Chapter 8 Infinite Series

8.1 Sequences

Concepts and Vocabulary

1. **False:** A sequence is a function whose domain is the set of positive *integers* and not the set of positive *real numbers.*

2. **False:** A sequence $\{s_n\}$ is convergent to a limit L if $\lim\limits_{n\to\infty} = L$. ($L = 0$ is a special case, but in general a sequence that is convergent will converge to a real number L.)

3. **True:** This is because if $f(x)$ is a related function of the sequence $\{s_n\}$ then if $\lim\limits_{x\to\infty} f(x) = L$ it follows that $\lim\limits_{n\to\infty} s_n = L$ (theorem, p.637).

4. $\boxed{\textbf{(b)}:}$ Bounded, because no term of $\{s_n\}$ will ever exceed in absolute value, K.

5. **False:** Consider the sequence $\{s_n\} = \left\{\sin(\frac{\pi}{2}n)\right\}$ (see p.642, fig.13). This sequence is bounded by ± 1, but it never converges as it oscillates between -1 and $+1$.

6. **True:** A sequence whose nth term cannot be bounded in absolute value by any finite real number cannot converge to a finite real number.

7. **False:** A sequence is decreasing if and only if $s_n > s_{n+1}$ for $n \geq 1$.

8. **False:** As an example, consider the sequence $\{t_n\} = \left\{1 + \frac{(-1)^n}{n^2}\right\}$ (see fig.14, p.642). This sequence is not monotonic, but it converges to the limit 1.

9. **False:** If the sequence $\{s_n\}$ is increasing then the algebraic ratio $\frac{s_{n+1}}{s_n} > 1$ for all $n \geq 1$.

10. $\boxed{\textbf{(b)}:}$ This follows from the properties of a monotonic function. A function whose derivative is < 0 at $x = x_0$ is decreasing in an open interval about $x = x_0$. Since the function is a related function of the sequence $\{s_n\}$ it follows that the sequence is also decreasing.

11. **True:** It is only the eventual (large n) terms of the sequence which determine its convergence or divergence properties.

12. **False:** A bounded, monotonic sequence will be a convergent sequence (see theorem, p.642).

Skill Building

13. $s_n = \frac{n+1}{n}$. Put $n = 1, 2, 3, 4$. Then $s_1 = 2$; $s_2 = \frac{3}{2}$; $s_3 = \frac{4}{3}$; $s_4 = \frac{5}{4}$. The first four terms of the sequence are $\boxed{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}.}$

14. $s_n = \frac{2}{n^2}$. Put $n = 1, 2, 3, 4$. Then $s_1 = 2$; $s_2 = \frac{2}{4} = \frac{1}{2}$; $s_3 = \frac{2}{9}$; $s_4 = \frac{2}{16} = \frac{1}{8}$. The first four terms of the sequence are $\boxed{2, \frac{1}{2}, \frac{2}{9}, \frac{1}{8}.}$

15. $s_n = \ln n$. Put $n = 1, 2, 3, 4$. Then $s_1 = \ln 1 = 0$; $s_2 = \ln 2$; $s_3 = \ln 3$; $s_4 = \ln 4$. The first four terms of the sequence are $\boxed{0, \ln 2, \ln 3, \ln 4.}$

16. $s_n = \frac{n}{\ln(n+1)}$. Put $n = 1, 2, 3, 4$. Then $s_1 = \frac{1}{\ln 2}$; $s_2 = \frac{2}{\ln 3}$; $s_3 = \frac{3}{\ln 4}$; $s_4 = \frac{4}{\ln 5}$. The first four terms of the sequence are $\boxed{\frac{1}{\ln 2}, \frac{2}{\ln 3}, \frac{3}{\ln 4}, \frac{4}{\ln 5}.}$

17. $s_n = \frac{(-1)^{n+1}}{2n+1}$. Put $n = 1, 2, 3, 4$. Then $s_1 = \frac{(-1)^2}{2+1} = \frac{1}{3}$; $s_2 = \frac{(-1)^3}{4+1} = -\frac{1}{5}$; $s_3 = \frac{(-1)^4}{6+1} = \frac{1}{7}$; $s_4 = \frac{(-1)^5}{8+1} = -\frac{1}{9}$. The first four terms of the sequence are $\boxed{\frac{1}{3}, -\frac{1}{5}, \frac{1}{7}, -\frac{1}{9}.}$

18. $s_n = \frac{1-(-1)^n}{2}$. Put $n = 1, 2, 3, 4$. Then $s_1 = \frac{1-(-1)^1}{2} = \frac{1+1}{2} = 1$; $s_2 = \frac{1-(-1)^2}{2} = \frac{1-1}{2} = 0$; $s_3 = \frac{1-(-1)^3}{2} = \frac{1+1}{2} = 1$; $s_4 = \frac{1-(-1)^4}{2} = \frac{1-1}{2} = 0$. The first four terms of the sequence are $\boxed{1, 0, 1, 0.}$

19. $s_n = \begin{cases} (-1)^{n+1} & \text{if n is even} \\ 1 & \text{if n is odd.} \end{cases}$ Put $n = 1, 2, 3, 4$. Then $s_1 = 1$; $s_2 = (-1)^{2+1} = (-1)^3 = -1$; $s_3 = 1$; $s_4 = (-1)^{4+1} = (-1)^5 = -1$. The first four terms of the sequence are $\boxed{1, -1, 1, -1.}$

20. $s_n = \begin{cases} n^2 + n & \text{if n is even} \\ 4n + 1 & \text{if n is odd.} \end{cases}$ Put $n = 1, 2, 3, 4$. Then $s_1 = 4(1) + 1 = 5$; $s_2 = 2^2 + 2 = 6$; $s_3 = 4(3) + 1 = 13$; $s_4 = 4^2 + 4 = 20$. The first four terms of the sequence are $\boxed{5, 6, 13, 20.}$

21. $s_n = \frac{n!}{2^n}$. Put $n = 1, 2, 3, 4$. Then $s_1 = \frac{1!}{2^1} = \frac{1}{2}$; $s_2 = \frac{2!}{2^2} = \frac{2\cdot 1}{4} = \frac{1}{2}$; $s_3 = \frac{3!}{2^3} = \frac{3\cdot 2\cdot 1}{8} = \frac{3}{4}$; $s_4 = \frac{4!}{2^4} = \frac{4\cdot 3\cdot 2\cdot 1}{16} = \frac{3}{2}$. The first four terms of the sequence are $\boxed{\frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{2}.}$

22. $s_n = \frac{n!}{n^2}$. Put $n = 1, 2, 3, 4$. Then $s_1 = \frac{1!}{1^2} = 1$; $s_2 = \frac{2!}{2^2} = \frac{2\cdot 1}{4} = \frac{1}{2}$; $s_3 = \frac{3!}{3^2} = \frac{3\cdot 2\cdot 1}{9} = \frac{2}{3}$; $s_4 = \frac{4!}{4^2} = \frac{4\cdot 3\cdot 2\cdot 1}{16} = \frac{3}{2}$. The first four terms of the sequence are $\boxed{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{2}.}$

23. $2, 4, 6, 8, 10, \ldots$ Observe that this is the set of even numbers. The nth of term of the sequence is $\boxed{2n.}$

24. $1, 3, 5, 7, 9, \ldots$ Observe that this is the set of odd numbers. The nth term of the sequence is $\boxed{2n - 1.}$ (We don't write $2n + 1$ because the first term, for $n = 1$, needs to be 1.)

25. $2, 4, 8, 16, 32, \ldots$ Notice that each term is two times the preceding term. So the nth term $= 2 \times (n-1)$st term $= 2 \times 2 \times (n-2)$nd term $= \cdots$ etc. The nth term of the sequence is $\boxed{2^n.}$

26. $1, 8, 27, 64, 125, \ldots$ Here, each term is a perfect cube. The nth term of the sequence is $\boxed{n^3.}$

27. $\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \ldots$ Notice that every even term appears with a minus sign. So we need a factor of $(-1)^{n+1}$ to appear in the nth term of the sequence, which is $\boxed{(-1)^{n+1}\frac{1}{n+1}.}$

28. $1, -2, 3, -4, 5, \ldots$ Notice that every even term appears with a minus sign. So we need a factor of $(-1)^{n+1}$ to appear in the nth term of the sequence, which is $\boxed{(-1)^{n+1}n.}$

29. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ Each term has both numerator and denominator increasing, with denominator being 1 more than the numerator. So the nth term of the sequence is $\boxed{\frac{n}{n+1}.}$

30. $\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \ldots$ The numerators are all perfect squares, while the denominators are just the consecutive positive integers plus 1. So the nth term of the sequence is $\boxed{\frac{n^2}{n+1}.}$

31. $1, 1, 2, 6, 24, 120, 720, \ldots$ The rapidly increasing sequence indicates that the factorial could be involved. The nth term of the sequence $\boxed{(n-1)!}$ generates this sequence because $(1-1)! = 0! = 1$; $(2-1)! = 1! = 1$; $(3-1)! = 2! = 2$, and so on.

32. $1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$ The rapidly increasing denominators could mean the factorial appears there. The nth term of the sequence $\boxed{\frac{1}{(n-1)!}}$ generates this sequence because $\frac{1}{(1-1)!} = \frac{1}{0!} = \frac{1}{1} = 1$; $\frac{1}{(2-1)!} = \frac{1}{1!} = 1$; $\frac{1}{(3-1)!} = \frac{1}{2}$, and so on.

33. The nth term of the sequence is $s_n = \frac{3}{n}$. We have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{3}{n} = 3 \cdot \lim_{n\to\infty} \frac{1}{n} = 3 \cdot 0 = 0,$$

where the constant multiple property of convergent sequences was used. The limit of the sequence is $\boxed{0.}$

34. The nth term of the sequence is $s_n = -\frac{2}{n}$. We have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} -\frac{2}{n} = (-2) \cdot \lim_{n\to\infty} \frac{1}{n} = (-2) \cdot 0 = 0,$$

where the constant multiple property of convergent sequences was used. The limit of the sequence is $\boxed{0.}$

35. The nth term of the sequence is $s_n = 1 - \frac{1}{n}$. We have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n} = 1 - 0 = 1,$$

where the sum and difference property of convergent sequences was used. The limit of the sequence is $\boxed{1.}$

36. The nth term of the sequence is $s_n = \frac{1}{n} + 4$. We have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{n}+4\right) = \lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty} 4 = 0+4 = 4,$$

where the sum and difference property of convergent sequences was used. The limit of the sequence is $\boxed{4.}$

37. The nth term of the sequence is $s_n = \frac{4n+2}{n} = 4 + \frac{2}{n}$. We have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(4+\frac{2}{n}\right) = \lim_{n\to\infty} 4 + \lim_{n\to\infty}\frac{2}{n} = \lim_{n\to\infty} 4 + 2\cdot\lim_{n\to\infty}\frac{1}{n} = 4+2\cdot 0 = 4,$$

where the sum and difference and constant multiple properties of convergent sequences were used. The limit of the sequence is $\boxed{4.}$

38. The nth term of the sequence is $s_n = \frac{2n+1}{n} = 2 + \frac{1}{n}$. We have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(2+\frac{1}{n}\right) = \lim_{n\to\infty} 2 + \lim_{n\to\infty}\frac{1}{n} = 2+0 = 2,$$

where the sum and difference property of convergent sequences was used. The limit of the sequence is $\boxed{2.}$

39. The nth term of the sequence is $s_n = \left(\frac{2-n}{n^2}\right)^4 = \left(\frac{2}{n^2}-\frac{1}{n}\right)^4$. We have $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{2}{n^2}-\frac{1}{n}\right)^4 = \left[\lim_{n\to\infty}\left(\frac{2}{n^2}-\frac{1}{n}\right)\right]^4 = \left[\lim_{n\to\infty}\frac{2}{n^2} - \lim_{n\to\infty}\frac{1}{n}\right]^4 =$ $\left[2\cdot\lim_{n\to\infty}\frac{1}{n^2} - \lim_{n\to\infty}\frac{1}{n}\right]^4 = [2\cdot 0 - 0]^4 = 0$, where the power, sum and difference, and constant multiple properties of convergent sequences were used. The limit of the sequence is $\boxed{0.}$

40. The nth term of the sequence is $s_n = \left(\frac{n^3-2n}{n^3}\right)^2 = \left(1-\frac{2}{n^2}\right)^2$. We have $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1-\frac{2}{n^2}\right)^2 = \left[\lim_{n\to\infty}\left(1-\frac{2}{n^2}\right)\right]^2 = \left[\lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{2}{n^2}\right]^2 =$ $\left[\lim_{n\to\infty} 1 - 2\cdot\lim_{n\to\infty}\frac{1}{n^2}\right]^2 = [1-2\cdot 0]^2 = 1^2 = 1$, where the power, sum and difference, and constant multiple properties of convergent sequences were used. The limit of the sequence is $\boxed{1.}$

41. The nth term of the sequence is $s_n = \sqrt{\frac{n+1}{n^2}} = \sqrt{\frac{1}{n}+\frac{1}{n^2}}$. We have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\sqrt{\frac{1}{n}+\frac{1}{n^2}} = \left[\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n^2}\right)\right]^{\frac{1}{2}} = \left[\lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}\frac{1}{n^2}\right]^{\frac{1}{2}} = [0+0]^{\frac{1}{2}} = 0,$$

where the power and sum and difference properties of convergent sequences were used. The limit of the sequence is $\boxed{0.}$

42. The nth term of the sequence is $s_n = \sqrt[3]{8-\frac{1}{n}} = \left(8-\frac{1}{n}\right)^{\frac{1}{3}}$. We have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(8-\frac{1}{n}\right)^{\frac{1}{3}} = \left[\lim_{n\to\infty}\left(8-\frac{1}{n}\right)\right]^{\frac{1}{3}} = \left[\lim_{n\to\infty} 8 - \lim_{n\to\infty}\frac{1}{n}\right]^{\frac{1}{3}} = [8-0]^{\frac{1}{3}} = 2,$$

where the power and sum and difference properties of convergent sequences were used. The limit of the sequence is $\boxed{2.}$

43. The nth term of the sequence is $s_n = \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)$. We have

$$\begin{aligned}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right) &= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)\cdot\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)\\ &= \left(1-\lim_{n\to\infty}\frac{1}{n}\right)\cdot\left(1-\lim_{n\to\infty}\frac{1}{n^2}\right)\\ &= (1-0)(1-0) = 1,\end{aligned}$$

where the product and sum and difference properties of convergent sequences were used. The limit of the sequence is $\boxed{1.}$

44. The nth term of the sequence is $s_n = \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{n^3}\right)$. We have

$$\begin{aligned}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{n^3}\right) &= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)\cdot\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)\cdot\lim_{n\to\infty}\left(1-\frac{1}{n^3}\right)\\ &= \left(1-\lim_{n\to\infty}\frac{1}{n}\right)\cdot\left(1-\lim_{n\to\infty}\frac{1}{n^2}\right)\cdot\left(1-\lim_{n\to\infty}\frac{1}{n^3}\right)\\ &= (1-0)(1-0)(1-0) = 1,\end{aligned}$$

where the product and sum and difference properties of convergent sequences were used. The limit of the sequence is $\boxed{1.}$

45. Consider a sequence $\{s_n\}$ with $s_n = \frac{n+1}{3n} = \frac{1}{3}+\frac{1}{3n}$. This sequence converges because

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{3n}\right) = \lim_{n\to\infty}\frac{1}{3}+\lim_{n\to\infty}\frac{1}{3n} = \lim_{n\to\infty}\frac{1}{3}+\frac{1}{3}\cdot\lim_{n\to\infty}\frac{1}{n} = \frac{1}{3}+0 = \frac{1}{3},$$

where the sum and difference and constant multiple properties of convergent sequences were used. Then by the theorem on p.637, since the function $f(x)=\ln x$ is continuous at $x=\frac{1}{3}$, the sequence $\{f(s_n)\}$ converges, and limit of the sequence $\{f(s_n)\} = \left\{\ln\left(\frac{n+1}{3n}\right)\right\}$ is $\boxed{\ln\left(\frac{1}{3}\right).}$

46. Consider a sequence $\{s_n\}$ with $s_n = \frac{n^2+2}{2n^2+3}$. This sequence converges because

$$\begin{aligned}\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{n^2+2}{2n^2+3} &= \lim_{n\to\infty}\frac{1+\frac{2}{n^2}}{2+\frac{3}{n^2}} = \frac{\lim\limits_{n\to\infty}\left(1+\frac{2}{n^2}\right)}{\lim\limits_{n\to\infty}\left(2+\frac{3}{n^2}\right)}\\ &= \frac{\lim\limits_{n\to\infty}1+\lim\limits_{n\to\infty}\frac{2}{n^2}}{\lim\limits_{n\to\infty}2+\lim\limits_{n\to\infty}\frac{3}{n^2}} = \frac{\lim\limits_{n\to\infty}1+2\cdot\lim\limits_{n\to\infty}\frac{1}{n^2}}{\lim\limits_{n\to\infty}2+3\cdot\lim\limits_{n\to\infty}\frac{1}{n^2}}\\ &= \frac{1+2\cdot 0}{2+3\cdot 0} = \frac{1}{2},\end{aligned}$$

where the quotient, sum and difference, and constant multiple properties of convergent sequences were used. The function $f(x)=\ln x$ is continuous at $x=\frac{1}{2}$. By the theorem on p. 637, the sequence $\{f(s_n)\}$ converges, and the limit of the sequence $\{f(s_n)\} = \left\{\ln\left(\frac{n^2+2}{2n^2+3}\right)\right\}$ is

$\boxed{\ln\left(\frac{1}{2}\right).}$

47. Consider the sequence $\{s_n\}$ with $s_n = \frac{4}{n} - 2$. This sequence converges because

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{4}{n} - 2\right) = \lim_{n\to\infty}\frac{4}{n} - \lim_{n\to\infty} 2 = 4 \cdot \lim_{n\to\infty}\frac{1}{n} - \lim_{n\to\infty} 2 = 4 \cdot 0 - 2 = -2,$$

where the sum and difference and constant multiple properties of convergent sequences were used. The function $f(x) = e^x$ is continuous at $x = -2$. By the theorem on p. 637, the sequence $\{f(s_n)\}$ converges, and the limit of the sequence $\{f(s_n)\} = \left\{e^{(4/n)-2}\right\}$ is $\boxed{e^{-2}.}$

