_1^e \frac{1}{x}\,dx = \frac{1}{2}\left[\ln x\right]_1^e = \frac{1}{2}.$$

The moments about the two axes are

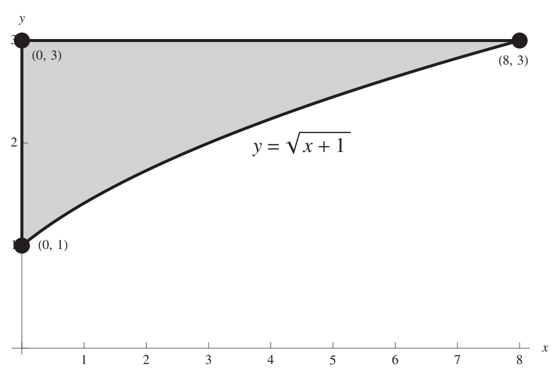
$$M_y = \int_a^b \rho(x) x f(x)\,dx = \int_1^e \frac{1}{x^3} \cdot x \cdot \frac{1}{2}x^2\,dx = \frac{1}{2}\int_1^e 1\,dx = \frac{1}{2}(e-1)$$

$$M_x = \frac{1}{2}\int_a^b \rho(x) f(x)^2\,dx = \frac{1}{2}\int_1^e \frac{1}{x^3} \cdot \frac{1}{4}x^4\,dx = \frac{1}{8}\int_1^e x\,dx = \frac{1}{8}\left[\frac{1}{2}x^2\right]_1^e = \frac{1}{16}(e^2-1).$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(e-1, \frac{1}{8}(e^2-1)\right)}.$$

47. The region is shown below:



From Problems 38 and 45, the mass of the lamina is

$$\begin{aligned} M &= \int_0^8 \rho(x)(3-\sqrt{x+1})\,dx \\ &= \int_0^8 \left(3x - x\sqrt{x+1}\right)\,dx \\ &= \int_0^8 \left(3x - (x+1)\sqrt{x+1} + \sqrt{x+1}\right)\,dx \\ &= \int_0^8 \left(3x - (x+1)^{3/2} + (x+1)^{1/2}\right)\,dx \\ &= \left[\frac{3}{2}x^2 - \frac{2}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2}\right]_0^8 \\ &= \frac{248}{15}. \end{aligned}$$

The moments about the two axes are

$$\begin{aligned} M_y &= \int_0^8 \rho(x)x(3-\sqrt{x+1})\,dx \\ &= \int_0^8 \left(3x^2 - x^2\sqrt{x+1}\right)\,dx \\ &= \int_0^8 \left(3x^2 - (x^2+2x+1)\sqrt{x+1} + (2x+1)\sqrt{x+1}\right)\,dx \\ &= \int_0^8 \left(3x^2 - (x+1)^{5/2} + 2(x+1)\sqrt{x+1} - \sqrt{x+1}\right)\,dx \\ &= \int_0^8 \left(3x^2 - (x+1)^{5/2} + 2(x+1)^{3/2} - (x+1)^{1/2}\right)\,dx \\ &= \left[x^3 - \frac{2}{7}(x+1)^{7/2} + \frac{4}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2}\right]_0^8 \\ &= \frac{6688}{105} \end{aligned}$$

and

$$M_x = \frac{1}{2}\int_0^8 \rho(x)\left(3^2 - \left(\sqrt{x+1}\right)^2\right)\,dx = \int_0^8 x(8-x)\,dx = \int_0^8 (8x-x^2)\,dx = \left[4x^2 - \frac{1}{3}x^3\right]_0^8 = \frac{128}{3}.$$

Therefore the center of mass is

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{836}{217}, \frac{80}{31}\right)}.$$

48. From Problem 45, the mass of the lamina is

$$M = \int_0^3 \rho(x)f(x)\,dx = \int_0^3 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^3 = 9.$$

The moments about the two axes are

$$\begin{aligned} M_y &= \int_0^3 \rho(x)xf(x)\,dx = \int_0^3 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^3 = \frac{81}{4} \\ M_x &= \frac{1}{2}\int_0^3 \rho(x)f(x)^2\,dx = \int_0^3 x^3\,dx = \frac{1}{2}\left[\frac{1}{4}x^4\right]_0^3 = \frac{81}{8}. \end{aligned}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{9}{4}, \frac{9}{8}\right)}.$$

49. From Problem 45, the mass of the lamina is

$$M = \int_1^4 \rho(x)f(x)\,dx = \int_1^4 x(3x-1)\,dx = \int_1^4 (3x^2 - x)\,dx = \left[x^3 - \frac{1}{2}x^2\right]_1^4 = \frac{111}{2}.$$

The moments about the two axes are

$$M_y = \int_1^4 \rho(x)xf(x)\,dx = \int_1^4 x^2(3x-1)\,dx = \int_1^4 (3x^3 - x^2)\,dx = \left[\frac{3}{4}x^4 - \frac{1}{3}x^3\right]_1^4 = \frac{681}{4}$$

$$M_x = \frac{1}{2}\int_1^4 \rho(x)f(x)^2\,dx = \int_1^4 x(3x-1)^2\,dx = \frac{1}{2}\int_1^4 \left(9x^3 - 6x^2 + x\right)\,dx$$

$$= \frac{1}{2}\left[\frac{9}{4}x^4 - 2x^3 + \frac{1}{2}x^2\right]_1^4 = \frac{1821}{8}.$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{227}{74}, \frac{607}{148}\right)}.$$

50. From Problem 45, the mass of the lamina is

$$M = \int_0^1 \rho(x)f(x)\,dx = \int_0^1 (x+1)\cdot 2x\,dx = \int_0^1 \left(2x^2 + 2x\right)\,dx = \left[\frac{2}{3}x^3 + x^2\right]_0^1 = \frac{5}{3}.$$

The moments about the two axes are

$$M_y = \int_0^1 \rho(x)xf(x)\,dx = \int_0^1 (x+1)\cdot x\cdot 2x\,dx = \int_0^1 \left(2x^3 + 2x^2\right)\,dx = \left[\frac{1}{2}x^4 + \frac{2}{3}x^3\right]_0^1 = \frac{7}{6}$$

$$M_x = \frac{1}{2}\int_0^1 \rho(x)f(x)^2\,dx = \frac{1}{2}\int_0^1 (x+1)(2x)^2\,dx = \frac{1}{2}\int_0^1 (4x^3 + 4x^2)\,dx = \frac{1}{2}\left[x^4 + \frac{4}{3}x^3\right]_0^1 = \frac{7}{6}.$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{7}{10}, \frac{7}{10}\right)}.$$

51. From Problem 45, the mass of the lamina is

$$M = \int_1^4 \rho(x)f(x)\,dx = \int_1^4 (x+1)x\,dx = \int_1^4 (x^2 + x)\,dx = \left[\frac{1}{3}x^3 + \frac{1}{2}x^2\right]_1^4 = \frac{57}{2}.$$

The moments about the two axes are

$$M_y = \int_1^4 \rho(x)xf(x)\,dx = \int_1^4 (x+1)\cdot x\cdot x\,dx = \int_1^4 \left(x^3 + x^2\right)\,dx = \left[\frac{1}{4}x^4 + \frac{1}{3}x^3\right]_1^4 = \frac{339}{4}$$

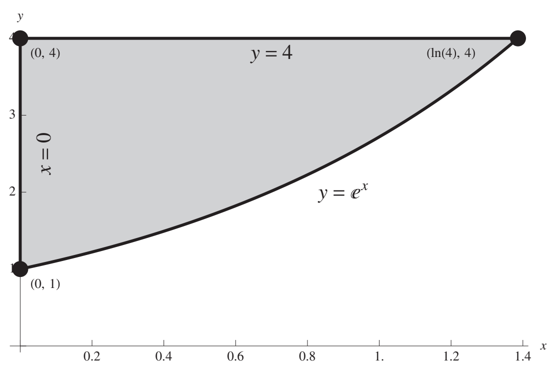
$$M_x = \frac{1}{2}\int_1^4 \rho(x)f(x)^2\,dx = \frac{1}{2}\int_1^4 (x+1)\cdot x^2\,dx = \frac{1}{2}\int_1^4 (x^3 + x^2)\,dx = \frac{1}{2}\left[\frac{1}{4}x^4 + \frac{1}{3}x^3\right]_1^4 = \frac{339}{8}.$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{113}{38}, \frac{113}{76}\right)}.$$

# Chapter 6 Review Exercises

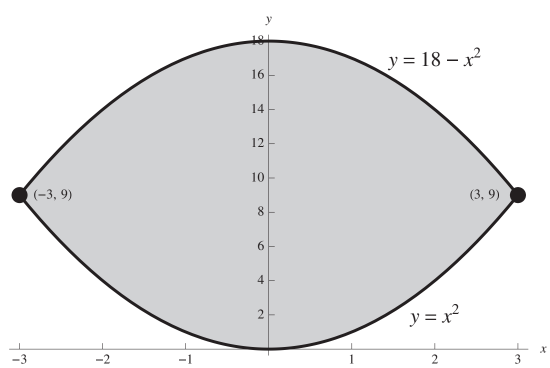
1. The region is shown below:




Partitioning along the $x$-axis, we have $e^x \leq 4$, so the area is

$$\int_0^{\ln 4} (4 - e^x)\, dx = [4x - e^x]_0^{\ln 4} = \boxed{4 \ln 4 - 3}.$$

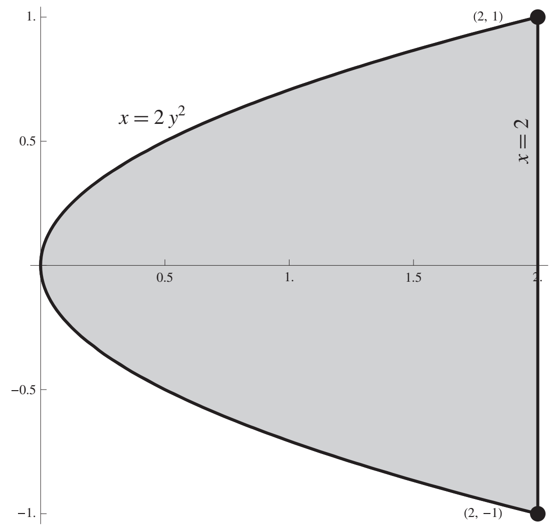
2. The region is shown below:




Partitioning along the $x$-axis, we have $x^2 \leq 18 - x^2$, so the area is

$$\int_{-3}^{3} \left(18 - x^2 - x^2\right)\, dx = \int_{-3}^{3} \left(18 - 2x^2\right)\, dx = \left[18x - \frac{2}{3}x^3\right]_{-3}^{3} = \boxed{72}.$$

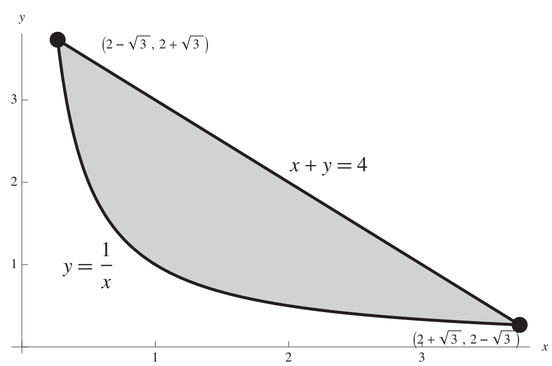
3. The region is shown below:




Since we are given $x$ in terms of $y$, it is more convenient to partition along the $y$-axis. There, we have $2y^2 \le 2$, so the area is

$$\int_{-1}^{1} \left(2 - 2y^2\right)\, dx = \left[2y - \frac{2}{3}y^3\right]_{-1}^{1} = \boxed{\frac{8}{3}}.$$

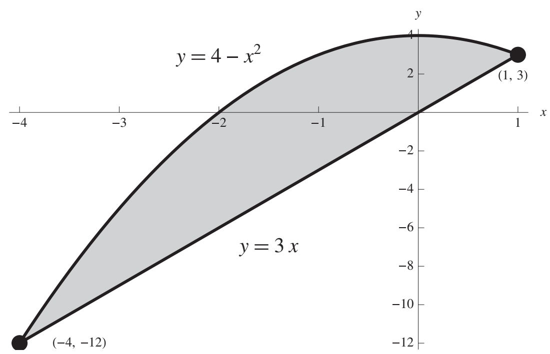
4. The region is shown below:




The equation $x + y = 4$ is the same as $y = 4 - x$. The graphs of $\frac{1}{x}$ and $4 - x$ meet when $\frac{1}{x} = 4 - x$, so when $4x - x^2 = 1$. This gives the quadratic $x^2 - 4x + 1 = 0$, which has roots $2 \pm \sqrt{3}$. Partitioning along the $x$-axis, we have $\frac{1}{x} \le 4 - x$, so the area is

$$\begin{aligned}\int_{2-\sqrt{3}}^{2+\sqrt{3}} \left(4 - x - \frac{1}{x}\right)\, dx &= \left[4x - \frac{1}{2}x^2 - \ln x\right]_{2-\sqrt{3}}^{2+\sqrt{3}} \\ &= \left(8 + 4\sqrt{3} - \left(\frac{7}{2} + 2\sqrt{3}\right) - \ln(2+\sqrt{3})\right) \\ &\quad - \left(8 - 4\sqrt{3} - \left(\frac{7}{2} - 2\sqrt{3}\right) - \ln(2-\sqrt{3})\right) \\ &= 4\sqrt{3} + \ln\frac{2-\sqrt{3}}{2+\sqrt{3}} = \boxed{4\sqrt{3} + \ln(7 - 4\sqrt{3})}.\end{aligned}$$

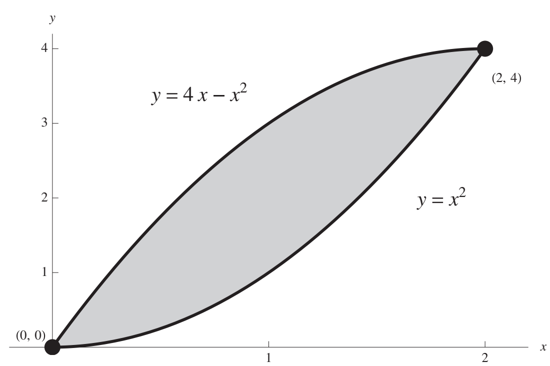
5. The region is shown below:




Partitioning along the $x$-axis, we have $4 - x^2 \geq 3x$, so the area is

$$\int_{-4}^{1} (4 - x^2 - 3x)\, dx = \left[4x - \frac{1}{3}x^3 - \frac{3}{2}x^2\right]_{-4}^{1} = \boxed{\frac{125}{6}}.$$

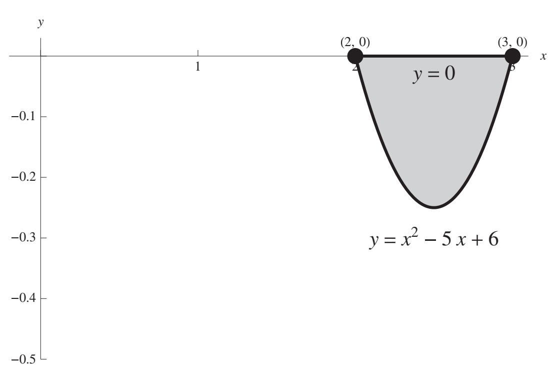
6. The region is shown below:




We use the washer method along the $x$-axis. Each outer radius is $4x - x^2$ and each inner radius is $x^2$, so the volume is

$$V = \pi \int_0^2 \left(\left(4x - x^2\right)^2 - \left(x^2\right)^2\right)\, dx = \pi \int_0^2 \left(16x^2 - 8x^3\right)\, dx = \pi\left[\frac{16}{3}x^3 - 2x^4\right]_0^2 = \boxed{\frac{32}{3}\pi}.$$

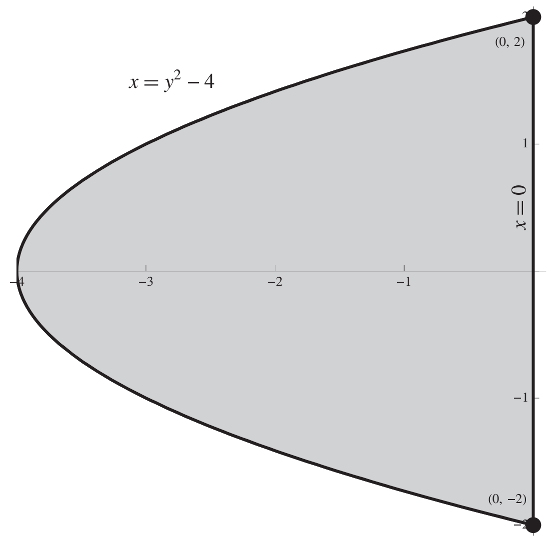
7. The region is shown below:



We use the shell method along the $x$-axis. Each height is $-x^2+5x-6$, and each radius is $x$, so the volume is

$$V = 2\pi \int_2^3 x\left(-x^2+5x-6\right)\,dx = 2\pi \int_2^3 \left(-x^3+5x^2-6x\right)\,dx = 2\pi\left[-\frac{1}{4}x^4+\frac{5}{3}x^3-3x^2\right]_2^3 = \boxed{\frac{5}{6}\pi}.$$

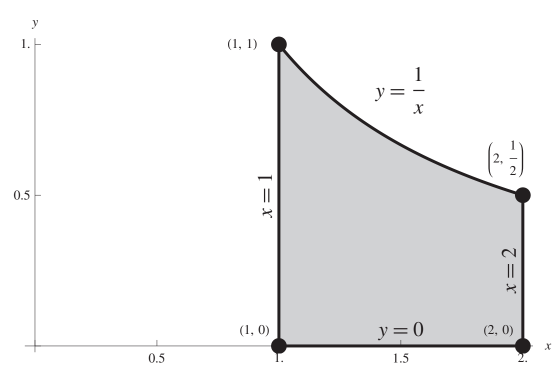
8. The region is shown below:




Since we are given equations expressing $x$ in terms of $y$, and we are revolving about the $y$-axis, it is easiest to use the disk method along the $y$-axis. each radius is $4-y^2$, so the volume is

$$V = \pi \int_{-2}^{2} \left(4-y^2\right)^2\,dy = \pi \int_{-2}^{2} \left(y^4-8y^2+16\right)\,dy = \pi\left[\frac{1}{5}y^5-\frac{8}{3}y^3+16y\right]_{-2}^{2} = \boxed{\frac{512}{15}\pi}.$$