48. Consider the sequence $\{s_n\}$ with $s_n = 3 + \frac{6}{n}$. This sequence converges because

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(3 + \frac{6}{n}\right) = \lim_{n\to\infty} 3 + \lim_{n\to\infty}\frac{6}{n} = \lim_{n\to\infty} 3 + 6 \cdot \lim_{n\to\infty}\frac{1}{n} = 3 + 6 \cdot 0 = 3,$$

where the sum and difference and constant multiple properties of convergent sequences were used. The function $f(x) = e^x$ is continuous at $x = 3$. By the theorem on p.637, the sequence $\{f(s_n)\}$ converges, and the limit of the sequence $\{f(s_n)\} = \left\{e^{3+(6/n)}\right\}$ is $\boxed{e^3.}$

49. Consider the sequence $\{s_n\}$ with $s_n = \frac{1}{n}$. This sequence converges because $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$. The function $f(x) = \sin x$ is continuous at $x = 0$. By the theorem on p.637, the sequence $\{f(s_n)\}$ converges, and the limit of the sequence $\{f(s_n)\} = \left\{\sin\left(\frac{1}{n}\right)\right\}$ will be $\sin(0) = \boxed{0.}$

50. Consider the sequence $\{s_n\}$ with $s_n = \frac{1}{n}$. This sequence converges because $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$. The function $f(x) = \cos x$ is continuous at $x = 0$. By the theorem on p.637, the sequence $\{f(s_n)\}$ converges, and the limit of the sequence $\{f(s_n)\} = \left\{\cos\left(\frac{1}{n}\right)\right\}$ will be $\cos(0) = \boxed{1.}$

51. The function $f(x) = \frac{x^2-4}{x^2+x-2}$ is a related function of the sequence $\left\{\frac{n^2-4}{n^2+n-2}\right\}$. Since

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty}\frac{x^2-4}{x^2+x-2} = \lim_{x\to\infty}\frac{1-\frac{4}{x^2}}{1+\frac{1}{x}-\frac{2}{x^2}}\\ &= \frac{\lim\limits_{x\to\infty}\left(1-\frac{4}{x^2}\right)}{\lim\limits_{x\to\infty}\left(1+\frac{1}{x}-\frac{2}{x^2}\right)} = \frac{\lim\limits_{x\to\infty} 1 - \lim\limits_{x\to\infty}\frac{4}{x^2}}{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty}\frac{1}{x} - \lim\limits_{x\to\infty}\frac{2}{x^2}}\\ &= \frac{1-0}{1+0-0} = 1,\end{aligned}$$

the sequence $\left\{\frac{n^2-4}{n^2+n-2}\right\}$ also converges and its limit is $\boxed{1.}$

52. The function $f(x) = \frac{x+2}{x^2+6x+8}$ is a related function of the sequence $\left\{\frac{n+2}{n^2+6n+8}\right\}$. Since

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty}\frac{x+2}{x^2+6x+8} = \lim_{x\to\infty}\frac{\frac{1}{x}+\frac{2}{x^2}}{1+\frac{6}{x}+\frac{8}{x^2}}\\ &= \frac{\lim\limits_{x\to\infty}\left(\frac{1}{x}+\frac{2}{x^2}\right)}{\lim\limits_{x\to\infty}\left(1+\frac{6}{x}+\frac{8}{x^2}\right)} = \frac{\lim\limits_{x\to\infty}\frac{1}{x} + \lim\limits_{x\to\infty}\frac{2}{x^2}}{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty}\frac{6}{x} + \lim\limits_{x\to\infty}\frac{8}{x^2}}\\ &= \frac{0+0}{1+0+0} = 0,\end{aligned}$$

the sequence $\left\{\frac{n+2}{n^2+6n+8}\right\}$ also converges and its limit is $\boxed{0.}$

53. We write the nth term of the sequence $\{s_n\} = \left\{\frac{n^2}{2n+1} - \frac{n^2}{2n-1}\right\}$ as

$$s_n = \frac{n^2}{2n+1} - \frac{n^2}{2n-1} = \frac{n^2[(2n-1)-(2n+1)]}{(2n)^2 - 1^2} = \frac{n^2(-2)}{4n^2-1} = -\frac{2n^2}{4n^2-1}.$$

The function $f(x) = -\frac{2x^2}{4x^2-1}$ is a related function of the sequence $\{s_n\}$. Since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} -\frac{2x^2}{4x^2-1} = \lim_{x\to\infty} \frac{-2}{4-\frac{1}{x^2}} = \frac{\lim\limits_{x\to\infty}(-2)}{\lim\limits_{x\to\infty}\left(4-\frac{1}{x^2}\right)} = \frac{\lim\limits_{x\to\infty}(-2)}{\lim\limits_{x\to\infty} 4 - \lim\limits_{x\to\infty}\frac{1}{x^2}} = \frac{-2}{4-0} = -\frac{1}{2},$$

the sequence $\{s_n\} = \left\{\frac{n^2}{2n+1} - \frac{n^2}{2n-1}\right\}$ also converges and its limit is $\boxed{-\frac{1}{2}.}$

54. The function $f(x) = \frac{6x^4-5}{7x^4+3}$ is a related function of the sequence $\left\{\frac{6n^4-5}{7n^4+3}\right\}$. Since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{6x^4-5}{7x^4+3} = \lim_{x\to\infty} \frac{6-\frac{5}{x^4}}{7+\frac{3}{x^4}} = \frac{\lim\limits_{x\to\infty}\left(6-\frac{5}{x^4}\right)}{\lim\limits_{x\to\infty}\left(7+\frac{3}{x^4}\right)} = \frac{\lim\limits_{x\to\infty} 6 - \lim\limits_{x\to\infty}\frac{5}{x^4}}{\lim\limits_{x\to\infty} 7 + \lim\limits_{x\to\infty}\frac{3}{x^4}} = \frac{6-0}{7+0} = \frac{6}{7},$$

the sequence $\left\{\frac{6n^4-5}{7n^4+3}\right\}$ also converges and its limit is $\boxed{\frac{6}{7}.}$

55. The function $f(x) = \frac{\sqrt{x}+2}{\sqrt{x}+5}$ is a related function of the sequence $\left\{\frac{\sqrt{n}+2}{\sqrt{n}+5}\right\}$. Since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{x}+2}{\sqrt{x}+5} = \lim_{x\to\infty} \frac{1+\frac{2}{\sqrt{x}}}{1+\frac{5}{\sqrt{x}}} = \frac{\lim\limits_{x\to\infty}\left(1+\frac{2}{\sqrt{x}}\right)}{\lim\limits_{x\to\infty}\left(1+\frac{5}{\sqrt{x}}\right)} = \frac{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty}\frac{2}{\sqrt{x}}}{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty}\frac{5}{\sqrt{x}}} = \frac{1+0}{1+0} = 1,$$

the sequence $\left\{\frac{\sqrt{n}+2}{\sqrt{n}+5}\right\}$ also converges and its limit is $\boxed{1.}$

56. The function $f(x) = \frac{\sqrt{x}}{e^x}$ is a related function of the sequence $\left\{\frac{\sqrt{n}}{e^n}\right\}$. Using L'Hôpital's rule, since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}e^x} = 0,$$

the sequence $\left\{\frac{\sqrt{n}}{e^n}\right\}$ also converges and its limit is $\boxed{0.}$

57. The function $f(x) = \frac{x^2}{3^x}$ is a related function of the sequence $\left\{\frac{n^2}{3^n}\right\}$. Using L'Hôpital's rule, since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{3^x} = \lim_{x\to\infty} \frac{2x}{3^x \ln 3} = \lim_{x\to\infty} \frac{2}{3^x(\ln 3)^2} = 0,$$

the sequence $\left\{\frac{n^2}{3^n}\right\}$ also converges and its limit is $\boxed{0.}$

58. The function $f(x) = \frac{(x-1)^2}{e^x}$ is a related function of the sequence $\left\{\frac{(n-1)^2}{e^n}\right\}$. Using L'Hôpital's rule, since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(x-1)^2}{e^x} = \lim_{x\to\infty} \frac{2(x-1)}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0,$$

the sequence $\left\{\frac{(n-1)^2}{e^n}\right\}$ also converges and its limit is $\boxed{0.}$

59. To see if the sequence $\{s_n\} = \left\{\frac{(-1)^n}{3n^2}\right\}$ converges, consider the absolute value of the nth term:

$$|s_n| = \left|\frac{(-1)^n}{3n^2}\right| \le \frac{1}{3n^2}.$$

That means $-\frac{1}{3n^2} \le \frac{(-1)^n}{3n^2} \le \frac{1}{3n^2}$. The sequence $\{s_n\}$ is bounded below by the sequence $\{a_n\} = \left\{-\frac{1}{3n^2}\right\}$ and bounded above by the sequence $\{b_n\} = \left\{\frac{1}{3n^2}\right\}$. Since $a_n \le s_n \le b_n$ for all n, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-1}{3n^2} = \left(-\frac{1}{3}\right) \cdot \lim_{n\to\infty} \frac{1}{n^2} = -\frac{1}{3} \cdot 0 = 0$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{3n^2} = \frac{1}{3} \cdot \lim_{n\to\infty} \frac{1}{n^2} = \frac{1}{3} \cdot 0 = 0$, then by the Squeeze Theorem, the sequence $\{s_n\} = \left\{\frac{(-1)^n}{3n^2}\right\}$ converges, and its limit is $\boxed{0.}$

60. To see if the sequence $\{s_n\} = \left\{\frac{(-1)^n}{\sqrt{n}}\right\}$ converges, consider the absolute value of the nth term:

$$|s_n| = \left|\frac{(-1)^n}{\sqrt{n}}\right| \le \frac{1}{\sqrt{n}}.$$

That means $-\frac{1}{\sqrt{n}} \le \frac{(-1)^n}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$. The sequence $\{s_n\}$ is bounded below by the sequence $\{a_n\} = \left\{-\frac{1}{\sqrt{n}}\right\}$ and bounded above by the sequence $\{b_n\} = \left\{\frac{1}{\sqrt{n}}\right\}$. Since $a_n \le s_n \le b_n$ for all n, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-1}{\sqrt{n}} = 0$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, then by the Squeeze Theorem, the sequence $\{s_n\} = \left\{\frac{(-1)^n}{\sqrt{n}}\right\}$ converges, and its limit is $\boxed{0.}$

61. To see if the sequence $\{s_n\} = \left\{\frac{\sin n}{n}\right\}$ converges, consider the absolute value of the nth term:

$$|s_n| = \left|\frac{\sin n}{n}\right| \le \frac{1}{n}.$$

That is, $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$. The sequence $\{s_n\}$ is bounded below by the sequence $\{a_n\} = \left\{-\frac{1}{n}\right\}$ and bounded above by the sequence $\{b_n\} = \left\{\frac{1}{n}\right\}$. Since $a_n \le s_n \le b_n$ for all n, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} -\frac{1}{n} = 0$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$, then by the Squeeze Theorem, the sequence $\{s_n\} = \left\{\frac{\sin n}{n}\right\}$ converges, and its limit is $\boxed{0.}$

62. To see if the sequence $\{s_n\} = \left\{\frac{\cos n}{n}\right\}$ converges, consider the absolute value of the nth term:

$$|s_n| = \left|\frac{\cos n}{n}\right| \le \frac{1}{n}.$$

That is, $-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$. The sequence $\{s_n\}$ is bounded below by the sequence $\{a_n\} = \left\{-\frac{1}{n}\right\}$ and bounded above by the sequence $\{b_n\} = \left\{\frac{1}{n}\right\}$. Since $a_n \le s_n \le b_n$ for all n, and

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{-1}{n} = 0$ and $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$, then by the Squeeze Theorem, the sequence $\{s_n\} = \left\{\frac{\cos n}{n}\right\}$ converges, and its limit is $\boxed{0.}$

63. The sequence $\{s_n\} = \{\cos(\pi n)\}$ oscillates between -1 and $+1$ for odd n or even n respectively. There is no single number that the terms of the sequence approach as $n \to \infty$. So the sequence $\{s_n\} = \{\cos(\pi n)\}$ $\boxed{\textit{diverges.}}$

64. The sequence $\{s_n\} = \left\{\cos\left(\frac{\pi}{2}n\right)\right\}$ oscillates between $0, -1$ or $+1$ depending on whether n is odd, a multiple of 2, or a multiple of 4, respectively. There is no single number that the terms of the sequence approach as $n \to \infty$. So the sequence $\{s_n\} = \left\{\cos\left(\frac{\pi}{2}n\right)\right\}$ $\boxed{\textit{diverges.}}$

65. The sequence is $\{s_n\} = \{\sqrt{n}\}$. The terms of the sequence are $1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots$ Given any positive number $M > 1$, we choose a positive integer $N > M^2 > 1$. Then whenever $n > N > 1$, we have

$$s_n = \sqrt{n} > \sqrt{N} > M.$$

That is, the sequence $\{s_n\} = \{\sqrt{n}\}$ $\boxed{\textit{diverges}}$ to infinity.

66. The sequence is $\{s_n\} = \{n^2\}$. The terms of the sequence are $1, 4, 9, 16, \ldots$ Given any positive number $M > 1$, we choose a positive integer $N > \sqrt{M} > 1$. Then whenever $n > N > 1$, we have

$$s_n = n^2 > N^2 > M.$$

That is, the sequence $\{s_n\} = \{n^2\}$ $\boxed{\textit{diverges}}$ to infinity.

67. The sequence $\{s_n\} = \left\{\left(-\frac{1}{3}\right)^n\right\}$ $\boxed{\textit{converges}}$ to the limit of $\boxed{0}$ because $-1 < -\frac{1}{3} < 1$.

68. The sequence $\{s_n\} = \left\{\left(\frac{1}{3}\right)^n\right\}$ $\boxed{\textit{converges}}$ to the limit of $\boxed{0}$ because $-1 < \frac{1}{3} < 1$.

69. The sequence $\{s_n\} = \left\{\left(\frac{5}{4}\right)^n\right\}$ $\boxed{\textit{diverges}}$ to infinity because $\frac{5}{4} > 1$.

70. The sequence $\{s_n\} = \left\{\left(\frac{\pi}{2}\right)^n\right\}$ $\boxed{\textit{diverges}}$ to infinity because $\frac{\pi}{2} > 1$.

71. The sequence is $\{s_n\} = \left\{\frac{n+(-1)^n}{n}\right\}$. The terms of the sequence are $0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \ldots$ The sequence appears to approach the limit $L = 1$ by oscillating on either side of 1. Let $\epsilon > 0$. Then

$$|s_n - L| = \left|\frac{n+(-1)^n}{n} - 1\right| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n}.$$

We have $|s_n - L| < \epsilon$ if $n > [|\frac{1}{\epsilon}|] = N$. This establishes that the sequence $\boxed{\textit{converges}}$ to the limit $\boxed{1.}$ (*Note:* $[|x|]$ means the integer part of x.)

72. The sequence is $\{s_n\} = \left\{\frac{1}{n} + (-1)^n\right\}$. The terms of the sequence are $0, \frac{1}{2} + 1 = \frac{3}{2}, \frac{1}{3} - 1 = -\frac{2}{3}, \frac{1}{4} + 1 = \frac{5}{4}, \frac{1}{5} - 1 = -\frac{4}{5}, \ldots$ The terms of the sequence oscillate between positive and negative numbers, with the positive terms tending to the limit of 1 from above, and the negative terms tending to a limit of -1 from above. Since there is no single number that the terms of the sequence approach as $n \to \infty$, the sequence $\{s_n\} = \left\{\frac{1}{n} + (-1)^n\right\}$ $\boxed{\textit{diverges.}}$

73. The sequence $\{s_n\} = \left\{\frac{\ln n}{n}\right\}$. The sequence has limit

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{x}}{1} = 0,$$

using L'Hôpital rule. This proves the sequence is convergent. By the theorem in the middle of p.640, the sequence being convergent is bounded, that is, it is bounded $\boxed{\textit{both}}$ from above and from below.

74. The sequence is $\{s_n\} = \left\{\frac{\sin n}{n}\right\}$. We have the following bound

$$|s_n| = \left|\frac{\sin n}{n}\right| \le \frac{1}{n}$$

which means $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$. The sequence $\{s_n\}$ is bounded below by the sequence $\left\{-\frac{1}{n}\right\}$ and bounded above by the sequence $\left\{\frac{1}{n}\right\}$. As $n \to \infty$, both these bounding sequences converge to 0. This means the sequence $\{s_n\} = \left\{\frac{\sin n}{n}\right\}$ is convergent to the same limit by the Squeeze Theorem. By the theorem in the middle of p.640, the sequence being convergent is bounded, that is, it is bounded $\boxed{\textit{both}}$ from above and from below.

75. The sequence is $\{s_n\} = \left\{n + \frac{1}{n}\right\}$. As $n \to \infty$, $\frac{1}{n} \to 0$, but n grows without limit. So there is no upper bound. However, a lower bound for the sequence is 2 (the value of s_n for $n = 1$), in that all terms of the sequence exceed 2. So the sequence $\{s_n\} = \left\{n + \frac{1}{n}\right\}$ is bounded only from $\boxed{\textit{below.}}$

76. The sequence $\{s_n\} = \left\{\frac{3}{n+1}\right\}$ is bounded from below by 0 (the limit of s_n as $n \to \infty$), and bounded from above by $\frac{3}{2}$ (the value of s_n for $n = 1$, since the sequence is decreasing). So the sequence $\{s_n\} = \left\{\frac{3}{n+1}\right\}$ is bounded $\boxed{\textit{both}}$ from above and from below.