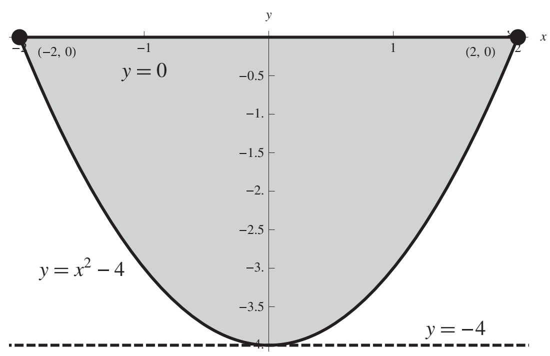
9. The region is shown below:




We use the disk method along the $x$-axis. Each radius is $\frac{1}{x}$, so the volume is

$$V = \pi \int_1^2 \left(\frac{1}{x}\right)^2\,dx = \pi \int_1^2 \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_1^2 = \boxed{\frac{1}{2}\pi}.$$

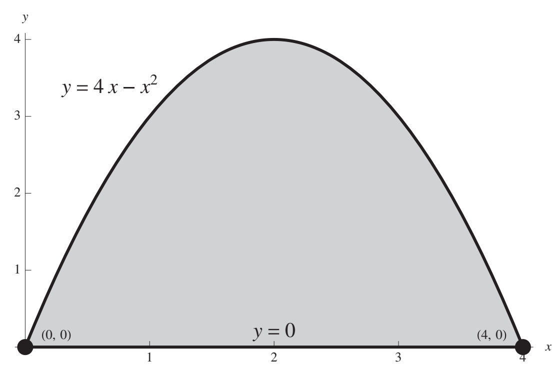
10. The region is shown below:




We use the washer method in the $x$ direction. Each outer radius is $0-(-4)=4$, and each inner radius is $x^2-4-(-4)=x^2$. Therefore the volume is

$$V=\pi\int_{-2}^{2}\left(4^2-\left(x^2\right)^2\right)\,dx=\pi\int_{-2}^{2}\left(16-x^4\right)\,dx=\pi\left[16x-\frac{1}{5}x^5\right]_{-2}^{2}=\boxed{\frac{256}{5}\pi}.$$

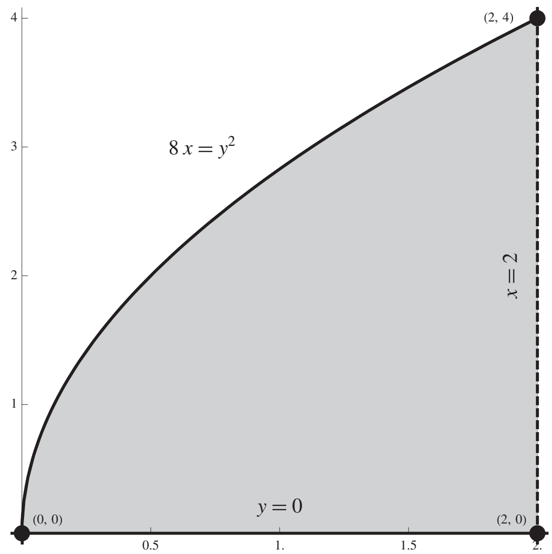
11. The region is shown below:




We use the disk method along the $x$-axis. Each radius is $4x-x^2$, so the volume is

$$V=\pi\int_{0}^{4}\left(4x-x^2\right)^2\,dx=\pi\int_{0}^{4}\left(16x^2-8x^3+x^4\right)\,dx=\pi\left[\frac{16}{3}x^3-2x^4+\frac{1}{5}x^5\right]_{0}^{4}=\boxed{\frac{512}{15}\pi}.$$

12. The region is shown below:

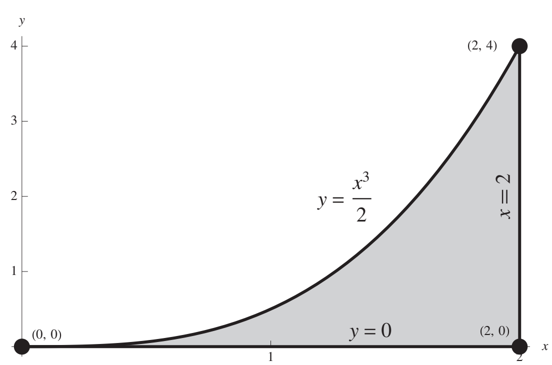



We use the shell method in the $x$ direction, so we must first solve for $y$, giving $y = \sqrt{8x}$ (note that from the diagram, we take the positive square root). Revolving about $x = 2$, each radius is $2 - x$, and each height is $\sqrt{8x}$, so that the volume is

$$V = 2\pi \int_0^2 (2-x)\sqrt{8x}\, dx = 4\pi\sqrt{2} \int_0^2 \left(2x^{1/2} - x^{3/2}\right)\, dx$$

$$= 4\pi\sqrt{2} \left[\frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^2 = 4\pi\sqrt{2}\left(\frac{8}{3}\sqrt{2} - \frac{8}{5}\sqrt{2}\right) = \boxed{\frac{128}{15}\pi}.$$

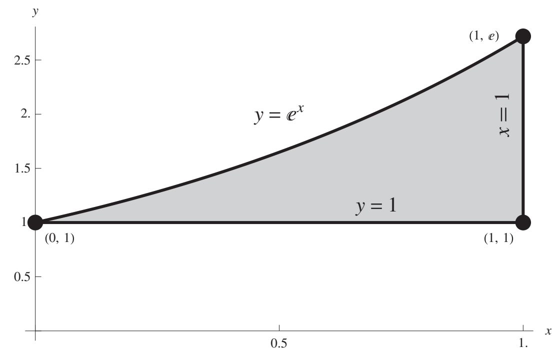
13. The region is shown below:




It is easiest to use the shell method along the $x$-axis. Each height is $\frac{x^3}{2}$, and each radius is $x$, so the volume is

$$V = 2\pi \int_0^2 x \cdot \frac{x^3}{2}\, dx = \pi \int_0^2 x^4\, dx = \pi \left[\frac{1}{5}x^5\right]_0^2 = \boxed{\frac{32}{5}\pi}.$$

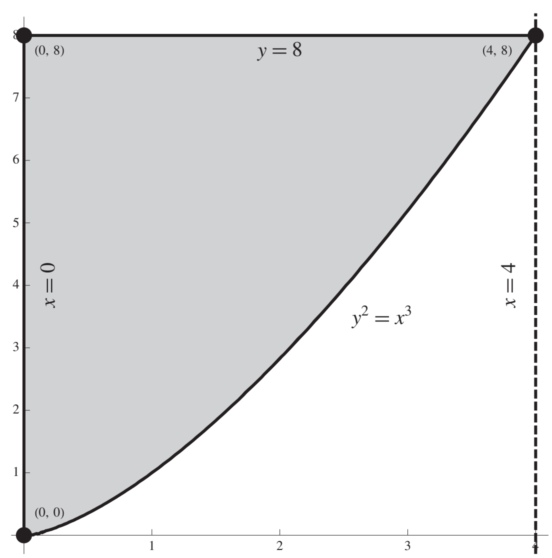
14. The region is shown below:




Use the washer method along the $x$-axis. Each outer radius is $e^x$ and each inner radius is 1, so the volume is

$$V = \pi \int_0^1 \left( (e^x)^2 - 1^2 \right) dx = \pi \int_0^1 \left( e^{2x} - 1 \right) dx = \pi \left[ \frac{1}{2} e^{2x} - x \right]_0^1 = \boxed{\frac{1}{2}\pi \left( e^2 - 3 \right)}.$$

15. The region is shown below:




To revolve about $x = 4$, we could use either shells along the $x$-axis or washers in the $y$ direction. We choose to use shells. Solving $y^2 = x^3$ for $y$ gives $y = x^{3/2}$ (we choose the positive square root since this is a first quadrant region). Then each height is $8 - x^{3/2}$, and each radius is $4 - x$. Therefore the volume is

$$V = 2\pi \int_0^4 (4 - x) \left( 8 - x^{3/2} \right) dx = 2\pi \int_0^4 \left( 32 - 8x - 4x^{3/2} + x^{5/2} \right) dx$$

$$= 2\pi \left[ 32x - 4x^2 - \frac{8}{5} x^{5/2} + \frac{2}{7} x^{7/2} \right]_0^4 = \boxed{\frac{3456}{35}\pi}.$$

16. Since $y' = \frac{3}{2} x^{1/2}$, the arc length is

$$L = \int_2^5 \sqrt{(y')^2 + 1}\, dx = \int_2^5 \sqrt{\frac{9}{4} x + 1}\, dx = \frac{1}{2} \int_2^5 \sqrt{9x + 4}\, dx.$$

Now use the substitution $u = 9x + 4$, so that $du = 9\,dx$; then $x = 2$ corresponds to $u = 22$ and $x = 5$ to $u = 49$, so that we get

$$L = \frac{1}{2}\int_2^5 \sqrt{9x+4}\,dx = \frac{1}{18}\int_{22}^{49} u^{1/2}\,du = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{22}^{49} = \boxed{\frac{1}{27}\left(343 - 22\sqrt{22}\right)}.$$

17. We have $y' = \frac{x^2}{2} - \frac{1}{2x^2}$, so that

$$(y')^2 + 1 = \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2 + 1 = \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^4} + 1 = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2.$$

Then the arc length is

$$L = \int_2^6 \sqrt{(y')^2 + 1}\,dy = \int_2^6 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_2^6 = \boxed{\frac{209}{6}}.$$