77. The sequence $\{s_n\} = \left\{\frac{n^2}{n+1}\right\}$ is bounded from below by $\frac{1}{2}$ (the value of s_n for $n = 1$), but it is not bounded from above since

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n^2}{n+1} = \lim_{x\to\infty} \frac{x^2}{x+1} = \lim_{x\to\infty} \frac{2x}{1} = \infty,$$

by using L'Hôpital's rule on a related function of the sequence. So the sequence $\{s_n\} = \left\{\frac{n^2}{n+1}\right\}$ is bounded only from $\boxed{\textit{below.}}$

78. The sequence $\{s_n\} = \left\{\frac{2^n}{n^2}\right\}$ is bounded from below. The terms of the sequence are $\frac{2}{1}, \frac{4}{4} = 1, \frac{8}{9}, \frac{16}{16} = 1, \frac{32}{25}, \ldots$ Note that the lower bound of the sequence is 1. It cannot be *less* than 1, since $2^n \ge n^2$ for all values of n. That this sequence has no upper bound is seen by examining

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2^n}{n^2} = \lim_{x\to\infty} \frac{2^x}{x^2} = \lim_{x\to\infty} \frac{2^x(\ln 2)}{2x} = \lim_{x\to\infty} \frac{2^x(\ln 2)^2}{2} = \infty,$$

by using L'Hôpital's rule on a related function of the sequence. So the sequence $\{s_n\} = \left\{\frac{2^n}{n^2}\right\}$ is bounded only from $\boxed{\textit{below.}}$

79. The terms of the sequence $\{s_n\} = \left\{\left(-\frac{1}{2}\right)^n\right\}$ are $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$ The sequence is bounded below by $-\frac{1}{2}$ and is bounded above by 0, since $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$, because $-1 \le -\frac{1}{2} \le 1$. So the sequence $\{s_n\} = \left\{\left(-\frac{1}{2}\right)^n\right\}$ is bounded $\boxed{\textit{both}}$ from above and from below.

80. The terms of sequence $\{s_n\} = \{\sqrt{n}\}$ are $1, \sqrt{2}, \sqrt{3}, \ldots$ The sequence is bounded from below by 1 but grows without limit as $n \to \infty$ (as was shown in Problem **65**), and is not bounded from above. So the sequence $\{s_n\} = \{\sqrt{n}\}$ is bounded only from $\boxed{\textit{below.}}$

81. The first three terms of the sequence $\{s_n\} = \left\{\frac{3^n}{(n+1)^3}\right\}$ are $s_1 = 0.375$, $s_2 \approx 0.333$, $s_3 \approx 0.422$. Since $s_1 > s_2$, but $s_2 < s_3$, the sequence is not monotonic. To see if it eventually increases, use the Algebraic Ratio Test on the sequence. We have

$$\frac{s_{n+1}}{s_n} = \frac{3^{n+1}/(n+2)^3}{3^n/(n+1)^3} = 3 \cdot \left(\frac{n+1}{n+2}\right)^3 > 1$$

when $\left(\frac{n+1}{n+2}\right)^3 > \frac{1}{3}$ or $\frac{n+1}{n+2} > \frac{1}{\sqrt[3]{3}}$. That is

$$\begin{aligned} \sqrt[3]{3}n + \sqrt[3]{3} &> n + 2 \\ (\sqrt[3]{3} - 1)n &> 2 - \sqrt[2]{3} \\ n &> \frac{2 - \sqrt[3]{3}}{\sqrt[3]{3} - 1} = \frac{0.5577...}{0.4422...} > 1. \end{aligned}$$

So for $n \geq 2$ the sequence $\{s_n\} = \left\{\frac{3^n}{(n+1)^3}\right\}$ will be a $\boxed{\textit{nonmonotonic}}$ and an eventually $\boxed{\textit{increasing}}$ one.

82. We use the Algebraic Difference Test on the sequence $\{s_n\} = \left\{\frac{2n+1}{n}\right\}$. We have

$$s_{n+1} - s_n = \frac{2n+3}{n+1} - \frac{2n+1}{n} = \frac{2n^2 + 3n - 2n^2 - 3n - 1}{n(n+1)} = -\frac{1}{n(n+1)} < 0,$$

whenever $n \geq 1$. This means the sequence $\{s_n\} = \left\{\frac{2n+1}{n}\right\}$ is a $\boxed{\textit{decreasing}}$ one for $n \geq 1$.

83. The sequence is $\left\{\frac{\ln n}{\sqrt{n}}\right\}$. A related function of the sequence is $f(x) = \frac{\ln x}{\sqrt{x}}$. Compute the derivative:

$$f'(x) = \frac{\sqrt{x} \cdot (\ln x)' - \ln x \cdot (\sqrt{x})'}{(\sqrt{x})^2} = \frac{\sqrt{x} \cdot \frac{1}{x} - \ln x \cdot \frac{1}{2\sqrt{x}}}{x} = \frac{1}{x^{3/2}} \left[1 - \frac{\ln x}{2}\right] < 0$$

when $1 - \frac{\ln x}{2} < 0$, or $\ln x > 2$ or $x > e^2 = 7.389....$ So the sequence $\left\{\frac{\ln n}{\sqrt{n}}\right\}$ is an eventually $\boxed{\textit{decreasing}}$ one for $n \geq 8$. This means that for $n < 8$, the sequence is increasing, so it is $\boxed{\textit{nonmonotonic.}}$

84. We use the Algebraic Ratio Test on the sequence $\{s_n\} = \left\{\frac{\sqrt{n+1}}{n}\right\}$. We have

$$\frac{s_{n+1}}{s_n} = \frac{\sqrt{n+2}/(n+1)}{\sqrt{n+1}/n} = \frac{\sqrt{n+2}}{\sqrt{n+1}} \cdot \frac{n}{(n+1)} < 1$$

whenever

$$\begin{aligned} \sqrt{n+2}(n) &< \sqrt{n+1}(n+1) \\ (n+2)n^2 &< (n+1)(n+1)^2 \\ n^3 + 2n^2 &< n^3 + 3n^2 + 3n + 1 \\ 0 &< n^2 + 3n + 1. \end{aligned}$$

This happens whenever $n \geq 1$. So the sequence $\{s_n\} = \left\{\frac{\sqrt{n+1}}{n}\right\}$ is a $\boxed{\textit{decreasing}}$ one for $n \geq 1$, and is $\boxed{\textit{monotonic}}$ as well.

85. The sequence is $\{s_n\} = \left\{\left(\frac{1}{3}\right)^n\right\}$. We use the Algebraic Ratio Test:

$$\frac{s_{n+1}}{s_n} = \frac{(1/3)^{n+1}}{(1/3)^n} = \frac{1}{3} < 1.$$

So the sequence $\{s_n\} = \left\{\left(\frac{1}{3}\right)^n\right\}$ is a $\boxed{\textit{decreasing}}$ one for $n \geq 1$, and is $\boxed{\textit{monotonic}}$ as well.

86. The sequence is $\{s_n\} = \left\{\frac{n^2}{5^n}\right\}$. We use the Algebraic Ratio Test:

$$\frac{s_{n+1}}{s_n} = \frac{(n+1)^2/5^{n+1}}{n^2/5^n} = \frac{(n+1)^2 5^n}{5^{n+1}n^2} = \frac{1}{5}\left(1+\frac{1}{n}\right)^2 < 1$$

whenever

$$\begin{aligned}\left(1+\frac{1}{n}\right)^2 &< 5\\ 1+\frac{1}{n} &< \sqrt{5}\\ \frac{1}{n} &< \sqrt{5}-1\\ n &> \frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4} = 0.809...\end{aligned}$$

So the sequence $\{s_n\} = \left\{\frac{n^2}{5^n}\right\}$ is a $\boxed{\textit{decreasing}}$ one for $n \geq 1$, and it is also $\boxed{\textit{monotonic.}}$

87. We use the Algebraic Ratio Test on the sequence $\{s_n\} = \left\{\frac{n!}{3^n}\right\}$. We have:

$$\frac{s_{n+1}}{s_n} = \frac{(n+1)!}{n!} \cdot \frac{3^n}{3^{n+1}} = \frac{1}{3} \cdot (n+1) \geq 1$$

whenever $n+1 \geq 3$, or $n \geq 2$. So the sequence $\{s_n\} = \left\{\frac{n!}{3^n}\right\}$ is an eventually $\boxed{\textit{nondecreasing}}$ one for $n \geq 2$. Since the sequence increases for $n < 2$, it is $\boxed{\textit{nonmonotonic.}}$

88. We use the Algebraic Ratio Test on the sequence $\{s_n\} = \left\{\frac{n!}{n^2}\right\}$. We have:

$$\frac{s_{n+1}}{s_n} = \frac{(n+1)!}{n!} \cdot \frac{n^2}{(n+1)^2} = n+1 \cdot \frac{n^2}{(n+1)^2} = \frac{n^2}{n+1} > 1$$

if $n^2 > n+1$. This happens whenever $n^2 - n - 1 > 0$ or $n > \frac{1+\sqrt{5}}{2} = 1.618....$ So the sequence $\{s_n\} = \left\{\frac{n!}{n^2}\right\}$ is an $\boxed{\textit{increasing}}$ one for $n \geq 2$. Since the sequence is decreasing for $n < 2$, it is $\boxed{\textit{nonmonotonic.}}$

89. The sequence $\{s_n\} = \{ne^{-n}\}$ is bounded from below by zero, since its terms are positive for all n. Examine the derivative of a related function $f(x) = xe^{-x}$ of the sequence :

$$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) < 0$$

if $x > 1$. This shows the sequence is a decreasing one for all $n \geq 1$. Since the sequence is bounded from below and is decreasing, it will $\boxed{\textit{converge.}}$

90. The sequence is $\{s_n\} = \{\tan^{-1} n\}$. Since $\tan^{-1} n < \frac{\pi}{2}$ for any positive integer n, the sequence is bounded from above by $\frac{\pi}{2}$. Consider a related function of the sequence $f(x) = \tan^{-1} x$. Since $f'(x) = \frac{1}{1+x^2} > 0$ when $x > 0$, this shows the sequence is increasing for $n \geq 1$. Since the sequence is increasing and is bounded from above, it will $\boxed{\textit{converge.}}$

91. The sequence is $\{s_n\} = \left\{\frac{n}{n+1}\right\}$. Since $n < n+1$ for all $n \geq 1$ the sequence is bounded from above by 1. Using the Algebraic Ratio Test,

$$\frac{s_{n+1}}{s_n} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} > 1$$

provided

$$\begin{aligned} (n+1)^2 &> n(n+2) \\ n^2 + 2n + 1 &> n^2 + 2n \\ 1 &> 0. \end{aligned}$$

So the sequence is always increasing for any $n \geq 1$. Since the sequence is bounded from above and is increasing, it will $\boxed{\textit{converge.}}$

92. The sequence is $\{s_n\} = \left\{\frac{n}{n^2+1}\right\}$. The sequence is bounded from below by 0 since the terms of the sequence are positive for all n. Consider a related function of the sequence, $f(x) = \frac{x}{x^2+1}$. Now its derivative

$$f'(x) = \frac{(x^2+1)x' - x(x^2+1)'}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \leq 0$$

for $x \geq 1$. Since a related function of the sequence is nonincreasing for $x \geq 1$ this shows the sequence is nonincreasing for $n \geq 1$. Since the sequence is bounded from below and is nonincreasing, it will $\boxed{\textit{converge.}}$

93. The sequence $\{s_n\} = \left\{2 - \frac{1}{n}\right\}$ is bounded from above by 2 since $\frac{1}{n} < 1$ for all values of n. Also, this sequence is increasing since the Algebraic Difference Test gives

$$s_{n+1} - s_n = 2 - \frac{1}{n+1} - 2 + \frac{1}{n} = \frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} > 0$$

for $n \geq 1$. Since the sequence is bounded from above and is increasing, it will $\boxed{\textit{converge.}}$

94. The sequence $\{s_n\} = \left\{\frac{n}{2^n}\right\}$ is bounded from below by 0 since its terms are positive for all n. To show the sequence is monotonic, we use the Algebraic Ratio Test:

$$\frac{s_{n+1}}{s_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2}\left(1 + \frac{1}{n}\right) \leq 1$$

whenever $1 + \frac{1}{n} \leq 2$, or $\frac{1}{n} \leq 1$, or $n \geq 1$. This shows the sequence is nonincreasing for $n \geq 1$. Since the sequence is bounded from below and is nonincreasing, it will $\boxed{\textit{converge.}}$

95. The sequence $\{s_n\} = \left\{\frac{3}{n} + 6\right\}$ is bounded from below by 6 since $\frac{3}{n} > 0$ for all n. To check the monotonicity of the sequence, we use the Algebraic Difference Test:

$$s_{n+1} - s_n = \frac{3}{n+1} + 6 - \frac{3}{n} - 6 = -\frac{3}{n(n+1)} < 0$$

whenever $n \geq 1$. This shows the sequence is decreasing. Since the sequence is decreasing and is bounded from below, it will $\boxed{\textit{converge.}}$ Next, compute

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{3}{n}+6\right) = \lim_{n\to\infty}\frac{3}{n} + \lim_{n\to\infty} 6 = 0+6=6,$$

where the sum and difference property of convergent sequences was used. So the limit to which the sequence will converge is $\boxed{6.}$

96. The sequence $\{s_n\} = \left\{2-\frac{4}{n}\right\}$ is bounded from above by 2 since $\frac{4}{n} > 0$ for all n. To check the monotonicity of the sequence, we use the Algebraic Difference Test:

$$s_{n+1} - s_n = 2 - \frac{4}{n+1} - 2 + \frac{4}{n} = \frac{4}{n(n+1)} > 0$$

for $n \geq 1$. This shows the sequence is an increasing one. Since the sequence is bounded from above by 2 and increasing, it will $\boxed{\textit{converge.}}$ Next, compute

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(2-\frac{4}{n}\right) = \lim_{n\to\infty} 2 - 4\cdot\lim_{n\to\infty}\frac{1}{n} = 2-4\cdot 0 = 2,$$

where the sum and difference and the constant multiple properties of convergent sequences have been used. So the limit to which the sequence will converge is $\boxed{2.}$

97. The sequence is $\{s_n\} = \left\{\ln\left(\frac{n+1}{3n}\right)\right\}$. The sequence $\{a_n\} = \left\{\frac{n+1}{3n}\right\} = \left\{\frac{1}{3}+\frac{1}{3n}\right\}$ is bounded from below by $\frac{1}{3}$ since $\frac{1}{3n} > 0$ for all n. Also, the sequence $\{a_n\}$ is decreasing as can be seen by the Algebraic Difference Test:

$$a_{n+1} - a_n = \frac{1}{3} + \frac{1}{3(n+1)} - \frac{1}{3} - \frac{1}{3n} = -\frac{1}{3n(n+1)} < 0$$

for $n \geq 1$. So the sequence $\{a_n\}$, being bounded from below and decreasing, converges. To find the limit of the sequence $\{a_n\}$, we compute:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{3n}\right) = \lim_{n\to\infty}\frac{1}{3} + \lim_{n\to\infty}\frac{1}{3n} = \frac{1}{3}+0 = \frac{1}{3},$$

where the sum and difference property of convergent sequences has been used. Since $f(x) = \ln x$ is continuous at $x = \frac{1}{3}$, then by the theorem on p.637, the sequence $\{f(a_n)\} = \{s_n\}$ will also $\boxed{\textit{converge}}$, and to the limit of $\ln\left(\frac{1}{3}\right) = \boxed{-\ln 3.}$

98. The sequence is $\{s_n\} = \left\{\cos\left(n\pi + \frac{\pi}{2}\right)\right\}$. For any value of n, the value of $s_n = 0$. This means the sequence is bounded from below, is nonincreasing, and so it $\boxed{\textit{converges}}$ to the limit of $\boxed{0.}$

99. The sequence $\{s_n\} = \{(-1)^n\sqrt{n}\}$ has terms $-1, \sqrt{2}, -\sqrt{3}, \sqrt{4}, -\sqrt{5}, \ldots$ As $n \to \infty$, the sequence oscillates between increasing positive and negative terms without bound. So the sequence $\boxed{\textit{diverges.}}$

100. The sequence is $\{s_n\} = \left\{\frac{(-1)^n}{2n}\right\}$. Consider

$$|s_n| = \left|\frac{(-1)^n}{2n}\right| \leq \frac{1}{2n}$$

$$\text{that is, } -\frac{1}{2n} \leq \frac{(-1)^n}{2n} \leq \frac{1}{2n}.$$

If $\{a_n\} = \left\{-\frac{1}{2n}\right\}$ and $\{b_n\} = \left\{\frac{1}{2n}\right\}$, we have $a_n \le s_n \le b_n$. Also, $\{a_n\}$ and $\{b_n\}$ are convergent sequences because 0 is an upper bound for the increasing sequence $\{a_n\}$ and 0 is a lower bound for the decreasing sequence $\{b_n\}$. By the Squeeze Theorem, this shows that the sequence $\{s_n\} = \left\{\frac{(-1)^n}{2n}\right\}$ also $\boxed{\textit{converges}}$ and to the same limit of $\boxed{0.}$

101. The sequence $\{s_n\} = \left\{\frac{3^n+1}{4^n}\right\} = \left\{\left(\frac{3}{4}\right)^n + \frac{1}{4^n}\right\}$ is bounded from below by 0 since the nth term is positive for all n. To check monotonicity, we use the Algebraic Difference Test:

$$s_{n+1} - s_n = \frac{3^{n+1}+1}{4^{n+1}} - \frac{3^n+1}{4^n} = \frac{3\cdot 3^n + 1 - 4\cdot 3^n - 4}{4^{n+1}} = \frac{-3^n - 3}{4^{n+1}} < 0$$

for $n \ge 1$. This means the sequence is decreasing. Since the sequence $\{s_n\} = \left\{\frac{3^n+1}{4^n}\right\}$ is bounded from below and decreasing, it $\boxed{\textit{converges.}}$ To find the limit, compute:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{3}{4}\right)^n + \lim_{n\to\infty}\frac{1}{4^n} = 0 + 0 = 0,$$

since $-1 \le \frac{3}{4} \le 1$. So the sequence converges to a limit of $\boxed{0.}$

102. The sequence is $\{s_n\} = \left\{n + \sin\frac{1}{n}\right\}$. As $\lim_{n\to\infty} n = \infty$, the sequence will $\boxed{\textit{diverge.}}$