18. Partition along the $y$-axis. Then $x = g(y) = \sqrt{2}\,y^{3/2}$, so that $g'(y) = \frac{3}{\sqrt{2}}\,y^{1/2}$. Then the arc length is

$$\begin{aligned} L &= \int_0^2 \sqrt{\left(g'(y)\right)^2 + 1}\,dy \\ &= \int_0^2 \sqrt{\frac{9}{2}x + 1}\,dy \\ &= \frac{1}{\sqrt{2}}\int_0^2 \sqrt{9x+2}\,dy \\ &= \frac{1}{\sqrt{2}}\left[\frac{2}{27}(9x+2)^{3/2}\right]_0^2 \\ &= \frac{\sqrt{2}}{27}\left(20\sqrt{20} - 2\sqrt{2}\right) \\ &= \boxed{\frac{4}{27}(10\sqrt{10} - 1)}. \end{aligned}$$

19. The center of mass is

$$\bar{x} = \frac{\sum_{i=1}^4 m_i x_i}{\sum_{i=1}^4 m_i} = \frac{1\cdot(-1) + 3\cdot 2 + 8\cdot 14 + 1\cdot 0}{1+3+8+1} = \boxed{9}.$$

20. The moments are

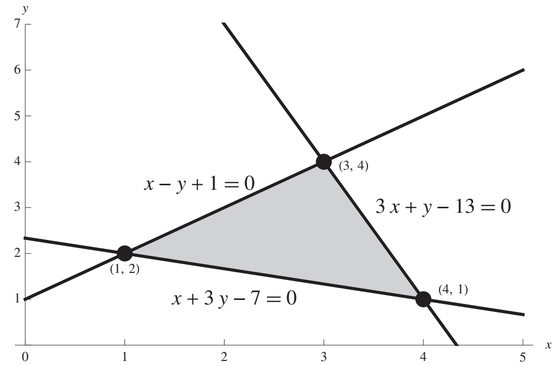
$$M_y = \sum_{i=1}^4 m_i x_i = 2\cdot(-4) + 2\cdot 2 + 3\cdot 4 + 2\cdot(-3) = 2$$

$$M_x = \sum_{i=1}^4 m_i y_i = 2\cdot 4 + 2\cdot 3 + 3\cdot 4 + 2\cdot(-5) = 16$$

and the mass is $M = 2 + 2 + 3 + 2 = 9$, so the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{2}{9}, \frac{16}{9}\right)}.$$

21. The region is shown below:




The lines $x - y + 1 = 0$ and $3x + y - 13 = 0$ intersect when

$$y = x + 1 = 13 - 3x, \text{ so that } 4x = 12, \text{ or } x = 3, \text{ which is the point } (3, 4).$$

The lines $x - y + 1 = 0$ and $x + 3y - 7 = 0$ intersect when

$$x = y - 1 = 7 - 3y, \text{ so that } 4y = 8, \text{ or } y = 2, \text{ which is the point } (1, 2).$$

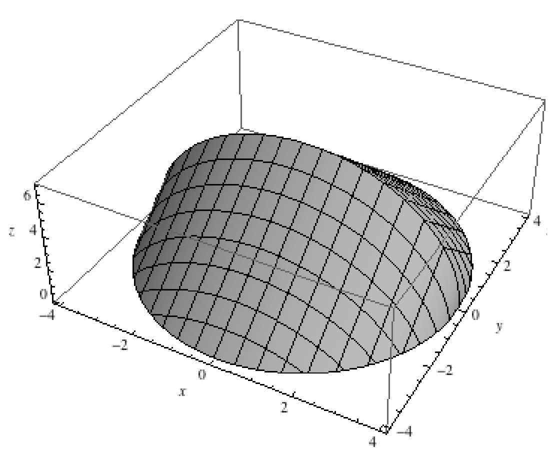
The lines $3x + y - 13 = 0$ and $x + 3y - 7 = 0$ intersect when

$$y = 13 - 3x = \frac{7}{3} - \frac{1}{3}x, \text{ so that } \frac{8}{3}x = \frac{32}{3}, \text{ or } x = 4, \text{ which is the point } (4, 1).$$

We partition along the $x$-axis, so we end up with two integrals, one from $x = 1$ to $x = 3$ and one from $x = 3$ to $x = 4$. The first has lower bound $y = \frac{7}{3} - \frac{1}{3}x$ and upper bound $y = x + 1$; the second has the same lower bound but has upper bound $y = 13 - 3x$. So the area is

$$\begin{aligned}
A &= \int_1^3 \left( (x+1) - \left( \frac{7}{3} - \frac{1}{3}x \right) \right) dx + \int_3^4 \left( (13 - 3x) - \left( \frac{7}{3} - \frac{1}{3}x \right) \right) dx \\
&= \int_1^3 \left( \frac{4}{3}x - \frac{4}{3} \right) dx + \int_3^4 \left( -\frac{8}{3}x + \frac{32}{3} \right) dx \\
&= \left[ \frac{2}{3}x^2 - \frac{4}{3}x \right]_1^3 + \left[ -\frac{4}{3}x^2 + \frac{32}{3}x \right]_3^4 \\
&= \frac{8}{3} + \frac{4}{3} = \boxed{4}.
\end{aligned}$$

22. The region is shown below:



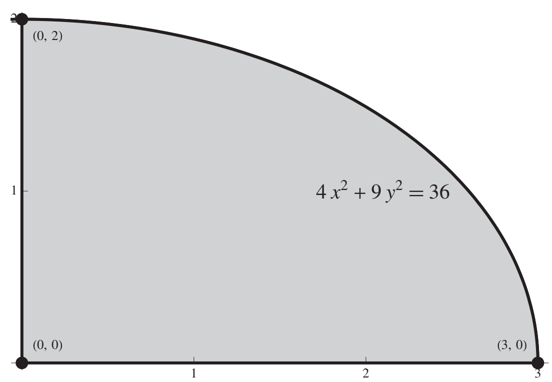
An equilateral triangle with side $a$ has height $a\sin 60^\circ = \frac{\sqrt{3}}{2}a$, so its area is $\frac{1}{2}\cdot a\cdot\frac{\sqrt{3}}{2}x = \frac{\sqrt{3}}{4}a^2$. With the fixed diameter as shown, along the $x$-axis, centered at the origin, the side length of the equilateral triangle at a given value of $x$ is $2\sqrt{16-x^2}$, so the area of the triangle is

$$\frac{\sqrt{3}}{4}\cdot\left(2\sqrt{16-x^2}\right)^2 = (16-x^2)\sqrt{3}.$$

Therefore the volume of the solid is

$$V = \int_{-4}^{4}(16-x^2)\sqrt{3}\,dx = \sqrt{3}\left[16x - \frac{1}{3}x^3\right]_{-4}^{4} = \boxed{\frac{256}{3}\sqrt{3}}.$$

23. The region is shown below:



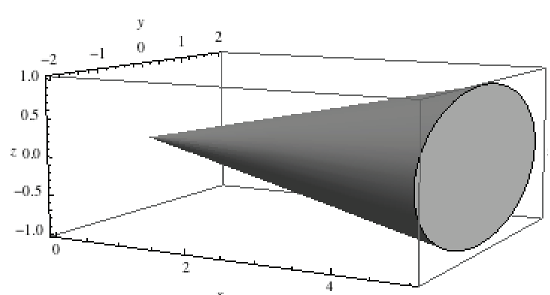

Using the shell method along the $y$-axis, we solve for $x$, giving $x = \sqrt{9-\frac{9}{4}y^2}$. Then the volume is

$$V = 2\pi\int_0^2 y\sqrt{9-\frac{9}{4}y^2}\,dy = \pi\int_0^2 y\sqrt{36-9y^2}\,dy.$$

Now use the substitution $u = 36-9y^2$, so that $du = -18y\,dy$. Then $y=0$ corresponds to $u=36$, and $y=2$ to $u=0$, and the integral becomes

$$V = \pi\int_0^2 y\sqrt{36-9y^2}\,dy = \pi\cdot\left(-\frac{1}{18}\right)\int_{36}^{0}u^{1/2}\,du = -\frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_{36}^{0} = \boxed{8\pi}.$$

24. The region is shown below:



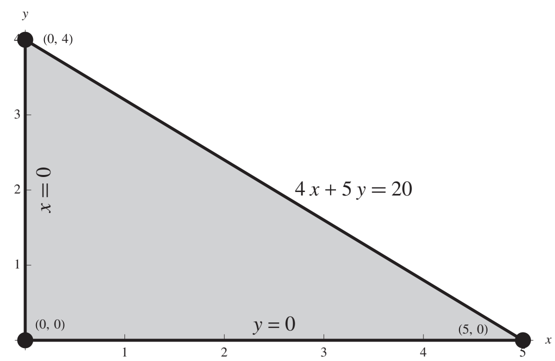

Place the origin at the tip of the cone, with the positive $x$-axis pointing in the direction of the base, so that the base of the cone is an ellipse with semi axes 2 and 1 at $x=5$. The cross section at each value of $x$ is an ellipse, and since both axes increase linearly, one from 0 to 2 and the other from 0 to 1 as $x$ increases from 0 to 5, it follows that for a given value of $x$, the cross section at $x$ is an ellipse with semi axes $2 \cdot \frac{x}{5} = \frac{2x}{5}$ and $\frac{x}{5}$. We must integrate over the area of these ellipses. Now by the hint, the area of each ellipse is

$$\pi \cdot \left(\frac{2x}{5}\right) \cdot \left(\frac{x}{5}\right) = \frac{2}{25}\pi x^2$$

so the volume is

$$V = \int_0^5 \left(\frac{2}{25}\pi x^2\right)\, dx = \frac{2}{25}\pi \left[\frac{1}{3}x^3\right]_0^5 = \boxed{\frac{10}{3}\pi}.$$