103. The sequence $\{s_n\} = \left\{\frac{\ln(n+1)}{n+1}\right\}$ is bounded from below by 0 since $\ln(n+1) > 0$ for all $n \ge 1$. A related function of the sequence is $f(x) = \frac{\ln(x+1)}{x+1}$. Now since

$$f'(x) = \frac{(x+1)\frac{1}{x+1} - \ln(x+1)\cdot 1}{(x+1)^2} = \frac{1 - \ln(x+1)}{(x+1)^2} < 0$$

for $x > e - 1$, it means the function is decreasing, and so the sequence is decreasing as well for $n \ge 2$. Since the sequence is bounded from below and decreasing, it $\boxed{\textit{converges}}$. To find the limit, compute:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{\ln(n+1)}{n+1} = \lim_{x\to\infty}\frac{\ln(x+1)}{x+1} = \lim_{x\to\infty}\frac{\frac{1}{x+1}}{1} = 0$$

using L'Hôpital's rule. So the sequence converges to a limit of $\boxed{0.}$

104. A related function of the sequence $\{s_n\} = \left\{\frac{\ln(n+1)}{\sqrt{n}}\right\}$ is $f(x) = \frac{\ln(x+1)}{\sqrt{x}}$. By the Theorem on p.637, we have

$$\lim_{n\to\infty} s_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{\ln(x+1)}{\sqrt{x}} = \lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty}\frac{2\sqrt{x}}{x+1} = \lim_{x\to\infty}\frac{2\cdot\frac{1}{\sqrt{x}}}{1} = 0,$$

using L'Hôpital's rule. So the sequence converges to a limit of $\boxed{0.}$

105. The sequence $\{s_n\} = \{0.5^n\}$ $\boxed{\textit{converges}}$ to a limit of $\boxed{0}$, since $-1 < 0.5 < 1$.

106. The sequence $\{s_n\} = \{(-2)^n\}$ $\boxed{\textit{diverges}}$, since $-2 < -1$.

107. The sequence $\{s_n\} = \left\{\cos\frac{\pi}{n}\right\}$ is bounded from above by 1 since $\left|\cos\frac{\pi}{n}\right| \le 1$. A related function of the sequence is $f(x) = \cos\frac{\pi}{x}$. Since

$$f'(x) = -\sin\frac{\pi}{x}\cdot\left(-\frac{\pi}{x^2}\right) = \frac{\pi}{x^2}\sin\frac{\pi}{x} > 0$$

for $x > 1$, the function is increasing, which means the sequence is increasing for $n \geq 1$. Since the sequence is bounded above and increasing, it $\boxed{\textit{converges.}}$ To find the limit, compute:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \cos\frac{\pi}{n} = \cos 0 = 1.$$

So the sequence converges to a limit of $\boxed{1.}$

108. The sequence $\{s_n\} = \left\{\sin\frac{\pi}{n}\right\}$ is bounded from below by 0 since $\sin\frac{\pi}{n} > 0$ for all $n \geq 1$. A related function of the sequence is $f(x) = \sin\frac{\pi}{x}$. Since

$$f'(x) = \cos\frac{\pi}{x} \cdot \left(-\frac{\pi}{x^2}\right) = -\frac{\pi}{x^2}\cos\frac{\pi}{x} < 0$$

for $x > 2$, the function is decreasing, which means the sequence is decreasing for $n \geq 2$. Since the sequence $\{s_n\} = \left\{\sin\frac{\pi}{n}\right\}$ is bounded below and decreasing, it $\boxed{\textit{converges.}}$ To find the limit, compute:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sin\frac{\pi}{n} = \sin 0 = 0.$$

So the sequence converges to a limit of $\boxed{0.}$

109. The sequence is $\{s_n\} = \left\{\cos\left(\frac{n}{e^n}\right)\right\}$. Consider the sequence $\{b_n\} = \left\{\frac{n}{e^n}\right\}$. The sequence $\{b_n\}$ is bounded from below by 0 since its terms are positive for all $n \geq 1$. A related function of the sequence $\{b_n\}$ is $f(x) = \frac{x}{e^x}$. We have

$$f'(x) = \frac{e^x - xe^x}{e^{2x}} = \frac{e^x(1-x)}{e^{2x}} < 0$$

if $x > 1$, which shows the function $f(x)$ is decreasing. So the sequence $\{b_n\}$ is decreasing for $n > 1$. This establishes that the sequence $\{b_n\}$ converges since it is bounded from below and is decreasing. To find the limit of the sequence $\{b_n\}$, compute:

$$\lim_{n\to\infty} b_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0,$$

using L'Hôpital's rule. Since $g(x) = \cos x$ is continuous at $x = 0$, from the theorem on p.636, it follows that the sequence $\{g(b_n)\} = \{s_n\} = \left\{\cos\left(\frac{n}{e^n}\right)\right\}$ also $\boxed{\textit{converges}}$, to the value $\cos 0 = \boxed{1.}$

110. The sequence is $\{s_n\} = \left\{\sin\left(\frac{(n+1)^3}{e^n}\right)\right\}$. Consider the sequence $\{b_n\} = \left\{\frac{(n+1)^3}{e^n}\right\}$. The sequence $\{b_n\}$ is bounded from below by 0 since its terms are positive for all $n \geq 1$. A related function of the sequence $\{b_n\}$ is $f(x) = \frac{(x+1)^3}{e^x}$. We have

$$f'(x) = \frac{e^x \cdot 3(x+1)^2 - (x+1)^3 \cdot e^x}{e^{2x}} = \frac{e^x(x+1)^2[3-(x+1)]}{e^{2x}} \leq 0$$

whenever $x \geq 2$, which shows the function $f(x)$ is a nonincreasing function. So $\{b_n\}$ is a nonincreasing sequence for $n \geq 2$. This establishes that the sequence $\{b_n\}$ converges, since it is bounded from below and nonincreasing. To find the limit of the sequence $\{b_n\}$, compute:

$$\lim_{n\to\infty} b_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(x+1)^3}{e^x} = \lim_{x\to\infty} \frac{3(x+1)^2}{e^x} = \lim_{x\to\infty} \frac{3\cdot 2(x+1)}{e^x} = \lim_{x\to\infty} \frac{3\cdot 2\cdot 1}{e^x} = 0,$$

using L'Hôpital's rule. Since $g(x) = \sin x$ is continuous at $x = 0$, from the theorem on p.636, the sequence $\{g(b_n)\} = \{s_n\} = \left\{\sin\left(\frac{(n+1)^3}{e^n}\right)\right\}$ also $\boxed{\textit{converges}}$, with a limit $\sin 0 = \boxed{0.}$

111. The sequence is $\{s_n\} = \left\{e^{1/n}\right\}$. Consider the sequence $\{b_n\} = \left\{\frac{1}{n}\right\}$. The sequence $\{b_n\}$ is bounded from below by 0 since its terms are positive for all $n \geq 1$. Since

$$\frac{b_{n+1}}{b_n} = \frac{n}{n+1} < 1$$

when $n \geq 1$, the sequence $\{b_n\}$ is decreasing according to the Algebraic Ratio Test. Because the sequence $\{b_n\}$ is bounded frrom below and decreasing, it converges. To find the limit of the sequence $\{b_n\}$, compute:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Now the function $f(x) = e^x$ is continuous at $x = 0$. So by the theorem on p.636, this means the sequence $\{f(b_n)\} = \{s_n\} = \left\{e^{1/n}\right\}$ also $\boxed{\textit{converges}}$, to a limit $e^0 = \boxed{1.}$

112. The sequence is $\{s_n\} = \left\{\frac{e^n}{n}\right\}$. Consider a related function of the sequence, $f(x) = \frac{e^x}{x}$. We have using L'Hôpital's rule:

$$\lim_{n\to\infty} s_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty.$$

This means the sequence $\{s_n\} = \left\{\frac{e^n}{n}\right\}$ $\boxed{\textit{diverges.}}$

113. The sequence $\{s_n\} = \left\{1 + \left(\frac{1}{2}\right)^n\right\}$ is bounded from below by 1, since $\left(\frac{1}{2}\right)^n > 0$ for all $n \geq 1$. Using the Algebraic Difference Test, we have

$$s_{n+1} - s_n = 1 + \frac{1}{2^{n+1}} - 1 - \frac{1}{2^n} = \frac{1}{2 \cdot 2^n} - \frac{1}{2^n} = -\frac{1}{2 \cdot 2^n} = -\frac{1}{2^{n+1}} < 0$$

for $n \geq 1$. This shows the sequence is a decreasing one. Since the sequence $\{s_n\} = \left\{1 + \left(\frac{1}{2}\right)^n\right\}$ is bounded from below and decreasing, it $\boxed{\textit{converges.}}$ To find the limit, compute:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left[1 + \left(\frac{1}{2}\right)^n\right] = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 1 + 0 = 1,$$

since $-1 < \frac{1}{2} < 1$. So the sequence converges to a limit of $\boxed{1.}$

114. The sequence $\{s_n\} = \left\{1 - \left(\frac{1}{2}\right)^n\right\}$ is bounded from above by 1, since $\left(\frac{1}{2}\right)^n > 0$ for all $n \geq 1$. Using the Algebraic Difference Test, we have

$$s_{n+1} - s_n = 1 - \frac{1}{2^{n+1}} - 1 + \frac{1}{2^n} = -\frac{1}{2 \cdot 2^n} + \frac{1}{2^n} = \frac{1}{2 \cdot 2^n} = \frac{1}{2^{n+1}} > 0$$

for $n \geq 1$. This shows the sequence is an increasing one. Since the sequence $\{s_n\} = \left\{1 - \left(\frac{1}{2}\right)^n\right\}$ is bounded from above and increasing, it $\boxed{\textit{converges.}}$ To find the limit, compute:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left[1 - \left(\frac{1}{2}\right)^n\right] = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 1 - 0 = 1,$$

since $-1 < \frac{1}{2} < 1$. So the sequence converges to a limit of $\boxed{1.}$

Applications and Extensions

115. A related function of the sequence $s_n = \frac{(\ln n)^2}{n}$ is $f(x) = \frac{(\ln x)^2}{x}$.
If $\lim\limits_{x\to\infty} f(x) = L$ then $\lim\limits_{x\to\infty} s_n = L$.

$$\lim_{x\to\infty} s_n = \lim_{x\to\infty} \frac{(\ln x)^2}{x}$$

This is in an indeterminate form of $\frac{\infty}{\infty}$ so L'Hôpital's Rule is applicable as follows.

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2(\ln x)}{x} = \lim_{x\to\infty} \frac{2}{x} = 0$$
$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} = \boxed{0}$$

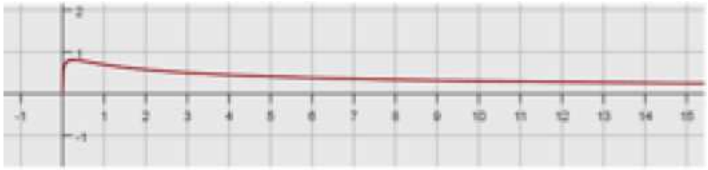
116. A related function of the sequence $s_n = \sqrt{n}\,\ln\frac{n+1}{n}$ is $f(x) = \sqrt{x}\,\ln\left(\frac{x+1}{x}\right)$.
If $\lim\limits_{x\to\infty} f(x) = L$ then $\lim\limits_{x\to\infty} s_n = L$.

$$\lim_{x\to\infty} s_n = \lim_{x\to\infty}\left[\sqrt{x}\,\ln\left(\frac{x+1}{x}\right)\right] = \lim_{x\to\infty}\frac{\ln\left(\frac{x+1}{x}\right)}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{\sqrt{x}}}$$

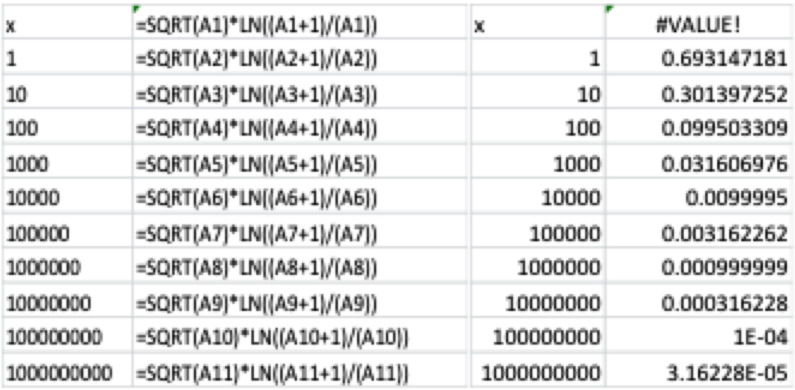
This is in an indeterminate form of $\frac{0}{0}$ so L'Hôpital's Rule is applicable as follows.

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\frac{-1}{x^2}}{\frac{-1}{2}\cdot x^{-3/2}} = \lim_{x\to\infty}\frac{-2\cdot\frac{1}{1+\frac{1}{x}}}{x^{1/2}} = -2\lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x}\right)\cdot x^{1/2}} = 0$$
$$\lim_{x\to\infty}\sqrt{n}\,\ln\frac{n+1}{n} = \boxed{0}$$

Below is a graph of $f(x) = \sqrt{x}\,\ln\left(\frac{x+1}{x}\right)$. Note the behavior of the graph: as the x-values approach infinity, the y-values approach the asymptote at $y = 0$.



Below is a numerical analysis of $\lim\limits_{x\to\infty}\left[\sqrt{x}\,\ln\left(\frac{x+1}{x}\right)\right]$. Note that as x values increase to large values, $\sqrt{x}\,\ln\left(\frac{x+1}{x}\right)$ becomes extremely small, approaching 0.



x	=SQRT(A1)*LN((A1+1)/(A1))	x	#VALUE!
1	=SQRT(A2)*LN((A2+1)/(A2))	1	0.693147181
10	=SQRT(A3)*LN((A3+1)/(A3))	10	0.301397252
100	=SQRT(A4)*LN((A4+1)/(A4))	100	0.099503309
1000	=SQRT(A5)*LN((A5+1)/(A5))	1000	0.031606976
10000	=SQRT(A6)*LN((A6+1)/(A6))	10000	0.0099995
100000	=SQRT(A7)*LN((A7+1)/(A7))	100000	0.003162262
1000000	=SQRT(A8)*LN((A8+1)/(A8))	1000000	0.000999999
10000000	=SQRT(A9)*LN((A9+1)/(A9))	10000000	0.000316228
100000000	=SQRT(A10)*LN((A10+1)/(A10))	100000000	1E-04
1000000000	=SQRT(A11)*LN((A11+1)/(A11))	1000000000	3.16228E-05

117. The sequence $\{s_n\} = \left\{\frac{n^2 \tan^{-1} n}{n^2+1}\right\}$ is bounded from above since

$$\frac{n^2 \tan^{-1} n}{n^2+1} < \tan^{-1} n < \frac{\pi}{2}$$

for all $n \geq 1$. A related function of the sequence is $f(x) = \frac{x^2 \tan^{-1} x}{x^2+1}$. Since

$$f'(x) = \frac{(x^2+1)[2x \tan^{-1} x + \frac{x^2}{1+x^2}] - x^2 \tan^{-1} x \cdot 2x}{(x^2+1)^2} = \frac{2x \tan^{-1} x + x^2}{(x^2+1)^2} > 0$$

for $x > 1$, the function is increasing, which means the sequence $\{s_n\}$ is increasing for $n \geq 1$. Since the sequence is bounded from above and increasing, it $\boxed{\textit{converges.}}$

118. The sequence $\{s_n\} = \left\{n \sin \frac{1}{n}\right\}$ has an upper bound of 1 since $\left|\frac{\sin n}{n}\right| \leq \frac{1}{n}$. A related function of the sequence is $f(x) = x \sin \frac{1}{x}$. We have

$$f'(x) = \sin \frac{1}{x} + x \cdot \cos \frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} > 0$$

whenever $\tan \frac{1}{x} > \frac{1}{x}$. This inequality is seen to be satisfied for all $x \geq 1$ as follows: Let $g(y) = \tan y - y$. Then $g'(y) = \sec^2 y - 1 \geq 0$ when $\frac{1}{\cos^2 y} \geq 1$ or $\cos^2 y \leq 1$, which is true for all $y \geq 0$. This shows that $g(y)$ is an increasing function for all $y \in [0, \frac{\pi}{2})$. Since $g(0) = 0$, this means that $g(y) \geq 0$ for all $y \in [0, \frac{\pi}{2})$, that is, $\tan y \geq y$. If $y = \frac{1}{x}$, then this means that $g(x) = \tan \frac{1}{x} - \frac{1}{x}$ is increasing (at least) in the interval $x \in [1, \infty)$, and since $g(1) = \tan 1 - 1 \approx 0.557 > 0$, it follows that $g(x) > 0$, or $\tan \frac{1}{x} > \frac{1}{x}$ for $x \in [1, \infty)$. That is, the function $f(x)$ is increasing, and the sequence in turn is increasing for $n \geq 1$. Since the sequence $\{s_n\} = \left\{n \sin \frac{1}{n}\right\}$ is bounded from above and increasing, it $\boxed{\textit{converges.}}$

<u>*Alternative Solution:*</u> We invoke the Theorem on p.637, and compute the limit on a related function $f(x) = x \sin \frac{1}{x}$ as follows:

$$\lim_{n\to\infty} s_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} x \sin \frac{1}{x} = \lim_{y\to 0} \frac{1}{y} \sin y = 1,$$

where $y = \frac{1}{x}$, and using the standard limit result that $\lim\limits_{y\to 0} \frac{\sin y}{y} = 1$. Since a related function of the sequence converges to a finite limit, it means the sequence also $\boxed{\textit{converges}}$ (and converges to the same limit).

119. The sequence is $\{s_n\} = \left\{\frac{n+\sin n}{n+\cos(4n)}\right\}$. Consider $\lim\limits_{n\to\infty} \frac{\sin n}{n}$. Since $\left|\frac{\sin n}{n}\right| \leq \frac{1}{n}$, we have $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$. Since $\lim\limits_{n\to\infty} \pm\frac{1}{n} = 0$, by the Squeeze theorem, $\lim\limits_{n\to\infty} \frac{\sin n}{n} = 0$. Similarly, since $\left|\frac{\cos(4n)}{n}\right| \leq \frac{1}{n}$, we have $-\frac{1}{n} \leq \frac{\cos(4n)}{n} \leq \frac{1}{n}$, and since $\lim\limits_{n\to\infty} \pm\frac{1}{n} = 0$, by the Squeeze theorem, $\lim\limits_{n\to\infty} \frac{\cos(4n)}{n} = 0$. We have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n+\sin n}{n+\cos(4n)} = \lim_{n\to\infty} \frac{1+\frac{\sin n}{n}}{1+\frac{\cos(4n)}{n}} = \frac{1+\lim\limits_{n\to\infty} \frac{\sin n}{n}}{1+\lim\limits_{n\to\infty} \frac{\cos(4n)}{n}} = \frac{1+0}{1+0} = 1.$$

The terms of the sequence oscillate on either side of 1, while this calculation shows that the sequence $\boxed{\textit{converges}}$ to a limit of 1.

120. The sequence is $\{s_n\} = \left\{\frac{n^2}{2n+1}\sin\frac{1}{n}\right\}$. A related function of the sequence is $f(x) = \frac{x^2}{2x+1}\sin\frac{1}{x}$. We have

$$\begin{aligned}\lim_{n\to\infty} s_n &= \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{x^2}{2x+1}\sin\frac{1}{x}\\ &= \lim_{x\to\infty}\frac{x}{2x+1}\frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{y\to 0}\frac{1}{2+y}\frac{\sin y}{y} \qquad (\text{Use } y=\tfrac{1}{x})\\ &= \lim_{y\to 0}\frac{1}{2+y}\cdot\lim_{y\to 0}\frac{\sin y}{y} = \frac{1}{2}\cdot 1 = \frac{1}{2}.\end{aligned}$$

So the sequence $\{s_n\} = \left\{\frac{n^2}{2n+1}\sin\frac{1}{n}\right\}$ $\boxed{\textit{converges}}$ to a limit of $\frac{1}{2}$.

121. The sequence is $\{s_n\} = \{\ln n - \ln(n+1)\} = \left\{\ln\frac{n}{n+1}\right\}$. The sequence $\{b_n\} = \left\{\frac{n}{n+1}\right\}$ is bounded from above by 1 since $\frac{n}{n+1} < 1$ for all $n \geq 1$.The sequence $\{b_n\}$ is increasing by the Algebraic Ratio Test:

$$\frac{b_{n+1}}{b_n} = \frac{n+1}{n+2}\frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n} > 1$$

for $n \geq 1$. So the sequence $\{b_n\}$, being both bounded from above and increasing, is convergent. To find the limit of the sequence $\{b_n\}$, compute:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1.$$

Since $f(x) = \ln x$ is a continuous function at $x = 1$, by the theorem on p.636, the sequence $\{f(b_n)\} = \{s_n\} = \{\ln n - \ln(n+1)\} = \left\{\ln\frac{n}{n+1}\right\}$ $\boxed{\textit{converges}}$ to the limit $\ln 1 = 0$.

122. The sequence is $\{s_n\} = \left\{\ln n^2 + \ln\frac{1}{n^2+1}\right\} = \left\{\ln\left(\frac{n^2}{n^2+1}\right)\right\}$. The sequence $\{b_n\} = \left\{\frac{n^2}{n^2+1}\right\}$ is bounded from above by 1 since $\frac{n^2}{n^2+1} < 1$ for all $n \geq 1$. The sequence $\{b_n\}$ is increasing by the Algebraic Ratio Test:

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)^2}{(n+1)^2+1}\cdot\frac{(n^2+1)}{n^2} = \frac{n^2(n+1)^2+(n+1)^2}{n^2[(n+1)^2+1]} = \frac{n^2(n+1)^2+n^2+2n+1}{n^2(n+1)^2+n^2} > 1$$

for $n \geq 1$. So the sequence $\{b_n\}$, being both bounded from above and increasing, is convergent. To find the limit of the sequence $\{b_n\}$, compute:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n^2}{n^2+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}} = \frac{1}{1+0} = 1.$$

Since $f(x) = \ln x$ is continuous at $x = 1$, by the theorem on p.636, the sequence $\{f(b_n)\} = \{s_n\} = \left\{\ln n^2 + \ln\frac{1}{n^2+1}\right\}$ $\boxed{\textit{converges}}$ to the limit $\ln 1 = 0$.

123. The sequence is $\{s_n\} = \left\{\frac{n^2}{\sqrt{n^2+1}}\right\}$. Since

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{n^2}{\sqrt{n^2+1}} = \lim_{n\to\infty}\frac{n}{\sqrt{1+\frac{1}{n^2}}} = \infty,$$

this sequence $\boxed{\textit{diverges.}}$

124. The sequence is $\{s_n\} = \left\{\frac{5^n}{(n+1)^2}\right\}$. Using the L'Hôpital rule on a related function of the sequence we have:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{5^n}{(n+1)^2} = \lim_{x\to\infty} \frac{5^x}{(x+1)^2} = \lim_{x\to\infty} \frac{5^x \ln 5}{2(x+1)} = \lim_{x\to\infty} \frac{5^x (\ln 5)^2}{2} = \infty.$$

This sequence $\boxed{\textit{diverges.}}$

125. The sequence is $\{s_n\} = \left\{\frac{2^n}{(2)(4)(6)\cdots(2n)}\right\} = \left\{\frac{2^n}{2^n(1\cdot 2\cdot 3\cdots n)}\right\} = \left\{\frac{1}{n!}\right\}$. It is bounded from below by 0 since $\frac{1}{n!} > 0$ for all $n \geq 1$. Using the Algebraic Ratio Test, we have

$$\frac{s_{n+1}}{s_n} = \frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1} < 1$$

for $n \geq 1$. This shows the sequence is decreasing. Since the sequence $\{s_n\} = \left\{\frac{2^n}{(2)(4)(6)\cdots(2n)}\right\}$ is bounded from below and decreasing, it will $\boxed{\textit{converge.}}$

126. The sequence is $\{s_n\} = \left\{\frac{3^{n+1}}{(3)(6)(9)\cdots(3n)}\right\} = \left\{\frac{3^{n+1}}{3^n\cdot n!}\right\} = \left\{\frac{3}{n!}\right\}$. It is bounded from below by 0 since $\frac{3}{n!} > 0$ for all $n \geq 1$.Using the algebraic ratio test, we have

$$\frac{s_{n+1}}{s_n} = \frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1} < 1$$

for $n \geq 1$. This shows the sequence is decreasing. Since the sequence $\{s_n\} = \left\{\frac{3^{n+1}}{(3)(6)(9)\cdots(3n)}\right\}$ is bounded from below and decreasing, it will $\boxed{\textit{converge.}}$

127. Let $f(x) = x^2 + x\cos x + 1$. Completing the square yields

$$f(x) = \left(x + \frac{\cos x}{2}\right)^2 + 1 - \frac{\cos^2 x}{4}$$

so that

$$f(x) \geq 1 - \frac{\cos^2 x}{4} \geq \frac{3}{4} > 0$$

since $\left(x + \frac{\cos x}{2}\right)^2 \geq 0$ and $\cos^2 x \leq 1$ for all x. Moreover,

$$\begin{aligned} f'(x) &= 2x - x\sin x + \cos x \\ &= x(2 - \sin x) + \cos x \\ &\geq x + \cos x && \text{since } 2 - \sin x \geq 1 \text{ because } |\sin x| \leq 1; \\ &\geq x - 1 && \text{since } -1 \leq \cos x \leq 1; \\ &> 0 \end{aligned}$$

for $x > 1$. This shows that $f(x)$ is a positive and increasing function of x for $x > 1$. It follows that

$$g(x) = \frac{1}{f(x)} = \frac{1}{x^2 + x\cos x + 1}$$

is a positive and decreasing function (because $g'(x) = -\frac{1}{f(x)^2} \cdot f'(x) < 0$ since $f'(x) > 0$) for $x > 1$. As $g(x)$ is a related function of the sequence $\{s_n\} = \left\{\frac{1}{n^2 + n\cos n + 1}\right\}$, the sequence $\{s_n\}$ is decreasing and bounded from below by 0, so it $\boxed{\textit{converges.}}$

128. **(a)** The first eight terms of the Fibonacci sequence are $\boxed{1, 1, 2, 3, 5, 8, 13, 21}$.

(b) We verify that the nth term given satisfies the recursive definition of the Fibonacci sequence:

1.
$$u_1 = \frac{(1+\sqrt{5})^1 - (1-\sqrt{5})^1}{2^1\sqrt{5}} = \frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1.$$

2.
$$u_2 = \frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2\sqrt{5}} = \frac{(1+2\sqrt{5}+5) - (1-2\sqrt{5}+5)}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1.$$

3.
$$\begin{aligned}
u_{n+2} - u_n &= \frac{(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}}{2^{n+2}\sqrt{5}} - \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^2(1+\sqrt{5})^n - (1-\sqrt{5})^2(1-\sqrt{5})^n}{4(2^n\sqrt{5})} - \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} \\
&= \frac{(1+5+2\sqrt{5})(1+\sqrt{5})^n - (1+5-2\sqrt{5})(1-\sqrt{5})^n}{4(2^n\sqrt{5})} - \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} \\
&= \frac{(3+\sqrt{5})(1+\sqrt{5})^n - (3-\sqrt{5})(1-\sqrt{5})^n}{2(2^n\sqrt{5})} - \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} \\
&= \frac{(3+\sqrt{5})(1+\sqrt{5})^n - (3-\sqrt{5})(1-\sqrt{5})^n - 2(1+\sqrt{5})^n + 2(1-\sqrt{5})^n}{2(2^n\sqrt{5})} \\
&= \frac{(3-2+\sqrt{5})(1+\sqrt{5})^n - (3-2-\sqrt{5})(1-\sqrt{5})^n}{2^{n+1}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} = u_{n+1}.
\end{aligned}$$

That is, $u_{n+2} = u_{n+1} + u_n$. This means that the nth term of the Fibonacci sequence is given by $u_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$, since we have verified that it satisfies the recursive definition of the Fibonacci sequence.

129. The fish population in Mirror Lake is $p_n = rp_{n-1} + h$. We have

$$\begin{aligned}
p_1 &= rp_0 + h; \\
p_2 &= rp_1 + h = r(rp_0 + h) + h \\
&= r^2p_0 + rh + h; \\
p_3 &= rp_2 + h = r(r^2p_0 + rh + h) + h \\
&= r^3p_0 + (r^2 + r + 1)h; \ \cdots \ \text{and so on.}
\end{aligned}$$

The pattern is such that

$$p_n = r^np_0 + (1 + r + r^2 + \cdots + r^{n-1})h.$$

Let $S_n = 1 + r + r^2 + \cdots + r^{n-1}$. Then $rS_n = r + r^2 + r^3 + \cdots + r^n$. Subtracting, we have $rS_n - S_n = r^n - 1$, or $S_n = \frac{r^n - 1}{r-1}$. So, the fish population for $p_0 = 3000$ is

$$\boxed{p_n = 3000r^n + \left(\frac{r^n - 1}{r - 1}\right)h.}$$

As $n \to \infty$, $r^n \to 0$ since $0 < r < 1$. We have

$$\lim_{n\to\infty} p_n = p_0 \lim_{n\to\infty} r^n + \left(\frac{1 - \lim_{n\to\infty} r^n}{1 - r}\right)h = 0 + \left(\frac{1-0}{1-r}\right)h = \frac{h}{1-r}.$$

So the sequence $\boxed{\textit{converges.}}$

130. **(a)** After one time constant, the charge remaining $Q_1 = \frac{Q_0}{e^1} = Q_0e^{-1}$. After two time constants, the charge remaining $Q_2 = \frac{Q_0}{e^2} = Q_0e^{-2}$. So after n time constants, the charge remaining is $\boxed{Q_n = Q_0e^{-n}.}$

(b) The sequence $\{Q_n\}$ is bounded from below by 0 since the charge Q_n is positive for all $n \geq 1$, and the sequence is decreasing as seen by the Algebraic Ratio Test:

$$\frac{Q_{n+1}}{Q_n} = \frac{Q_0e^{-n+1}}{Q_0e^{-n}} = \frac{1}{e} < 1.$$

Since the sequence $\{Q_n\}$ is bounded from below and is decreasing, it $\boxed{\textit{converges.}}$ Since the sequence converges, we compute the limit: $\lim\limits_{n\to\infty} Q_n = \lim\limits_{n\to\infty} Q_0e^{-n} = \boxed{0.}$

131. **(a)** Intensity after the first reflection $I_1 = (0.95)I_0$. Intensity after the second reflection $I_2 = (0.95)I_1 = (0.95)(0.95)I_0 = (0.95)^2I_0$. So, proceeding this way, the intensity after n reflections will be $\boxed{I_n = (0.95)^nI_0.}$

(b) We require $I_n \leq (0.02)I_0$ for the intensity after n reflections to have decreased by at least 98%. So

$$\begin{aligned} I_n &\leq (0.02)I_0 \\ n\log(0.95) &\leq \log(0.02) \\ n &\geq \frac{\log(0.02)}{\log(0.95)} \approx 76.3 \end{aligned}$$

(*Note:* The last inequality was reversed because for numbers less than 1, the log function is negative, so we are dividing by a negative number, which reverses inequalities.) We want at least $\boxed{77}$ reflections.

132. **(a)** Let r be the average number of neutrons released per fission event. Each of these neutrons causes another fission event, each releasing on average r neutrons, for a total of r^2 neutrons on average after the second fission event. So after n fission events, r^n neutrons on average are released. For $r = 2.5$, the sequence that describes the average number of neutrons after n fission events is $\{r^n\} = \boxed{\{(2.5)^n\}.}$

(b) Since $r = 2.5 > 1$, $\lim\limits_{n\to\infty} r^n = \infty$, so the sequence $\boxed{\textit{diverges.}}$

(c) The number of fission events in the fission chain reaction of Uranium-235 increases without bound, leading to a nuclear reactor meltdown or an explosion in a nuclear bomb.

133. A related function of the sequence $s_n = \left(1 + \frac{2}{n}\right)^n$ is $f(x) = \left(1 + \frac{2}{x}\right)^x$.
If $\lim\limits_{x\to\infty} f(x) = L$ then $\lim\limits_{n\to\infty} s_n = L$.
Because $\lim\limits_{x\to\infty} \left(1 + \frac{2}{x}\right) = 1$ and $\lim\limits_{x\to\infty} x = \infty$, the expression $\left(1 + \frac{2}{x}\right)^x$ is an indeterminate form at ∞ of the type 1^∞. Let $y = \left(1 + \frac{2}{x}\right)^x$.
Then $\ln y = \ln\left(1 + \frac{2}{x}\right)^x = x\ln\left(1 + \frac{2}{x}\right)$, which is an indeterminate form at ∞ of the type $0 \cdot \infty$.
Rewrite $x\ln\left(1 + \frac{2}{x}\right)$ as $\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}$ which is now an indeterminate form at ∞ of the type $\frac{0}{0}$.

Using L'Hôpital's Rule,

$$\lim_{x\to\infty}(\ln y) = \lim_{x\to\infty}\left[x\ln\left(1+\frac{2}{x}\right)\right] = \lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{d}{dx}\left[\ln\left(1+\frac{2}{x}\right)\right]}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$
$$= \lim_{x\to\infty}\frac{\left(\frac{1}{1+\frac{2}{x}}\right)\left(-\frac{2}{x^2}\right)}{\frac{-1}{x^2}} = \lim_{x\to\infty}\frac{2}{1+\frac{2}{x}} = \frac{2}{1} = 2$$

Finally, because $\lim\limits_{x\to\infty}(\ln y) = 2$, it follows that

$$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(1-\frac{4}{x}\right)^x = e^2$$

So, $\lim\limits_{n\to\infty}\left(1-\frac{4}{n}\right)^n = \boxed{e^2}$.

134. A related function of the sequence $s_n = \left(1-\frac{4}{n}\right)^n$ is $f(x) = \left(1-\frac{4}{x}\right)^x$.
If $\lim\limits_{x\to\infty} f(x) = L$ then $\lim\limits_{x\to\infty} s_n = L$.
Because $\lim\limits_{x\to\infty}\left(1-\frac{4}{x}\right) = 1$ and $\lim\limits_{x\to\infty} x = \infty$, the expression $\left(1-\frac{4}{x}\right)^x$ is an indeterminate form at ∞ of the type 1^∞. Let $y = \left(1-\frac{4}{x}\right)^x$.
Then $\ln y = \ln\left(1-\frac{4}{x}\right)^x = x\ln\left(1-\frac{4}{x}\right)$, which is an indeterminate form at ∞ of the type $0\cdot\infty$.
Rewrite $x\ln\left(1-\frac{4}{x}\right)$ as $\frac{\ln\left(1-\frac{4}{x}\right)}{\frac{1}{x}}$ which is now an indeterminate form at ∞ of the type $\frac{0}{0}$.
Using L'Hôpital's Rule,

$$\lim_{x\to\infty}(\ln y) = \lim_{x\to\infty}\left[x\ln\left(1-\frac{4}{x}\right)\right] = \lim_{x\to\infty}\frac{\ln\left(1-\frac{4}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{d}{dx}\left[\ln\left(1-\frac{4}{x}\right)\right]}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$
$$= \lim_{x\to\infty}\frac{\left(\frac{1}{1-\frac{4}{x}}\right)\cdot\frac{4}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty}\frac{-4}{1-\frac{4}{x}} = \frac{-4}{1} = -4$$

Finally, because $\lim\limits_{x\to\infty}(\ln y) = -4$, it follows that

$$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(1-\frac{4}{x}\right)^x = e^{-4}$$

So $\lim\limits_{n\to\infty}\left(1-\frac{4}{n}\right)^n = \boxed{e^{-4}}$.

135. A related function of the sequence $s_n = \left(1+\frac{1}{n}\right)^{3n}$ is $f(x) = \left(1+\frac{1}{x}\right)^{3x}$.
If $\lim\limits_{x\to\infty} f(x) = L$ then $\lim\limits_{n\to\infty} s_n = L$.
Because $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right) = 1$ and $\lim\limits_{x\to\infty} 3x = \infty$, the expression $\left(1+\frac{1}{x}\right)^{3x}$ is an indeterminate form at ∞ of the type 1^∞. Let $y = \left(1+\frac{1}{x}\right)^{3x}$.
Then $\ln y = \ln\left(1+\frac{1}{x}\right)^{3x} = 3x\ln\left(1+\frac{1}{x}\right)$, which is an indeterminate form at ∞ of the type $0\cdot\infty$. Rewrite $3x\ln\left(1+\frac{1}{x}\right)$ as $3\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}$ which is now an indeterminate form at ∞ of the type $\frac{0}{0}$.