25. The base is shown below:




Solving $4x + 5y = 20$ for $y$ gives $y = 4 - \frac{4}{5}x$, so for a given value of $x$, the area of the semicircular cross section is

$$\frac{1}{2}\pi\left(\frac{d}{2}\right)^2 = \frac{1}{2}\pi\left(\frac{4 - \frac{4}{5}x}{2}\right)^2 = \frac{1}{8}\pi\left(16 - \frac{32}{5}x + \frac{16}{25}x^2\right).$$

Therefore the volume of the solid is

$$V = \int_0^5 \frac{1}{8}\pi\left(16 - \frac{32}{5}x + \frac{16}{25}x^2\right)\, dx = \frac{1}{8}\pi\left[16x - \frac{16}{5}x^2 + \frac{16}{75}x^3\right]_0^5 = \boxed{\frac{10}{3}\pi}.$$

26. We have $y' = x^{1/2}$, which is continuous for $x > 0$. So using the Arc Length Formula, the arc length from 0 to $k$ is

$$L = \int_0^k \sqrt{(y')^2 + 1}\, dx = \int_0^k \sqrt{x + 1}\, dx = \left[\frac{2}{3}(x + 1)^{3/2}\right]_0^k = \frac{2}{3}\left((k + 1)^{3/2} - 1\right).$$

We want this length to be $\frac{52}{3}$, so setting the two equal and multiplying through by 3 gives

$$2((k + 1)^{3/2} - 1) = 52, \text{ or } (k + 1)^{3/2} - 1 = 26.$$

Add 1 to both sides and simplify to get $k = 8$. Therefore the point $P$ is $\left(8, \frac{2}{3} \cdot 8^{3/2}\right) = \boxed{\left(8, \frac{32}{3}\sqrt{2}\right)}$.

27. The anchor must be lifted 150 ft; since it weighs 800 lb, the work involved in lifting the anchor itself is $800 \cdot 150 = 120000$ ft-lbs. The remainder of the work expended is for the chain itself. The weight density of the chain is 20 lb/ft, and the portion of the chain $x$ ft below the boat must be lifted $x$ ft. Therefore the total work expended in lifting the chain is

$$\int_0^{150} 20x\, dx = \left[10x^2\right]_0^{150} = 225000 \text{ ft-lbs}.$$

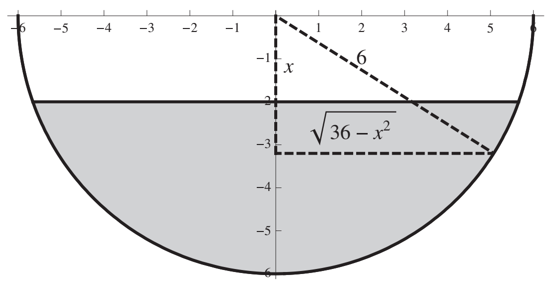
So the work required to lift the anchor and chain is $120000 + 225000 = \boxed{345000 \text{ ft-lbs}}$.

28. The container must be lifted 1200 m; since its mass is 1000 kg, its weight is 9800 N. Then the work involved in lifting the container itself is $1200 \cdot 9800 = 1.176 \times 10^7$ J. The remainder of the work expended is for the cable itself. The density of the cable is 3 kg/m, and the portion of the cable $x$ ft below the surface must be lifted $x$ ft. Therefore the total work expended in lifting the cable is

$$9.8 \int_0^{1200} 3x\,dx = 9.8 \left[\frac{3}{2}x^2\right]_0^{1200} = 2.1168 \times 10^7 \text{ J}.$$

So the work required to lift the container and the cable is $1.176 \times 10^7 + 2.1168 \times 10^7 = \boxed{3.2928 \times 10^7 \text{ J}}$.

29. To determine the radius of the cross section $x$ m below the top of the bowl, consider the following diagram, which shows a vertical cross section through the center of the bowl:




At a height $x$ m below the top, the radius of the cross section is $\sqrt{36 - x^2}$, so that the area of the cross section at that height is $(36 - x^2)\pi$. That cross section must be lifted $x$ m, so the work performed is

$$9.8 \int_2^6 1000x(36 - x^2)\pi\,dx = 9800\pi \int_2^6 (36h - x^3)\,dx = 9800\pi \left[18x^2 - \frac{1}{4}x^4\right]_2^6 = \boxed{2.5088 \times 10^6 \pi \text{ J}}.$$

30. To determine $k$, note that a displacement of 0.2 m requires 4 N, so that $-4 = k \cdot 0.2$, or $k = -20$. So the equation of the spring is $F = -20x$. Then stretching it to 1.4 m, or 0.8 m from its equilibrium point, takes

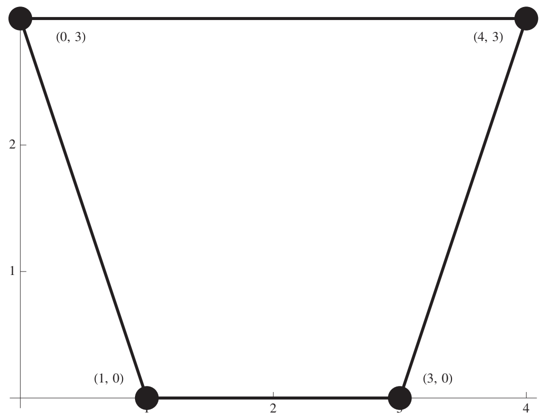
$$\int_0^{0.8} (-20x)\,dx = \left[-10x^2\right]_0^{0.8} = \boxed{-6.4 \text{ J}}.$$

31. Let the equilibrium length of the spring be $s$, and the spring constant be $k$. Then

$$\begin{aligned}
\int_{1-s}^{1.4-s} (-kx)\,dx &= \frac{1}{2}\int_{1.2-s}^{1.8-s} (-kx)\,dx \\
\left[-\frac{1}{2}kx^2\right]_{1-s}^{1.4-s} &= \frac{1}{2}\left[-\frac{1}{2}kx^2\right]_{1.2-s}^{1.8-s} \\
\left[x^2\right]_{1-s}^{1.4-s} &= \frac{1}{2}\left[x^2\right]_{1.2-s}^{1.8-s} \\
2((1.4 - s)^2 - (1 - s)^2) &= (1.8 - s)^2 - (1.2 - s)^2 \\
-1.6s + 1.92 &= -1.2s + 1.8 \\
0.4s &= 0.12 \\
s &= 0.3.
\end{aligned}$$

The equilibrium length of the spring is $\boxed{0.3 \text{ m}}$.

32. Place the trapezoid on a Cartesian grid:




Then the equations of the two diagonal lines are:

$$y - 0 = \frac{3-0}{0-1}(x-1), \text{ or } y = -3x+3, \text{ or } x = -\frac{1}{3}y+1$$
$$y - 0 = \frac{3-0}{4-3}(x-3), \text{ or } y = 3x-9, \text{ or } x = \frac{1}{3}y+3.$$

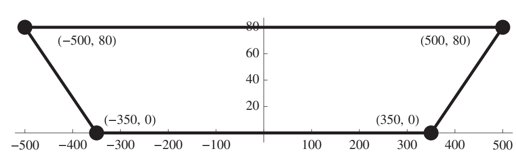
So at any height $y$, the height of the water above a slice at that point is $3-y$, and the width of the slice is $\left(\frac{1}{3}y+3\right) - \left(-\frac{1}{3}y+1\right) = \frac{2}{3}y+2$. For water, $\rho g = 62.5$, so the force on the end of the trough is

$$\int_0^3 \rho g(3-y)\left(\frac{2}{3}y+2\right)\,dy = \rho g\int_0^3\left(-\frac{2}{3}y^2+6\right)\,dy = \rho g\left[-\frac{2}{9}y^3+6y\right]_0^3 = 12{\cdot}62.5 = \boxed{750 \text{ lbs}}.$$

33. Since the tank's radius is 5 m, it is half-full. Position the $y$-axis at the center of one end of the tank, with the positive $y$-axis pointing down. Then at a given $y$-coordinate (depth), the width of the cross section of that end is $2\sqrt{5^2-y^2}$, and the cross section lies $y$ m below the surface. So the total force is

$$\int_0^5 \rho g y\cdot 2\sqrt{5^2-y^2}\,dy = 2\rho g\left[-\frac{1}{3}(5^2-y^2)^{3/2}\right]_0^5 = -\frac{2}{3}\rho g\left((5^2-5^2)^{3/2}-(5^2-0^2)^{3/2}\right)$$
$$= -\frac{2}{3}\rho g\cdot(-125) = \frac{2}{3}\cdot 737\cdot 9.8\cdot 125 \boxed{\approx 601883 \text{ N}}.$$

34. Placing the origin at the center of the bottom of the dam, the face of the dam is




Note that for water, $\rho g = 62.5$. The two diagonal lines have equations

$$y - 0 = \frac{80-0}{-500-(-350)}(x-(-350)) \quad\text{or}\quad y = -\frac{8}{15}(x+350)$$
$$y - 0 = \frac{80-0}{500-350}(x-350) \quad\text{or}\quad y = \frac{8}{15}(x-350).$$

Solving for $x$ gives

$$x = -\frac{15}{8}y - 350, \qquad x = \frac{15}{8}y + 350.$$

Therefore the width of a cross section at height $y$ is

$$\left(\frac{15}{8}y + 350\right) - \left(-\frac{15}{8}y - 350\right) = \frac{15}{4}y + 700.$$