Using L'Hôpital's Rule,

$$\lim_{x\to\infty}(\ln y) = \lim_{x\to\infty}\left[3x\ln\left(1+\frac{1}{x}\right)\right] = 3\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = 3\lim_{x\to\infty}\frac{\frac{d}{dx}\left[\ln\left(1+\frac{1}{x}\right)\right]}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$

$$= 3\lim_{x\to\infty}\frac{\left(\frac{1}{1+\frac{1}{x}}\right)\left(-\frac{1}{x^2}\right)}{\frac{-1}{x^2}} = 3\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 3$$

Finally, because $\lim\limits_{x\to\infty}(\ln y) = 3$, it follows that

$$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(1-\frac{4}{x}\right)^x = e^3$$

So, $\lim\limits_{n\to\infty}\left(1-\frac{4}{n}\right)^n = \boxed{e^3}$.

136. A related function of the sequence $s_n = \left(1+\frac{1}{n}\right)^{-2n}$ is $f(x) = \left(1+\frac{1}{x}\right)^{-2x}$.
If $\lim\limits_{x\to\infty} f(x) = L$ then $\lim\limits_{x\to\infty} s_n = L$.
Because $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right) = 1$ and $\lim\limits_{x\to\infty} 2x = \infty$, the expression $\left(1+\frac{1}{x}\right)^{-2x}$ is an indeterminate form at ∞ of the type 1^∞. Let $y = \left(1+\frac{1}{x}\right)^{-2x}$.
Then $\ln y = \ln\left(1+\frac{1}{x}\right)^{-2x} = -2x\ln\left(1+\frac{1}{x}\right)$, which is an indeterminate form at ∞ of the type $0\cdot\infty$. Rewrite $-2x\ln\left(1+\frac{1}{x}\right)$ as $\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{-2}{x}}$ which is now an indeterminate form at ∞ of the type $\frac{0}{0}$.
Using L'Hôpital's Rule,

$$\lim_{x\to\infty}(\ln y) = \lim_{x\to\infty}\left[-2x\ln\left(1+\frac{1}{x}\right)\right] = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{-2}{x}} = \lim_{x\to\infty}\frac{\frac{d}{dx}\left[\ln\left(1+\frac{1}{x}\right)\right]}{\frac{d}{dx}\left(\frac{-2}{x}\right)}$$

$$= \lim_{x\to\infty}\frac{\left(\frac{1}{1+\frac{1}{x}}\right)\cdot\frac{-1}{x^2}}{\frac{2}{x^2}} = \lim_{x\to\infty}\frac{-2}{1+\frac{1}{x}} = \frac{-2}{1} = -2$$

Finally, because $\lim\limits_{x\to\infty}(\ln y) = -2$, it follows that

$$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-2x} = e^{-2}$$

So, $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n} = e^{-2}$.
Also, by reference, consider the conclusion of exercise 103 in Section 4.5 which shows that $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^x$. Consequently, here

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-2x} = \lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^{-2} = \left(e^1\right)^{-2} = \boxed{e^{-2}}.$$

137. To use the Squeeze Theorem, we need to find a_n and b_n such that $a_n \le (-1)^n\frac{1}{n!} \le b_n$.
We know that

$$(-1)^n\frac{1}{n!} \le \left|(-1)^n\frac{1}{n!}\right| = |(-1)^n|\left|\frac{1}{n!}\right| = \left|\frac{1}{n!}\right|$$

Thus,

$$-\frac{1}{n!} \leq (-1)^n \frac{1}{n!} \leq \frac{1}{n!}$$

Since $\lim\limits_{x\to\infty} \left(-\frac{1}{n!}\right) = \lim\limits_{n\to\infty} \frac{1}{n!} = 0$, $\lim\limits_{n\to\infty} \left[(-1)^n \frac{1}{n!}\right] = 0$.

Challenge Problems

138. We need to show that when $0 < r < 1$, $\lim\limits_{n\to\infty} r^n = 0$. Let $r = \frac{1}{1+p}$, where $p > 0$. Expanding by the Binomial Theorem,

$$|r|^n = \frac{1}{|(1+p)^n|} = \frac{1}{\left|1 + np + \frac{n(n-1)}{2}p^2 + \cdots + p^n\right|} < \frac{1}{np}.$$

This means $-\frac{1}{np} < r^n < \frac{1}{np}$. Since $\lim\limits_{n\to\infty} \left(-\frac{1}{np}\right) = 0 = \lim\limits_{n\to\infty} \left(\frac{1}{np}\right)$, by the Squeeze Theorem, we have $\lim\limits_{n\to\infty} r^n = 0$, which was to be proved.

139. Suppose $-1 < r < 0$, and let $s = -r$. Then $0 < s < 1$. Moreover

$$r^n = (-1)^n s^n$$

so

$$-s^n \leq r^n \leq s^n$$

for all n. From Problem 138, $\lim\limits_{n\to\infty} s^n = 0$, therefore $\lim\limits_{n\to\infty} r^n = 0$ by the Squeeze Theorem.

140. We need to show that if $r > 1$, then $\lim\limits_{n\to\infty} r^n = \infty$. Let $r = 1 + p$, where $p > 0$. Then, by the Binomial Theorem, we have

$$r^n = (1+p)^n = 1 + np + \frac{n(n-1)}{2}p^2 + \cdots + p^n > np,$$

or $r^n > np$. Since $\lim\limits_{n\to\infty} np = \infty$ if $p > 0$, it follows than $\lim\limits_{n\to\infty} r^n = \infty$, which was to be proved.

141. If $r < -1$, then $r = 1 + p$ means that $p < -2$. If n is even, then $r^n > 0$, and $r^n > n|p|$, from the calculation of the previous problem 140. If n is odd, then $r^n < 0$ and $r^n < -n|p|$. As $n \to \infty$, r^n oscillates without bound between large positive and negative values, so $\lim\limits_{n\to\infty} r^n$ does not exist for $r < -1$, as was to be shown.

142. Recall that a function f is continuous at $x_0 = L$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |x - L| < \delta$ implies $|f(x) - f(L)| < \epsilon$. Let $\{s_n\}$ be a sequence with a limit L. Then $\lim\limits_{n\to\infty} s_n = L$ means that for every $\epsilon' > 0$. there exists an $n > N$ such that $|s_n - L| < \epsilon'$. Choose x to be s_n. Set $\delta = \epsilon'$. Then for $0 < |s_n - L| < \epsilon'$, continuity of f means for $n > N$, $|f(x) - f(L)| < \epsilon$. But this means that $\lim\limits_{n\to\infty} f(s_n) = f(L)$, which was to be shown.

143. We have $|s_n| = |s_n - L + L| \leq |s_n - L| + |L|$, using the Triangle Inequality. Similarly, $|L| = |L - s_n + s_n| \leq |L - s_n| + |s_n| = |s_n - L| + |s_n|$. Together these can be written as $||s_n| - |L|| \leq |s_n - L|$. If $|s_n - L| < \epsilon$ for $n > N$, then $||s_n| - |L|| \leq |s_n - L| < \epsilon$. This means that $\lim\limits_{n\to\infty} |s_n| = |L|$.

The converse is not true. For example, the sequence $\{|(-1)^n|\}$ converges to 1, while the sequence $\{(-1)^n\}$ is nonconvergent as its terms oscillate between -1 and $+1$.

144. Suppose the limit of a sequence is *not* unique. Then we would have $\lim\limits_{n\to\infty} s_n = L_1$, and $\lim\limits_{n\to\infty} s_n = L_2$. Then we have:

$$\begin{aligned} \lim_{n\to\infty} (s_n - s_n) &= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_n \\ \lim_{n\to\infty} 0 = 0 &= L_1 - L_2 \\ \text{or,} \qquad L_1 &= L_2. \end{aligned}$$

This proves that the limit of a convergent sequence is indeed unique, since our calculation shows $L_1 = L_2$.

145. The definition of a limit at infinity of a function $f(x)$ is: $\lim\limits_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$, there is an N such that $|f(x) - L| < \epsilon$ for $x > N$.
The definition of a limit of a sequence $\{s_n\}$ is: $\lim\limits_{n\to\infty} s_n = L$ if for every $\epsilon > 0$ there is an N such that $|s_n - L| < \epsilon$ for $n > N$.
The definitions appear identical, even though in the case of functions, the choice of N is not necessarily an integer as it is in the case of sequences.

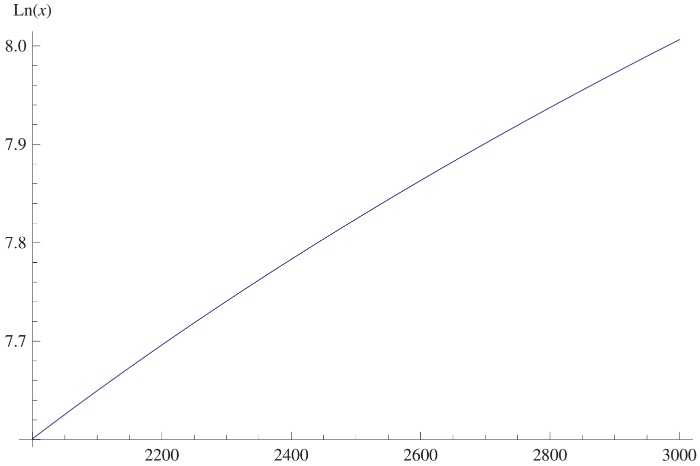
146. **(a)** A related function of the sequence $\{\ln n\}$ is $f(x) = \ln x$. Since $f'(x) = \frac{1}{x} > 0$ for $x > 0$, the related function of the sequence is increasing, which means the sequence $\{\ln n\}$ is increasing for $n \geq 1$.

(b) Using the algebraic ratio test, we have $\frac{\ln(n+1)}{\ln n} > 1$, since $\ln n$ is an increasing function of n. This shows that the sequence $\{\ln n\}$ is unbounded from above.

(c) The sequence $\{\ln n\}$ is an increasing sequence that is unbounded from above, and so it will diverge.

(d) We need an N such that $N > e^{20} \approx 4.852 \times 10^8$. So the smallest N so that $\ln N > 20$ is $N = e^{20} + 1$.

(e) The zoomed-in graph of $\ln x$ for large values of x appears below:



(f) The graph suggests that the function $y = \ln x$ increases as x increases, hence the sequence $\{\ln n\}$ will diverge without bound as well, confirming the result in (c).

147. **(a)** For $a, b > 0$,

$$\frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}(a - 2\sqrt{ab} + b) = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 > 0.$$

So $\frac{a+b}{2} > \sqrt{ab}$. If $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = \sqrt{a_n b_n}$, then using the result we have proved, namely $\frac{a_n+b_n}{2} > \sqrt{a_n b_n}$ for all n, we have $a_{n+1} > b_{n+1}$ for all n, or, equivalently, $a_n > b_n$ for all n. Beacuse of this, we can write $a_{n+1} = \frac{a_n+b_n}{2} < \frac{a_n+a_n}{2} = a_n$, i.e., $a_{n+1} < a_n$. So by the Algebraic Difference Test, the sequence $\{a_n\}$ *decreases.*

Now $a_n > b_n$ also implies $b_{n+1} = \sqrt{a_n b_n} > \sqrt{b_n b_n} = b_n$, i.e., $b_{n+1} > b_n$. So by the algebraic ratio test, the sequence $\{b_n\}$ *increases.* We have, since $\{a_n\}$ is decreasing

$a_n < a_{n-1} < a_{n-2} < \cdots < a_1$ and since $\{b_n\}$ is increasing, $b_n < b_{n+1} < b_{n+2} < \cdots$. Combining this with $b_n < a_n$, or equivalently, $b_{n+1} < a_{n+1}$, we have:

$$b_1 < b_2 < \cdots < b_n < b_{n+1} < a_{n+1} < a_n < \cdots < a_1$$

for all n, and, in particular, $\boxed{b_n < b_{n+1} < a_1}$ for all n.

(b) From the chain of inequalities that appear above, it follows that $\boxed{b_1 < a_{n+1} < a_n}$ for all n.

(c) Let's first prove $\sqrt{a} - \sqrt{b} < a - b$ when $a > b > 0$:

$$\begin{aligned}
\sqrt{a} - \sqrt{b} &< a - b \\
\text{is the same as} \quad \sqrt{a} - a &< \sqrt{b} - b \\
\sqrt{a}(1 - \sqrt{a}) &< \sqrt{b}(1 - \sqrt{b}). \\
\text{Squaring,} \quad a(1 + a - 2\sqrt{a}) &< b(1 + b - 2\sqrt{b}) \\
&< a(1 + a - 2\sqrt{b}), \qquad \text{since } b < a \\
a(1 + a) - 2a\sqrt{a} &< a(1 + a) - 2a\sqrt{b} \\
-2a\sqrt{a} &< -2a\sqrt{b} \\
a\sqrt{a} &> a\sqrt{b} \\
\sqrt{a} &> \sqrt{b} \\
a &> b
\end{aligned}$$

which is the case, ending the proof. Consider

$$0 < a_{n+1} - b_{n+1} = \frac{a_n + b_n}{2} - \sqrt{a_n b_n} = \frac{1}{2}(a_n - 2\sqrt{a_n b_n} + b_n) = \frac{1}{2}(\sqrt{a_n} - \sqrt{b_n}) < \frac{1}{2}(a_n - b_n) < \cdots < \frac{a_1 - b_1}{2^n}$$

by repeating the process n times, and using the result proved above in the form $\sqrt{a_n} - \sqrt{b_n} < a_n - b_n$ since $b_n < a_n$. This shows that $0 < a_{n+1} - b_{n+1} < \frac{a_1 - b_1}{2^n}$.

(d) The result of part (c) shows that as $n \to \infty$, the difference between a_{n+1} and b_{n+1}, gets arbitrarily small. Since $\lim\limits_{n\to\infty} \frac{a_1 - b_1}{2^n} = (a_1 - b_1) \lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$, since $-1 < \frac{1}{2} < 1$, and $\lim\limits_{n\to\infty} 0 = 0$, by the Squeeze Theorem, we have $\lim\limits_{n\to\infty} (a_{n+1} - b_{n+1}) = 0$, which means that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$, as required to be proved. (Note that each limit separately exists because the sequence $\{a_n\}$ is bounded from below by 0 and is decreasing, so it converges to some limit; the sequence $\{b_n\}$ is bounded from above by a_1—see the string of inequalities in part (a)— and is increasing, so it also converges to some limit.)

148. $s_n = \frac{2^{n-1}4^n}{n!}$. Using the algebraic ratio test, we have

$$\frac{s_{n+1}}{s_n} = \frac{2^n 4^{n+1}}{(n+1)!} \frac{n!}{2^{n-1}4^n} = \frac{2 \cdot 4}{n+1} = \frac{8}{n+1} \leq 1$$

for $n \geq 7$. This shows the sequence $\{s_n\}$ is nonincreasing. Since each term of the sequence is positive, $s_n > 0$ for $n \geq 1$, the sequence $\{s_n\}$ is bounded from below. Since the sequence $\{s_n\}$ is nonincreasing and bounded from below, it $\boxed{\textit{converges.}}$

149. $s_n = \frac{n!}{3^n \cdot 4^n}$. Using the algebraic ratio test, we have

$$\frac{s_{n+1}}{s_n} = \frac{(n+1)!}{3^{n+1} \cdot 4^{n+1}} \frac{3^n \cdot 4^n}{n!} = \frac{n+1}{12} \geq 1$$

for $n \geq 11$. The sequence is nondecreasing, so there is no upper bound. So the sequence $\boxed{\textit{diverges.}}$

150. To see if the sequence $\{s_n\}$ is monotonic, we find the ratio $\frac{s_{n+1}}{s_n}$.

$$\frac{s_{n+1}}{s_n} = \frac{\frac{(n+1)!}{3^{n+1}+8(n+1)}}{\frac{n!}{3^n+8n}} = \frac{(n+1)!}{n!}\frac{3^n+8n}{3^{n+1}+8(n+1)} = (n+1)\frac{3^n+8n}{3^{n+1}+8(n+1)}.$$

Using L'Hôpital's rule on a related function of the fractional term,

$$\begin{aligned}\lim_{n\to\infty}\frac{3^n+8n}{3^{n+1}+8(n+1)} &= \lim_{x\to\infty}\frac{3^x+8x}{3^{x+1}+8(x+1)} = \lim_{x\to\infty}\frac{(\ln 3)3^x+8}{(\ln 3)3^{x+1}+8} \\ &= \lim_{x\to\infty}\frac{(\ln 3)^2 3^x}{(\ln 3)^2 3^{x+1}} = \lim_{x\to\infty}\frac{3^x}{3^{x+1}} = \frac{1}{3}.\end{aligned}$$

For large n, we have $\lim\limits_{n\to\infty}\frac{s_{n+1}}{s_n} = \lim\limits_{n\to\infty}\frac{(n+1)}{3} = \infty$. The increasing sequence $\{s_n\}$ is not bounded from above, so it will diverge.

151. The sequence is $\{s_n\} = \left\{(3^n+5^n)^{1/n}\right\} = \left\{5\left(1+\left(\frac{3}{5}\right)^n\right)^{1/n}\right\}$. The sequence is bounded from below by 5 since $\left(\frac{3}{5}\right)^n > 0$ and $\left(1+\left(\frac{3}{5}\right)^n\right)^{1/n} > 1$. (The nth root of a number greater than 1 is greater than 1.) A related function of the sequence is $f(x) = 5\left(1+\left(\frac{3}{5}\right)^x\right)^{1/x}$. Note that $f(x) > 0$ when $x > 0$. Taking the logarithm of both sides and differentiating with respect to x, we get

$$\ln f = \ln 5 + \frac{1}{x}\ln\left(1+\left(\frac{3}{5}\right)^x\right);$$

$$\frac{f'}{f} = 0 - \frac{1}{x^2}\ln\left(1+\left(\frac{3^x}{5}\right)\right) + \frac{1}{x}\cdot\frac{1}{\left(1+\left(\frac{3}{5}\right)^x\right)}\cdot\left(\frac{3}{5}\right)^x\cdot\ln\left(\frac{3}{5}\right) < 0$$

whenever $x > 0$ since $\ln\left(\frac{3}{5}\right) < 0$. Since $f(x) > 0$, we have shown that $f'(x) < 0$, or the function is decreasing for $x > 0$. This means the sequence is decreasing for $n \geq 1$. Since the sequence is bounded from below and is decreasing, it converges, as was to be shown.