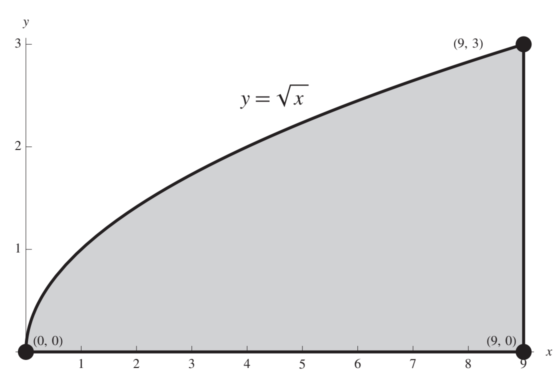
**(a)** If the dam is full, then the cross section at $y$ lies $80 - y$ ft below the surface, and the total force is

$$\begin{aligned}\int_0^{80} \rho g(80-y)\left(\frac{15}{4}y + 700\right) dy &= \rho g \int_0^{80} \left(-\frac{15}{4}y^2 - 400y + 56000\right) dy \\ &= \rho g \left[-\frac{5}{4}y^3 - 200y^2 + 56000y\right]_0^{80} \\ &= 2.56 \times 10^6 \rho g = \boxed{1.6 \times 10^8 \text{ lbs}}.\end{aligned}$$

**(b)** If the water has a depth of 60 ft, then the cross section at $y$ lies only $60 - y$ ft below the surface, and the integral goes from 0 to 60, so the total force is

$$\begin{aligned}\int_0^{60} \rho g(60-y)\left(\frac{15}{4}y + 700\right) dy &= \rho g \int_0^{60} \left(-\frac{15}{4}y^2 - 475y + 42000\right) dy \\ &= \rho g \left[-\frac{5}{4}y^3 - \frac{475}{2}y^2 + 42000y\right]_0^{60} \\ &= 1.395 \times 10^6 \rho g \boxed{\approx 8.719 \times 10^7 \text{ lbs}}.\end{aligned}$$

35. The region is shown below:




The mass of the region is

$$M = \rho \int_0^9 \sqrt{x}\,dx = \rho \left[\frac{2}{3}x^{3/2}\right]_0^9 = 18\rho,$$

and the moments about the axes are

$$\begin{aligned}M_y &= \rho \int_0^9 x\sqrt{x}\,dx = \rho \int_0^9 x^{3/2}\,dx = \rho \left[\frac{2}{5}x^{5/2}\right]_0^9 = \frac{486}{5}\rho \\ M_x &= \rho \cdot \frac{1}{2} \int_0^9 \left(\sqrt{x}\right)^2 dx = \frac{1}{2}\rho \int_0^9 x\,dx = \frac{1}{2}\rho \left[\frac{1}{2}x^2\right]_0^9 = \frac{81}{4}\rho.\end{aligned}$$

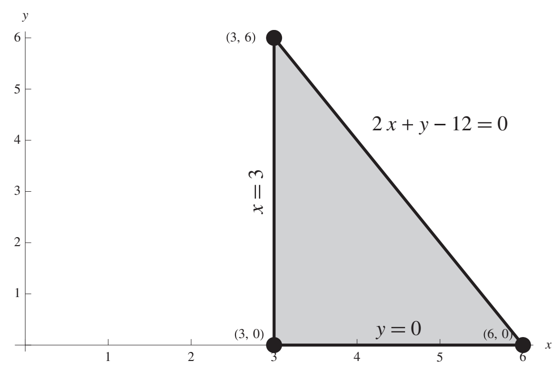
Therefore the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \boxed{\left(\frac{27}{5}, \frac{9}{8}\right)}.$$

36. This torus is the solid of revolution of the circle of radius $\frac{5-2}{2} = \frac{3}{2}$ centered at $\left(\frac{5+2}{2} = \frac{7}{2}, 0\right)$ about the $y$-axis. By symmetry, the centroid of this circle is at $\left(\frac{7}{2}, 0\right)$, which is $\frac{7}{2}$ cm from the axis of revolution. So by Pappus' Theorem, the volume of the torus is

$$2\pi A d = 2\pi \cdot \pi \left(\frac{3}{2}\right)^2 \cdot \frac{7}{2} = \boxed{\frac{63}{4}\pi^2 \text{ cm}^2}.$$

37. The region is shown below:




Since we are dealing with areas, we may assume that the region is a homogeneous lamina, and we must compute its centroid. The diagonal line has equation $y = 12 - 2x$, so we have

$$A = \int_3^6 (12 - 2x)\,dx = \left[12x - x^2\right]_3^6 = 36 - 27 = 9$$

$$\bar{x} = \frac{1}{A}\int_3^6 x(12 - 2x)\,dx = \frac{1}{9}\int_3^6 (12x - 2x^2)\,dx = \frac{1}{9}\left[6x^2 - \frac{2}{3}x^3\right]_3^6 = \frac{1}{9}(72 - 36) = 4$$

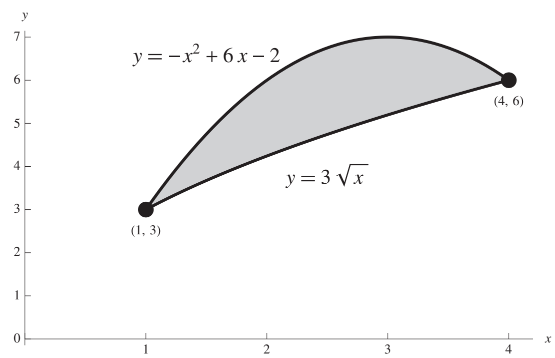
$$\bar{y} = \frac{1}{2A}\int_3^6 (12 - 2x)^2\,dx = \frac{1}{18}\int_3^6 (144 - 48x + 4x^2)\,dx = \frac{1}{18}\left[144x - 24x^2 + \frac{4}{3}x^3\right]_3^6$$

$$= \frac{1}{18}(288 - 252) = 2.$$

Then the centroid is at $(4, 2)$. We are revolving about the $y$-axis, so the distance from the centroid to the line of revolution is 4. The area of the region is 9. So by Pappus' Theorem, the volume of the solid of revolution is

$$V = 2\pi A d = 2\pi \cdot 9 \cdot 4 = \boxed{72\pi}.$$

38. The region is shown below:




The two graphs intersect when $3\sqrt{x} = -x^2+6x-2$. Looking at the graph, the intersection points appear to be at $x = 1$ and $x = 4$; substituting those values into the equation above reveals that $3\sqrt{1} = -1^2 + 6 \cdot 1 - 2$ and $3\sqrt{4} = -4^2 + 6 \cdot 4 - 2$, so that those are in fact the correct $x$-coordinates, and the two points of intersection are $(1, 3)$ and $(4, 6)$.

**(a)** To revolve about $y = 0$ (the $x$-axis), use the washer method. Each outer radius is $-x^2 + 6x - 2$ and each inner radius is $3\sqrt{x}$, so the volume is

$$\begin{aligned} V &= \pi \int_1^4 \left( \left(-x^2 + 6x - 2\right)^2 - \left(3\sqrt{x}\right)^2 \right) dx \\ &= \pi \int_1^4 \left(x^4 - 12x^3 + 40x^2 - 33x + 4\right) dx \\ &= \pi \left[ \frac{1}{5}x^5 - 3x^4 + \frac{40}{3}x^3 - \frac{33}{2}x^2 + 4x \right]_1^4 \\ &= \boxed{\frac{441}{10}\pi}. \end{aligned}$$

**(b)** To revolve about $x = 0$ (the $y$-axis), use the shell method. The radius of each shell is $x$, and the height of each shell is $-x^2 + 6x - 2 - 3\sqrt{x}$, so the volume is

$$\begin{aligned} V &= 2\pi \int_1^4 x\left(-x^2 + 6x - 2 - 3\sqrt{x}\right) dx \\ &= 2\pi \int_1^4 \left(-x^3 + 6x^2 - 2x - 3x^{3/2}\right) dx \\ &= 2\pi \left[ -\frac{1}{4}x^4 + 2x^3 - x^2 - \frac{6}{5}x^{5/2} \right]_1^4 \\ &= \boxed{\frac{201}{10}\pi}. \end{aligned}$$

**(c)** To revolve about $y = -2$, use the washer method. Outer radii are $-x^2+6x-2-(-2) = -x^2 + 6x$, and inner radii are $3\sqrt{x} - (-2) = 3\sqrt{x} + 2$. Therefore the volume is

$$\begin{aligned} V &= \pi \int_1^4 \left( \left(-x^2 + 6x\right)^2 - \left(3\sqrt{x} + 2\right)^2 \right) \, dx \\ &= \pi \int_1^4 \left( x^4 - 12x^3 + 36x^2 - 9x - 12x^{1/2} - 4 \right) \, dx \\ &= \pi \left[ \frac{1}{5}x^5 - 3x^4 + 12x^3 - \frac{9}{2}x^2 - 8x^{3/2} - 4x \right]_1^4 \\ &= \boxed{\frac{601}{10}\pi}. \end{aligned}$$

**(d)** To revolve about $y = 8$, use the washer method. Each outer radius is $8 - 3\sqrt{x}$ and each inner radius is $8 - (-x^2 + 6x - 2) = x^2 - 6x + 10$, so the volume is

$$\begin{aligned} V &= \pi \int_1^4 \left( \left(8 - 3\sqrt{x}\right)^2 - \left(x^2 - 6x + 10\right)^2 \right) \, dx \\ &= \pi \int_1^4 \left( -x^4 + 12x^3 - 56x^2 + 129x - 48x^{1/2} - 36 \right) \, dx \\ &= \pi \left[ -\frac{1}{5}x^5 + 3x^4 - \frac{56}{3}x^3 + \frac{129}{2}x^2 - 32x^{3/2} - 36x \right]_1^4 \\ &= \boxed{\frac{199}{10}\pi}. \end{aligned}$$