152. (a) Let a_1 and N be positive numbers. Then

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{N}{a_n}\right) = \frac{a_n^2+N}{2a_n} > 0$$

for all $n \geq 1$. Since

$$a_n^2 - 2a_n\sqrt{n} + N = (a_n - N)^2 \geq 0,$$

it follows that

$$a_n^2 + N \geq 2a_n\sqrt{n}, \quad\text{or}\quad \frac{a_n^2+N}{2a_n} \geq \sqrt{N}.$$

So we have

$$a_{n+1} = \frac{a_n^2+N}{2a_n} \geq \sqrt{N} \text{ for } n \geq 1,$$

that is, $a_n \geq \sqrt{N}$ for $n \geq 2$. By the Algebraic Ratio Test, we get

$$\frac{a_{n+1}}{a_n} = \frac{1}{2}\left(1+\frac{N}{a_n^2}\right) \leq \frac{1}{2}\left(1+\frac{N}{N}\right) = 1$$

for $n \geq 2$ since $a_n \geq \sqrt{N}$ for $n \geq 2$. We have shown that the sequence $\{a_n\}$ is a nonincreasing sequence that is bounded from below, so it converges. Suppose the limit of the sequence is L. Taking the limit as $n \to \infty$ of both sides of the equation

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{N}{a_n}\right)$$

yields $L = \frac{1}{2}\left(L + \frac{N}{L}\right)$ or $2L^2 = L^2 + N$, or $L = \sqrt{N}$.

(b) We want to approximate $\sqrt{28}$, so $N = 28$. Let $a_1 = 1$ (Any other value may be chosen, in which case the answers will vary for a_3, but for a_6, the answers will be much closer if not identical to three decimal places.) Then $a_2 = \frac{1}{2}\left[a_1 + \frac{28}{a_1}\right] = \frac{1}{2}[1 + 28] = \frac{29}{2}$. Then $a_3 = \frac{1}{2}\left[\frac{29}{2} + \frac{28}{(29/2)}\right] = \frac{1}{2}\left[\frac{29}{2} + \frac{56}{29}\right] = \frac{29}{4} + \frac{28}{29} \approx 8.216$. Next, $a_3 = \frac{1}{2}\left[8.216 + \frac{28}{8.216}\right] \approx 5.812$; $a_4 = \frac{1}{2}\left[5.812 + \frac{28}{5.812}\right] \approx 5.315$; $a_5 = \frac{1}{2}\left[5.315 + \frac{28}{5.315}\right] \approx 5.292$; $a_6 = \frac{1}{2}\left[5.292 + \frac{28}{5.292}\right] \approx 5.292$. To three decimal places, we have using a calculator, $\sqrt{28} \approx 5.292$. We see that a_6 is 100% accurate and a_3 is about 91% accurate.

153. The given sequence can be rewritten as $\{s_n\} = \left\{\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\right\} = \left\{\frac{1\cdot2\cdot3\cdot4\cdot5\cdot\cdots(2n-2)\cdot(2n-1)}{(2\cdot4\cdot6\cdots(2n-2))^2(2n)}\right\} = \left\{\frac{(2n-1)!}{(2^{n-1}(1\cdot2\cdot3\cdots(n-1)))^2\cdot2n}\right\} = \left\{\frac{(2n-1)!}{2^{2n-2}\cdot(n-1)!(n-1)!\cdot2n}\right\} = \left\{\frac{(2n-1)!}{2^{2n-1}n!(n-1)!}\right\}$. Using the algebraic ratio test, we have

$$\frac{s_{n+1}}{s_n} = \frac{(2n+1)!}{2^{2n+1}(n+1)!n!} \cdot \frac{2^{2n-1}n!(n-1)!}{(2n-1)!} = \frac{(2n+1)(2n)(2n-1)!(n-1)!2^{2n-1}}{(2n-1)!(n+1)n(n-1)!2^2 2^{2n-1}} = \frac{2n+1}{2n+2} < 1$$

for $n \geq 1$ which means the sequence $\{s_n\}$ is decreasing. Since the sequence is decreasing, the upper bound on the terms of the sequence is $s_1 = \frac{1}{2}$. So the sequence is bounded below by 0 and above by $\frac{1}{2}$. Hence the sequence is both monotonic and bounded as needed to be shown.

154. Expanding $\left(1 + \frac{1}{n}\right)^n$ using the Binomial Theorem, we obtain

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \binom{n}{0} + \binom{n}{1}\left(\frac{1}{n}\right)^1 + \binom{n}{2}\left(\frac{1}{n}\right)^2 + \cdots + \binom{n}{k}\left(\frac{1}{n}\right)^k + \cdots + \binom{n}{n}\left(\frac{1}{n}\right)^n \\ &= \underbrace{1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\left(\frac{1}{n}\right)^k + \cdots + \left(\frac{1}{n}\right)^n}_{n+1\text{terms}}.\end{aligned}$$

Simplifying, we have

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1\left(1-\frac{1}{n}\right)}{2!} + \cdots + \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{k!} + \\ \cdots + \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)}{n!}. \qquad (1)\end{aligned}$$

Next, we expand $\left(1 + \frac{1}{n+1}\right)^{n+1}$:

$$\begin{aligned}\left(1+\frac{1}{n+1}\right)^{n+1} &= \binom{n+1}{0} + \binom{n+1}{1}\left(\frac{1}{n+1}\right)^1 + \binom{n+1}{2}\left(\frac{1}{n+1}\right)^2 + \cdots + \binom{n+1}{n+1}\left(\frac{1}{n+1}\right)^{n+1} \\ &= \underbrace{1 + 1 + \frac{1\left(1-\frac{1}{n+1}\right)}{2!} + \frac{1\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)}{3!} + \cdots + \frac{1\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)}{n!}}_{n+1\text{terms}} \\ &\quad + \frac{1\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)}{(n+1)!}. \qquad (2)\end{aligned}$$

We make two observations from equations (1) and (2):

- After the first two terms, each term of (2) is greater than the corresponding term of (1).
- The expansion of (2) has one more positive term than the expansion of (1).

So this demonstrates that

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$$

which means the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is increasing.
To show that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is bounded from above, we first note that

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1+1+\frac{1\left(1-\frac{1}{n}\right)}{2!}+\cdots+\frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)}{n!}\\ &< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}. \qquad (3)\end{aligned}$$

Now, for $n \geq 2$, we have

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n(n-1)(n-2)\cdots 3\cdot 2 \geq \underbrace{2\cdot 2\cdot 2\cdots 2\cdot 2}_{n-1\text{terms}} = 2^{n-1},$$

that is,

$$\frac{1}{n!} \leq \frac{1}{2^{n-1}}$$

for $n \geq 2$. Going back to inequality (3), we have

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &< 1+\left(1+\frac{1}{2!}+\frac{3!}{+}\cdots+\frac{1}{n!}\right)\\ &\leq 1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}\right). \qquad (4)\end{aligned}$$

Let

$$S_n = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}. \qquad (5)$$

Then, multiplying through by $\frac{1}{2}$, we get

$$\frac{1}{2}S_n = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}+\frac{1}{2^n}. \qquad (6)$$

Subtracting (6) from (5) gives

$$S_n - \frac{1}{2}S_n = \frac{1}{2}S_n = 1-\frac{1}{2^n},$$

so that

$$S_n = \frac{1-\left(\frac{1}{2}\right)^n}{\frac{1}{2}} = 2\left[1-\left(\frac{1}{2}\right)^n\right] < 2.$$

We continue the estimation of the inequality (4):

$$\left(1+\frac{1}{n}\right)^n < 1+S_n < 1+2 = 3,$$

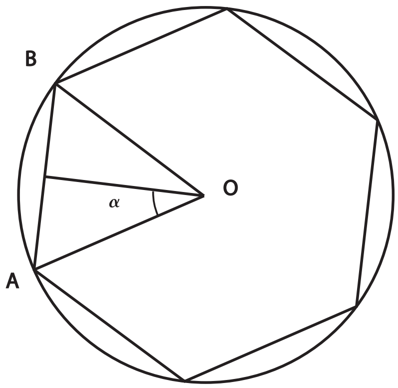
which proves that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is bounded from above.

155. Let the limit of the convergent sequence $\{s_n\}$ be S. The nth term of the arithmetic mean sequence $\{a_n\}$ is $a_n = \frac{1}{n}[s_1 + s_2 + \cdots + s_n]$. Let $\epsilon > 0$. Since s_n is convergent to the limit S, then there exists an N such that for all $n \geq N$ $|s_n - S| < \frac{\epsilon}{2}$. Then we have

$$\begin{aligned} |a_n - S| &= \left| \frac{1}{n} \sum_{k=1}^{n} (s_k) - S \right| = \frac{1}{n} \left| \sum_{k=1}^{n} (s_k) - ns \right| = \frac{1}{n} \left| \sum_{k=1}^{n} (s_k - S) \right| \\ &\leq \frac{1}{n} \sum_{k=1}^{N} |s_k - S| + \frac{1}{n} \sum_{k=N+1}^{n} |s_k - S| \end{aligned}$$

Since for $n \geq N$, $|s_n - S| < \frac{\epsilon}{2}$, the sum in the second term satisfies $\sum\limits_{k=N+1}^{n} |s_k - S| < n\frac{\epsilon}{2}$. Let $\sum\limits_{k=1}^{N} |s_k - S| = M$. Choose $n \geq \max\{N, \frac{2}{\epsilon}M\}$. Then we have $|a_n - S| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, proving the result that the sequence of arithmetic means $\{a_n\}$ converges, and converges to the same limit as the original sequence $\{s_n\}$.

156. **(a)** From the figure below, which shows a circle of radius R circumscribing a regular polygon of n sides, we see that area of the triangle with the angle $\alpha = \frac{\pi}{n}$ indicated is $A_\triangle = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2} \cdot R\sin\alpha \cdot R\cos\alpha$. Area of the polygon, $A_n = 2n \cdot A_\triangle = 2n \cdot \frac{1}{2}R^2 \sin\alpha\cos\alpha = \frac{n}{2}R^2\sin 2\alpha$, or $\boxed{A_n = \frac{n}{2}R^2 \sin\left(\frac{2\pi}{n}\right)}$.

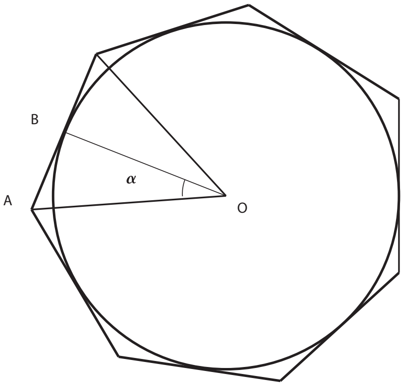


(b) We have, as the number of sides of the polygon grows without limit,

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{n}{2}R^2 \sin\left(\frac{2\pi}{n}\right) = \lim_{n\to\infty} \pi R^2 \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} = \pi R^2 \lim_{y\to 0} \frac{\sin y}{y} = \pi R^2,$$

where the substitution of $y = \frac{2\pi}{n}$ was made, and the standard result $\lim\limits_{y\to 0} \frac{\sin y}{y} = 1$ was used. This shows that as the polygon's sides increase, its area approaches that of the circumscribing circle.

157. **(a)** From the figure below, which shows a circle of radius r inscribed in a polygon of n sides, we have $\alpha = \frac{\pi}{n}$. Now, the area of the $\triangle$ ABO,
$A_\triangle = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}r \cdot r\tan\alpha = \frac{1}{2}r^2\tan\left(\frac{\pi}{n}\right)$. The area of the polygon,
$A_n = 2n \times A_\triangle = \boxed{nr^2\tan\left(\frac{\pi}{n}\right).}$

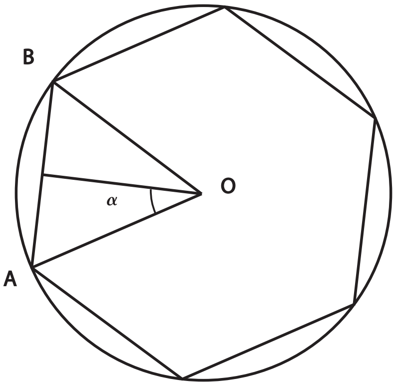


(b) As the number of sides of the polygon increases without limit, we have:

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} nr^2\tan\frac{\pi}{n} = \pi r^2 \lim_{n\to\infty}\frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}} = \pi r^2\lim_{y\to 0}\frac{\tan y}{y} = \pi r^2,$$

where the substitution of $y = \frac{\pi}{n}$ was made, and the standard result $\lim\limits_{y\to 0}\frac{\tan y}{y} = 1$ was used. This shows that as the polygon's sides increase, its area approaches that of the inscribed circle.

158. **(a)** From the figure below, which shows a circle of radius R circumscribing a regular polygon of n sides, we see that the length of the segment AB is $2R\sin\alpha = 2\left(R\sin\left(\frac{\pi}{n}\right)\right)$. The polygon has n sides, so its perimeter will be n times the length of the segment AB, or
$\boxed{P_n = 2nR\sin\left(\frac{\pi}{n}\right).}$



(b) As the number of sides is increased without limit, we expect the perimeter of inscribed polygon to approach the circumference of the circle.

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} 2nR\sin\left(\frac{\pi}{n}\right) = 2\pi R\lim_{n\to\infty}\frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} = 2\pi R\lim_{y\to 0}\frac{\sin y}{y} = 2\pi R,$$

where the substitution of $y = \frac{\pi}{n}$ was made, and the standard result $\lim\limits_{y\to 0}\frac{\sin y}{y} = 1$ was used.

159. According to the definition of a convergent sequence (p.645) a sequence $\{s_l\}$ converges to a real number L if, for any number $\epsilon > 0$, there is a positive integer N so that $|s_l - L| < \epsilon$ for all integers $l > N$. Consider the absolute value of the difference of s_n and s_m:

$$|s_n - s_m| = |s_n - L + L - s_m| \leq |s_n - L| + |L - s_m|,$$

where we have used the triangle inequality. Since both sequences $\{s_n\}$ and $\{s_m\}$ converge to the same limit L, applying the definition, we can find positive integers N_1 and N_2 such that for any number $\epsilon > 0$, $|s_n - L| < \epsilon/2$ for all $n > N_1$ and $|s_m - L| < \epsilon/2$ for all $m > N_2$. Then, since $|L - s_m| = |s_m - L|$, we have

$$|s_n - s_m| \leq |s_n - L| + |s_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

i.e., $|s_n - s_m| < \epsilon$, for $n, m > N$ where N is the larger of $\{N_1, N_2\}$. This shows that every convergent sequence must be a Cauchy sequence.

160. **(a)** The sequence is $\{s_n\} = \left\{e^{n/(n+2)}\right\}$. Consider the sequence $\{b_n\} = \left\{\frac{n}{n+2}\right\}$. This sequence is bounded from above by 1 since $\frac{n}{n+2} < 1$ for all $n \geq 1$.Also, the sequence $\{b_n\}$ is increasing since by the Algebraic Ratio Test:

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)(n+2)}{(n+3)n} = \frac{n^2+3n+2}{n^2+3n} > 1$$

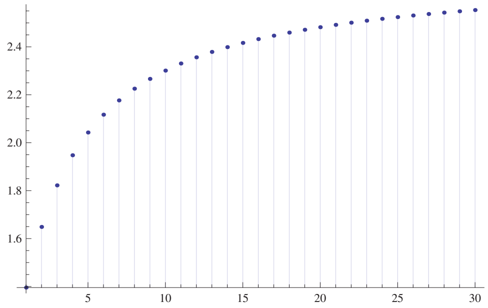
for all $n \geq 1$. So $\{b_n\}$ is a convergent sequence since it is bounded from above and increasing. To find the limit of the sequence $\{b_n\}$, compute:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n+2} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} = \frac{1}{1+0} = 1.$$

Since $f(x) = e^x$ is continuous at $x = 1$, by the theorem on p.637, the sequence $\{f(b_n)\} = \{s_n\} = \left\{e^{n/(n+2)}\right\}$ also $\boxed{\textit{converges.}}$

(b) $\lim\limits_{n\to\infty} e^{n/(n+2)} = \lim\limits_{n\to\infty} f(b_n) = f(\lim\limits_{n\to\infty} b_n) = f(1) = e^1 = \boxed{e.}$

(c) As seen by plotting the sequence $\{s_n\} = \left\{e^{n/(n+2)}\right\}$, the points seem to reach the asymptotic value of e. See figure below:



AP$^{®}$ Practice Problems

1. $(-1)^{n+1}\frac{(n-1)^2}{n}$

   CHOICE C

2. $b_n = \left(\frac{-2}{3}\right)^n (n-3)$

$$b_1 = \left(\frac{-2}{3}\right)^1 (1-3) = \boxed{\frac{4}{3}}$$

$$b_2 = \left(\frac{-2}{3}\right)^2 (2-3) = \boxed{-\frac{4}{9}}$$

$$b_3 = \left(\frac{-2}{3}\right)^3 (3-3) = \boxed{0}$$

Enough terms have been generated to identify CHOICE B.

3. $a_n = \frac{2}{3n+1}$. A related function to $a_n = \frac{2}{3n+1}$ is $y = \frac{2}{3x+1}$ $y' = -2(3x+1)^{-2}(3) = \frac{-6}{(3x+1)^2}$. y' is negative for all x.