**(e)** To revolve about $x = -3$, use the shell method along the $x$-axis. The height of each shell is $-x^2 + 6x - 2 - 3\sqrt{x}$ and its radius is $x - (-3) = x + 3$, so the volume is

$$\begin{aligned} V &= 2\pi \int_1^4 (x + 3) \left(-x^2 + 6x - 2 - 3\sqrt{x}\right) \, dx \\ &= 2\pi \int_1^4 \left( -x^3 + 3x^2 + 16x - 3x^{3/2} - 9x^{1/2} - 6 \right) \, dx \\ &= 2\pi \left[ -\frac{1}{4}x^4 + x^3 + 8x^2 - \frac{6}{5}x^{5/2} - 6x^{3/2} - 6x \right]_1^4 \\ &= \boxed{\frac{441}{10}\pi}. \end{aligned}$$

**(f)** To revolve about $x = 5$, use the shell method along the $x$-axis. The height of each shell is $-x^2 + 6x - 2 - 3\sqrt{x}$ and its radius is $5 - x$, so the volume is

$$\begin{aligned} V &= 2\pi \int_1^4 (5 - x) \left(-x^2 + 6x - 2 - 3\sqrt{x}\right) \, dx \\ &= 2\pi \int_1^4 \left( x^3 - 11x^2 + 32x + 3x^{3/2} - 15x^{1/2} - 10 \right) \, dx \\ &= 2\pi \left[ \frac{1}{4}x^4 - \frac{11}{3}x^3 + 16x^2 + \frac{6}{5}x^{5/2} - 10x^{3/2} - 10x \right]_1^4 \\ &= \boxed{\frac{199}{10}\pi}. \end{aligned}$$

39. If $f(x) = 4\sqrt{2x} = 4(2x)^{1/2}$, then $f'(x) = 4 \cdot \frac{1}{2}(2x)^{-1/2} \cdot 2 = \frac{4}{\sqrt{2x}}$. Therefore the surface area is

$$S = 2\pi \int_1^4 4\sqrt{2x}\sqrt{1 + \left(\frac{4}{\sqrt{2x}}\right)^2} = 2\pi \int_1^4 4\sqrt{2x}\sqrt{1 + \frac{16}{2x}}\, dx = 2\pi \int_1^4 4\sqrt{2x}\sqrt{1 + \frac{8}{x}}\, dx$$
$$= 2\pi \int_1^4 4\sqrt{2x}\frac{\sqrt{x+8}}{\sqrt{x}}\, dx = 8\pi\sqrt{2} \int_1^4 \sqrt{x+8}\, dx$$

Let $u = x + 8$, so $du = dx$, $u(1) = (1) + 8 = 9$, and $u(4) = (4) + 8 = 12$. Then

$$S = 8\pi\sqrt{2} \int_9^{12} u^{1/2}\, du = 8\pi\sqrt{2}\left[\frac{2}{3}u^{3/2}\right]_9^{12} = \frac{16\pi\sqrt{2}}{3}\left(12^{3/2} - 9^{3/2}\right)$$
$$= \boxed{\frac{16\sqrt{2}}{3}\pi\left(12^{3/2} - 27\right) \quad \text{or} \quad 16\sqrt{2}(8\sqrt{3} - 9)\pi}$$

40.
$$S = 2\pi \int_0^4 \frac{5x^3}{3}\sqrt{1 + 25x^4}\, dx$$
$$= \frac{10\pi}{3} \int_0^4 x^3\sqrt{1 + 25x^4}\, dx$$

Let $u = 1 + 25x^4$, so $du = 100x^3\, dx$, $x^3\, dx = \frac{du}{100}$, $u(0) = 1$, and $u(4) = 6401$.

$$S = \frac{10\pi}{3} \int_1^{6401} \frac{1}{100}\left(\sqrt{u}\right) du$$
$$= \left[\frac{\pi}{30}\left(\frac{2}{3}\right)u^{3/2}\right]_1^{6401} = \frac{\pi}{45}\left[6401^{3/2} - 1\right] \approx \boxed{35{,}752.651}$$

## AP® Review Problems

1. $$V = \frac{\pi}{2} \int_1^4 \left(\frac{2x}{2}\right)^2 dx$$
$$= \frac{\pi}{2} \int_1^4 x^2\, dx$$
$$= \frac{\pi}{2}\left(\frac{x^3}{3}\right)_1^4$$
$$= \frac{\pi}{2}\left(\frac{64}{3} - \frac{1}{3}\right)$$
$$= \boxed{\frac{21}{2}\pi}$$

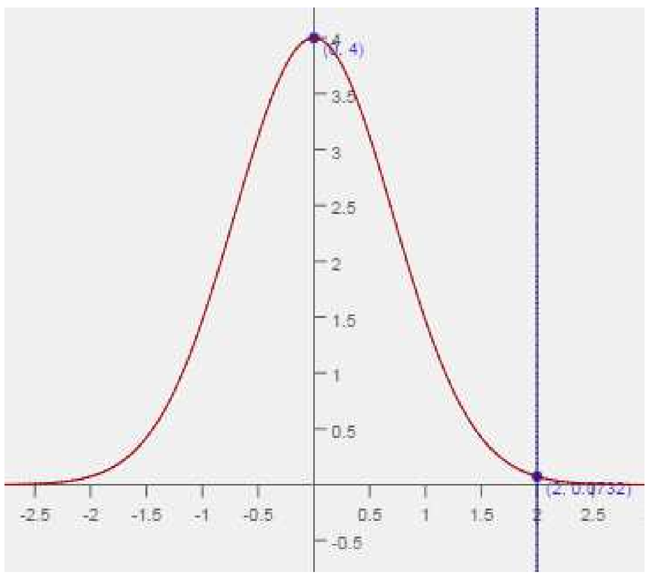
CHOICE A

2. $$V = \int_0^1 (2x + 4 - e^x)\, dx$$
$$= \left[x^2 + 4x - e^x\right]_0^1$$
$$= (1 + 4 - e) - \left(0 + 0 - e^0\right)$$
$$= \boxed{6 - e}$$

CHOICE C

3. By the Shell Method

$$
\begin{aligned}
V &= 2\pi \int_0^2 x(4e^{-x^2})\,dx \\
&= \left[2\pi\left(\frac{-1}{2}\right)(4e^{-x^2})\right]_0^2 \\
&= \left[-4\pi e^{-x^2}\right]_0^2 \\
&= -4\pi\big(e^{-4} - e^0\big) \\
&= \boxed{4\pi\big(1 - e^{-4}\big)}
\end{aligned}
$$




By the disk method

$$
\begin{aligned}
y &= 4e^{-x^2} \\
\frac{y}{4} &= e^{-x^2} \\
\ln\left(\frac{y}{4}\right) &= \ln\left(e^{-x^2}\right) \\
\ln\left(\frac{y}{4}\right) &= -x^2(\ln e) \\
x^2 &= \ln\left(\frac{4}{y}\right)
\end{aligned}
$$

$$
\begin{aligned}
V &= \pi \int_{4/e^4}^{4} x^2 \, dy + 4\pi \int_0^{4/e^4} dy \\
&= \pi \int_{4/e^4}^{4} \ln\left(\frac{4}{y}\right) dy + 4\pi \int_0^{4/e^4} dy \\
&= \pi \int_{4/e^4}^{4} (\ln 4 - \ln y) \, dy + 4\pi \int_0^{4/e^4} dy \\
&= \pi \ln 4 \int_{4/e^4}^{4} dy - \pi \int_{4/e^4}^{4} \ln y \, dy + 4\pi \int_0^{4/e^4} dy \\
&= \left[(\pi \ln 4)(y)\right]_{4/e^4}^{4} - \pi \left[ [y \ln y]_{4/e^4}^{4} - \int_{4/e^4}^{4} dy \right] + 4\pi \int_0^{4/e^4} dy
\end{aligned}
$$

[By integration by parts on $\int_{4/e^4}^{4} \ln y \, dy$]

$$
\begin{aligned}
&= \pi \ln 4 \left(4 - \frac{4}{e^4}\right) - \pi \left(4 \ln 4 - \frac{4}{e^4} \ln\left(\frac{4}{e^4}\right)\right) + \pi[y]_{4/e^4}^{4} + 4\pi[y]_0^{4/e^4} \\
&= 4\pi \ln 4 - \frac{4\pi \ln 4}{e^4} - 4\pi \ln 4 + \frac{4\pi}{e^4} \ln\left(\frac{4}{e^4}\right) + 4\pi - \frac{4\pi}{e^4} + \frac{16\pi}{e^4} \\
&= 4\pi \ln 4 - \frac{4\pi \ln 4}{e^4} - 4\pi \ln 4 + \frac{4\pi}{e^4}\left(\ln 4 - \ln e^4\right) + 4\pi + \frac{12\pi}{e^4} \\
&= 4\pi \ln 4 - \frac{4\pi \ln 4}{e^4} - 4\pi \ln 4 + \frac{4\pi \ln 4}{e^4} - \frac{16\pi}{e^4} + 4\pi + \frac{12\pi}{e^4} \\
&= 4\pi - \frac{4\pi}{e^4} \\
&= \boxed{4\pi\left(1 - e^{-4}\right)}
\end{aligned}
$$

CHOICE D

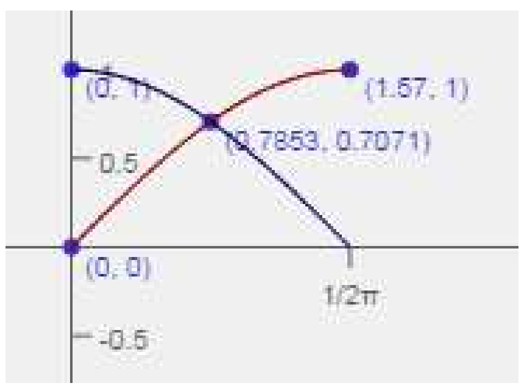
4. $$
\begin{aligned}
V &= \pi \int_0^2 (y - (-1))^2 \, dx \\
&= \pi \int_0^2 (y + 1)^2 \, dx \\
&= \pi \int_0^2 \left(x^2 + 1\right)^2 dx \\
&= \pi \int_0^2 \left(x^4 + 2x^2 + 1\right) dx \\
&= \pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_0^2 \\
&= \pi \left[\frac{32}{5} + \frac{16}{3} + 2 - 0\right] = \pi \left[\frac{96 + 80 + 30}{15}\right] = \boxed{\frac{206}{15}\pi}
\end{aligned}
$$

CHOICE C

5. Determine the point of intersection in the 1st quadrant of

$$\begin{aligned}
y &= \sin x \quad \text{and} \quad y = \cos x: \\
y &= \sin x = \cos x \\
\tan x &= 1 \\
x &= \frac{\pi}{4}
\end{aligned}$$

$$\begin{aligned}
V &= \pi \int_0^{\pi/4} (\cos x - 0)^2\, dx - \pi \int_0^{\pi/4} (\sin x - 0)^2\, dx \\
&= \pi \int_0^{\pi/4} \left(\cos^2 x - \sin^2 x\right) dx \\
&= \pi \int_0^{\pi/4} \left(1 - \sin^2 x - \sin^2 x\right) dx \\
&= \pi \int_0^{\pi/4} \left(1 - 2\sin^2 x\right) dx \\
&= \pi \int_0^{\pi/4} \cos(2x)\, dx \\
&= \left[\frac{\pi \sin 2x}{2}\right]_0^{\pi/4} \\
&= \frac{\pi}{2}\left[\sin\left(\frac{\pi}{2}\right) - \sin 0\right] \\
&= \frac{\pi}{2}[1 - 0] = \boxed{\frac{1}{2}\pi}
\end{aligned}$$




CHOICE B

6. By the Shell Method:

$$\begin{aligned}
V &= 2\pi \int_0^2 x\left(x^3\right) dx \\
&= 2\pi \int_0^2 \left(x^4\right) dx
\end{aligned}$$

By the Shell Method:

$$\begin{aligned}V &= 2\pi \int_0^2 x\left(x^3\right) dx \\ &= 2\pi \int_0^2 \left(x^4\right) dx \\ &= 2\pi \left[\frac{x^5}{5}\right]_0^2 = \boxed{\frac{64\pi}{5}}\end{aligned}$$

By the Disk Method:

For $y = x^3$, $x = y^{1/3}$

$$\begin{aligned}V &= \pi \int_0^8 2^2\, dy - \pi \int_0^8 x^2\, dy \\ &= 4\pi \int_0^8 dy - \pi \int_0^8 \left(y^{1/3}\right)^2 dy \\ &= 4\pi \int_0^8 dy - \pi \int_0^8 (y)^{2/3}\, dy \\ &= 4\pi [y]_0^8 - \frac{3\pi}{5}\left[y^{5/3}\right]_0^8 \\ &= 32\pi - \frac{3\pi}{5}\left[8^{5/3} - 0\right] \\ &= 32\pi - \frac{96\pi}{5} \\ &= \boxed{\frac{64\pi}{5}}\end{aligned}$$

CHOICE B

7. **(a)** For the limits of the Area Integral, determine the points of intersection of $f(x) = \sqrt{x}$ and $g(x) = \frac{x}{2}$.

$$\begin{aligned}f(x) = g(x) = \sqrt{x} &= \frac{x}{2} \\ x &= \frac{x^2}{4} \\ 4x &= x^2 \\ x^2 - 4x &= 0 \\ x(x-4) &= 0 \\ x &= 0 \quad x = 4\end{aligned}$$

$$A = \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx$$
$$= \left[\frac{2x^{3/2}}{3} - \frac{x^2}{4}\right]_0^4$$
$$= \frac{2}{3}(4)^{3/2} - \frac{16}{4} = \frac{16}{3} - \frac{12}{3} = \boxed{\frac{4}{3}}$$

**(b)** $$V = \pi \int_0^4 \left(\sqrt{x}\right)^2 dx - \pi \int_0^4 \left(\frac{x}{2}\right)^2 dx$$
$$= \pi \int_0^4 x\,dx - \pi \int_0^4 \frac{x^2}{4}\,dx$$
$$= \pi \left[\frac{x^2}{2}\right]_0^4 - \frac{\pi}{4}\left[\frac{x^3}{3}\right]_0^4$$
$$= 8\pi - \frac{16\pi}{3} = \boxed{\frac{8}{3}\pi}$$

**(c)** By the Shell Method

$$V = 2\pi \int_0^4 x\left(\sqrt{x} - \frac{x}{2}\right) dx$$
$$= 2\pi \int_0^4 \left(x^{3/2} - \frac{x^2}{2}\right) dx$$
$$= 2\pi \left[\frac{2x^{5/2}}{5} - \frac{x^3}{6}\right]_0^4$$
$$= 2\pi \left[\frac{2(4)^{5/2}}{5} - \frac{4^3}{6} - 0\right]$$
$$= 2\pi\left(\frac{64}{5} - \frac{32}{3}\right) = 2\pi\left(\frac{192 - 160}{15}\right) = \boxed{\frac{64}{5}\pi}$$

By the Disk Method:
For $f(x) = y = \sqrt{x}$, $x = y^2$
For $g(x) = y = \frac{x}{2}$, $x = 2y$

$$V = \pi \int_0^2 (2y)^2\,dy - \pi \int_0^2 \left(y^2\right)^2 dy$$
$$= \pi \int_0^2 \left(4y^2\right) dy - \pi \int_0^2 \left(y^4\right) dy$$
$$= \left[\frac{4\pi y^3}{3}\right]_0^2 - \left[\frac{\pi y^5}{5}\right]_0^2$$
$$= \left(\frac{32\pi}{3} - 0\right) - \left(\frac{32\pi}{5} - 0\right) = \frac{160\pi - 96\pi}{15} = \boxed{\frac{64\pi}{15}}$$

**(d)** $$\begin{aligned} V &= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)^2 dx \\ &= \int_0^4 \left(x - x\sqrt{x} + \frac{x^2}{4}\right) dx \\ &= \int_0^4 \left(x - x\sqrt{x} + \frac{x^2}{4}\right) dx \\ &= \int_0^4 \left(x - x^{3/2} + \frac{x^2}{4}\right) dx \\ &= \left[\frac{x^2}{2} - \frac{2x^{5/2}}{5} + \frac{x^3}{12}\right]_0^4 \\ &= \left(8 - \frac{64}{5} + \frac{16}{3} - 0\right) = \boxed{\frac{8}{15}} \end{aligned}$$

8. $$\begin{aligned} F &= \int_c^d \rho g(H - y)[f(y) - h(y)]\, dy \\ &= \int_{-1}^0 (840)(9.8)(0 - y)\left[\frac{\sqrt{1-y^2}}{2} - \left(\frac{-\sqrt{1-y^2}}{2}\right)\right] dy \\ &= \boxed{(840)(9.8)\int_{-1}^0 (-y)\sqrt{1-y^2}\, dy} \end{aligned}$$

CHOICE C

9. **(a)** $$\begin{aligned} W &= \rho g(100\pi)\int_0^{27} (36 - x)\, dx \\ &= (1000)(9.8)(100\pi)\left[36x - \frac{x^2}{2}\right]_0^{27} \\ &= 980{,}000\pi\left[36(27) - \frac{27^2}{2}\right] \\ &= 980{,}000\pi(607.5) \\ &= 595{,}350{,}000\pi \\ &\approx 1{,}870{,}347{,}186 \approx \boxed{1.870 \times 10^9 \text{ J}} \end{aligned}$$

**(b)** $$\begin{aligned} W &= \rho g(100\pi)\int_0^{27} (40 - x)\, dx \\ &= (1{,}000)(9.8)(100\pi)\left[40x - \frac{x^2}{2}\right]_0^{27} \\ &= 980{,}000\pi\left[40(27) - \frac{27^2}{2}\right] \\ &= 980{,}000\pi(715.5) \\ &= 701{,}190{,}000\pi \approx 2{,}202{,}853{,}353 \approx \boxed{2.203 \times 10^9 \text{ J}} \end{aligned}$$

10. $y = \sqrt{x} = x^{1/2}$

$$y' = \frac{1}{2}x^{-1/2}$$

$$L = \int_a^b \sqrt{1 + (y')^2}\,dx$$

$$= \int_2^5 \sqrt{1 + \left(\frac{1}{2}x^{-1/2}\right)^2}\,dx$$

$$= \boxed{\int_2^5 \sqrt{1 + \frac{1}{4x}}\,dx}$$

CHOICE A