So $y(x)$ is a decreasing function for all x, so $a_n = \frac{2}{3n+1}$ is a $\boxed{\text{decreasing}}$ sequence. Since $a_n = \frac{2}{3n+1}$ is a decreasing sequence, $a_1 = \frac{2}{3(1)+1} = \frac{1}{2} > a_n$ for all $n > 1$.

So the upper bound for a_n is $\frac{1}{2}$. The graph of the related function $y = \frac{2}{3x+1}$ has a horizontal asymptote at $y = 0$. Consequently $a_n = \frac{2}{3n+1}$ is $\boxed{\text{bounded}}$ as follows: $0 < a_n = \frac{2}{3n+1} \leq \frac{1}{2}$.

CHOICE B

4. The geometric sequence $s_n = 3x^n$ converges to 0 if $-1 < x < 1$ and converges to 3 if $x = 1$.

Therefore $s_n = 3x^n$ converges for $\boxed{-1 < x \leq 1}$.

CHOICE A

5. $\frac{-1}{n^3} \leq \frac{\sin n}{n^3} \leq \frac{1}{n^3}$

$s_n = \frac{\sin n}{n^3}$ is bounded by $a_n = \frac{-1}{n^3}$ and $b_n = \frac{1}{n^3}$.

Since $a_n \leq s_n \leq b_n$ and both $\lim\limits_{n\to\infty}(a_n) = 0$ and $\lim\limits_{n\to\infty}(b_n) = 0$, then, by the Squeeze Theorem for sequences, the sequence $s_n = \frac{\sin n}{n^3}$ converges and $\lim\limits_{n\to\infty}(s_n) = \boxed{0}$.

The sequence $\boxed{\text{converges}}$.

8.2 Infinite Series

Concepts and Vocabulary

1. $\boxed{\textbf{(b)}:}$ $\sum\limits_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_n + \cdots$ is called an infinite series.

2. $\boxed{\textbf{(d)}:}$ $S_n = \sum\limits_{k=1}^{n} a_k$ is the nth partial sum, and the sequence $\{S_n\}$ is called the sequence of partial sums.

3. **True:** If a series converges then the sequence of partial sums converges. Conversely, if the sequence of partial sums converges, then the series also converges.

4. **False:** The geometric series $\sum_{k=1}^{\infty} ar^{k-1}$, $a \neq 0$, converges if $|r| < 1$, but diverges when $|r| = 1$.

5. $S = \boxed{\dfrac{a}{1-r}}$ is the sum of the convergent geometric series $\sum_{k=1}^{\infty} ar^{k-1}$, $a \neq 0$.

6. **False:** The limit of the nth term of a series as $n \to \infty$ does not tell us if the series itself will converge. In this instance, the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

Skill Building

7. $S_4 = \sum_{k=1}^{4} \left(\frac{3}{4}\right)^{k-1} = \left(\frac{3}{4}\right)^0 + \left(\frac{3}{4}\right)^1 + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 = 1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} = \boxed{\dfrac{175}{64}.}$

8. $S_4 = \sum_{k=1}^{4} \frac{(-1)^{k+1}}{3^{k-1}} = \frac{(-1)^2}{3^0} + \frac{(-1)^3}{3^1} + \frac{(-1)^4}{3^2} + \frac{(-1)^5}{3^3} = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} = \boxed{\dfrac{20}{27}.}$

9. $S_4 = \sum_{k=1}^{4} k = 1 + 2 + 3 + 4 = \boxed{10.}$

10. $S_4 = \sum_{k=1}^{4} \ln k = \ln 1 + \ln 2 + \ln 3 + \ln 4 = \ln(1 \cdot 2 \cdot 3 \cdot 4) = \boxed{\ln 24.}$

11.
$S_n = \sum_{k=1}^{n} \left(\frac{1}{k+2} - \frac{1}{k+3}\right) = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) = \frac{1}{3} - \frac{1}{n+3}$.
We have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{3} - \frac{1}{n+3}\right) = \lim_{n\to\infty} \frac{1}{3} - \lim_{n\to\infty} \frac{1}{n+3} = \frac{1}{3} - 0 = \frac{1}{3}.$$

Since the sequence of partial sums converges to $\frac{1}{3}$, this means the sum of the telescoping series is $\boxed{\dfrac{1}{3}.}$

12. $S_n = \sum_{k=1}^{n} \left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right] = \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \cdots + \left(\frac{1}{(n-1)^2} - \frac{1}{n^2}\right) + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = 1 - \frac{1}{(n+1)^2}$.
We have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{(n+1)^2}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{(n+1)^2} = 1 - 0 = 1.$$

Since the sequence of partial sums converges to 1, this means the sum of the telescoping series is $\boxed{1.}$

13.
$S_n = \sum_{k=1}^{n} \left(\frac{1}{3^{k+1}} - \frac{1}{3^k}\right) = \left(\frac{1}{3^2} - \frac{1}{3}\right) + \left(\frac{1}{3^3} - \frac{1}{3^2}\right) + \cdots + \left(\frac{1}{3^n} - \frac{1}{3^{n-1}}\right) + \left(\frac{1}{3^{n+1}} - \frac{1}{3^n}\right) = \frac{1}{3^{n+1}} - \frac{1}{3}$.
We have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{3^{n+1}} - \frac{1}{3}\right) = \lim_{n\to\infty} \frac{1}{3^{n+1}} - \lim_{n\to\infty} \frac{1}{3} = 0 - \frac{1}{3} = -\frac{1}{3}.$$

Since the sequence of partial sums converges to $-\frac{1}{3}$, this means the sum of the telescoping series is $\boxed{-\dfrac{1}{3}.}$

14.
$S_n = \sum\limits_{k=1}^{n} \left(\frac{1}{4^{k+1}} - \frac{1}{4^k}\right) = \left(\frac{1}{4^2} - \frac{1}{4}\right) + \left(\frac{1}{4^3} - \frac{1}{4^2}\right) + \cdots + \left(\frac{1}{4^n} - \frac{1}{4^{n-1}}\right) + \left(\frac{1}{4^{n+1}} - \frac{1}{4^n}\right) = \frac{1}{4^{n+1}} - \frac{1}{4}.$
We have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{4^{n+1}} - \frac{1}{4}\right) = \lim_{n\to\infty}\frac{1}{4^{n+1}} - \lim_{n\to\infty}\frac{1}{4} = 0 - \frac{1}{4} = -\frac{1}{4}.$$

Since the sequence of partial sums converges to $-\frac{1}{4}$, this means the sum of the telescoping series is $\boxed{-\frac{1}{4}.}$

15. We write the general term as a difference of terms:
$S_n = \sum\limits_{k=1}^{n} \frac{1}{4k^2-1} = \sum\limits_{k=1}^{n} \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) =$
$\frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \frac{1}{2}\left(\frac{1}{2n-3} - \frac{1}{2n-1}\right) + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2n+1}\right).$
We have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2}\left(\frac{1}{2n-1}\right)\right) = \lim_{n\to\infty}\frac{1}{2} - \lim_{n\to\infty}\frac{1}{2}\left(\frac{1}{2n+1}\right) = \frac{1}{2} - 0 = \frac{1}{2}.$$

Since the sequence of partial sums converges to $\frac{1}{2}$, this means the sum of the telescoping series is $\boxed{\frac{1}{2}.}$

16. We write the general term as a difference of terms:

$$\begin{aligned}
S_n &= \sum_{k=2}^{n} \frac{2}{k^2-1} = \sum_{k=2}^{n} \frac{2}{(k-1)(k+1)} = \sum_{k=2}^{n}\left(\frac{1}{k-1} + \frac{1}{k+1}\right) \\
&= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n-4} - \frac{1}{n-2}\right) \\
&\quad + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) \\
&= 1 + \frac{1}{2} + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) + \left(-\frac{1}{5} + \frac{1}{5}\right) + \cdots + \left(-\frac{1}{n-2} + \frac{1}{n-2}\right) \\
&\quad + \left(-\frac{1}{n-1} + \frac{1}{n-1}\right) - \frac{1}{n} - \frac{1}{n+1} \\
&= \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}.
\end{aligned}$$

We have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}\right) = \frac{3}{2}.$$

Since the sequence of partial sums converges to $\frac{3}{2}$, this means that the sum of the telescoping series converges to $\boxed{\frac{3}{2}}$.

17. $\sum\limits_{k=1}^{\infty} (\sqrt{2})^{k-1}$. This has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \sqrt{2}$.
The series will $\boxed{\textit{diverge}}$ since $|r| > 1$.

18. $\sum\limits_{k=1}^{\infty}(0.33)^{k-1}$. This has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = 0.33$. Since $|r| < 1$, this series will $\boxed{converge}$ to a sum of $\frac{a}{1-r} = \frac{1}{1-0.33} = \frac{1}{0.67} = \boxed{\frac{100}{67}.}$

19. $\sum\limits_{k=1}^{\infty} 5\left(\frac{1}{6}\right)^{k-1}$. This has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = 5$ and $r = \frac{1}{6}$. Since $|r| < 1$, this series will $\boxed{converge}$ to a sum of $\frac{1}{1-r} = \frac{5}{1-\frac{1}{6}} = \frac{5}{\frac{5}{6}} = \boxed{6.}$

20. $\sum\limits_{k=1}^{\infty} 4(1.1)^{k-1}$. This has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = 4$ and $r = 1.1$. The series will $\boxed{diverge}$ since $|r| > 1$.

21. $\sum\limits_{k=0}^{\infty} 7\left(\frac{1}{3}\right)^{k} = 7\left(\frac{1}{3}\right)^{0} + \sum\limits_{k=1}^{\infty} 7\left(\frac{1}{3}\right)^{k} = 7 + \sum\limits_{k=1}^{\infty} \frac{7}{3}\left(\frac{1}{3}\right)^{k-1}$. The second term has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{7}{3}$ and $r = \frac{1}{3}$. Since $|r| < 1$, this series will converge to a sum of $\frac{a}{1-r} = \frac{\frac{7}{3}}{1-\frac{1}{3}} = \frac{7/3}{2/3} = \frac{7}{2}$. The original series will $\boxed{converge}$ to a sum of $7 + \frac{7}{2} = \boxed{\frac{21}{2}.}$

22. $\sum\limits_{k=0}^{\infty} \left(\frac{7}{4}\right)^{k} = \left(\frac{7}{4}\right)^{0} + \sum\limits_{k=1}^{\infty} \left(\frac{7}{4}\right)^{k} = 1 + \sum\limits_{k=1}^{\infty} \frac{7}{4}\left(\frac{7}{4}\right)^{k-1}$. The second term has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{7}{4}$ and $r = \frac{7}{4}$. Since $|r| > 1$, this series will diverge, which means the original series will $\boxed{diverge}$ as well.

23. $\sum\limits_{k=1}^{\infty}(-0.38)^{k-1}$. This has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = -0.38$. Since $|r| < 1$, this series will $\boxed{converge}$ to a sum of $\frac{a}{1-r} = \frac{1}{1-(-0.38)} = \frac{1}{1.38} = \frac{100}{138} = \boxed{\frac{50}{69}.}$

24. $\sum\limits_{k=1}^{\infty}(-0.38)^{k} = \sum\limits_{k=1}^{\infty}(-0.38)(-0.38)^{k-1}$. This has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = -0.38$ and $r = -0.38$. Since $|r| < 1$, this series will $\boxed{converge}$ to a sum of $\frac{a}{1-r} = \frac{-0.38}{1-(-0.38)} = -\frac{0.38}{1.38} = -\frac{38}{138} = \boxed{-\frac{19}{69}.}$

25. $\sum\limits_{k=0}^{\infty} \frac{2^{k+1}}{3^{k}} = \frac{2^{0+1}}{3^{0}} + \sum\limits_{k=1}^{\infty} 2\cdot\left(\frac{2}{3}\right)^{k} = 2 + \sum\limits_{k=1}^{\infty} 2\cdot\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{k-1} = 2 + \sum\limits_{k=1}^{\infty} \frac{4}{3}\left(\frac{2}{3}\right)^{k-1}$. The second term has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{4}{3}$ and $r = \frac{2}{3}$. Since $|r| < 1$, this series will converge to a sum of $\frac{a}{1-r} = \frac{4/3}{1-\frac{2}{3}} = \frac{4/3}{1/3} = 4$. The original series will $\boxed{converge}$ to a sum of $2 + 4 = \boxed{6.}$

26. $\sum\limits_{k=0}^{\infty} \frac{5^k}{6^{k+1}} = \frac{5^0}{6^{0+1}} + \sum\limits_{k=1}^{\infty} \frac{5^k}{6^{k+1}} = \frac{1}{6} + \sum\limits_{k=1}^{\infty} \frac{5}{6^2} \cdot \frac{5^{k-1}}{6^{k-1}} = \frac{1}{6} + \sum\limits_{k=1}^{\infty} \frac{5}{36} \cdot \left(\frac{5}{6}\right)^{k-1}$.

The second term has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{5}{36}$ and $r = \frac{5}{6}$. Since $|r| < 1$, this series will converge to a sum of $\frac{a}{1-r} = \frac{5/36}{1-\frac{5}{6}} = \frac{5/36}{1/6} = \frac{5}{6}$. The original series $\boxed{\textit{converges}}$ to a sum of $\frac{1}{6} + \frac{5}{6} = \boxed{1.}$

27. $\sum\limits_{k=0}^{\infty} \frac{1}{4^{k+1}} = \frac{1}{4^{0+1}} + \sum\limits_{k=1}^{\infty} \frac{1}{4^{k+1}} = \frac{1}{4} + \sum\limits_{k=1}^{\infty} \frac{1}{4^2} \cdot \left(\frac{1}{4}\right)^{k-1}$.

The second term has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{1}{4^2}$ and $r = \frac{1}{4}$. Since $|r| < 1$, this series will converge to a sum of $\frac{a}{1-r} = \frac{1/4^2}{1-\frac{1}{4}} = \frac{1/16}{3/4} = \frac{1}{12}$. The original series $\boxed{\textit{converges}}$ to a sum of $\frac{1}{4} + \frac{1}{12} = \boxed{\frac{1}{3}.}$

28. $\sum\limits_{k=0}^{\infty} \frac{4^{k+1}}{3^k} = \frac{4^{0+1}}{3^0} + \sum\limits_{k=1}^{\infty} \frac{4^{k+1}}{3^k} = 4 + \sum\limits_{k=1}^{\infty} \frac{4^2}{3} \left(\frac{4}{3}\right)^{k-1}$.

The second term has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{4^2}{3}$ and $r = \frac{4}{3}$. Since $|r| > 1$, this series $\boxed{\textit{diverges}}$, and so does the original series.

29. $\sum\limits_{k=1}^{\infty} \sin^{k-1}\left(\frac{\pi}{2}\right)$ has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \sin\frac{\pi}{2} = 1$. Since $|r| = 1$, this series $\boxed{\textit{diverges.}}$

30. $\sum\limits_{k=1}^{\infty} \tan^{k-1}\left(\frac{\pi}{4}\right)$ has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \tan\frac{\pi}{4} = 1$. Since $|r| = 1$, this series $\boxed{\textit{diverges.}}$

31. $\sum\limits_{k=1}^{\infty} \left(-\frac{3}{2}\right)^{k-1}$ has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = -\frac{3}{2}$. Since $|r| > 1$, this series $\boxed{\textit{diverges.}}$

32. $\sum\limits_{k=1}^{\infty} \left(-\frac{2}{3}\right)^{k-1}$ has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = -\frac{2}{3}$. Since $|r| < 1$, this series will $\boxed{\textit{converge}}$ to a sum of $\frac{a}{1-r} = \frac{1}{1-\left(-\frac{2}{3}\right)} = \frac{1}{5/3} = \boxed{\frac{3}{5}.}$

33. $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \left(\frac{1}{3}\right)^n + \cdots = \sum\limits_{k=1}^{\infty} \left(\frac{1}{3}\right)^{k-1}$ has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \frac{1}{3}$. Since $|r| < 1$, this series $\boxed{\textit{converges}}$ to a sum of $\frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{1}{2/3} = \boxed{\frac{3}{2}.}$

34. $1 + \frac{1}{4} + \frac{1}{16} + \cdots + \left(\frac{1}{4}\right)^n + \cdots = \sum\limits_{k=1}^{\infty} \left(\frac{1}{4}\right)^{k-1}$ has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \frac{1}{4}$. Since $|r| < 1$, this series $\boxed{\textit{converges}}$ to a sum of $\frac{a}{1-r} = \frac{1}{1-\frac{1}{4}} = \frac{1}{3/4} = \boxed{\frac{4}{3}.}$

35. $1+2+4+\cdots+2^n+\cdots=\sum\limits_{k=1}^{\infty}2^{k-1}$ has the form of a geometric series $\sum\limits_{k=1}^{\infty}ar^{k-1}$ with $a=1$ and $r=2$. Since $|r|>1$, this series will $\boxed{\textit{diverge.}}$

36. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+\frac{(-1)^{n-1}}{2^{n-1}}+\cdots=\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}}{2^{k-1}}=\sum\limits_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1}$ has the form of a geometric series $\sum\limits_{k=1}^{\infty}ar^{k-1}$ with $a=1$ and $r=-\frac{1}{2}$. Since $|r|<1$, this series $\boxed{\textit{converges}}$ to a sum of $\frac{a}{1-r}=\frac{1}{1-(-\frac{1}{2})}=\frac{1}{3/2}=\boxed{\frac{2}{3}.}$

37. $\left(\frac{1}{7}\right)^2+\left(\frac{1}{7}\right)^3+\cdots+\left(\frac{1}{7}\right)^n+\cdots=\sum\limits_{k=1}^{\infty}\left(\frac{1}{7}\right)^{k-1}-1-\frac{1}{7}.$

The first term has the form of a geometric series $\sum\limits_{k=1}^{\infty}ar^{k-1}$ with $a=1$ and $r=\frac{1}{7}$. Since $|r|<1$, this series will converge to a sum of $\frac{a}{1-r}=\frac{1}{1-\frac{1}{7}}=\frac{1}{6/7}=\frac{7}{6}$. The original series $\boxed{\textit{converges}}$ to a sum of $\frac{7}{6}-1-\frac{1}{7}=\boxed{\frac{1}{42}.}$

38. $\left(\frac{3}{4}\right)^5+\left(\frac{3}{4}\right)^6+\cdots+\left(\frac{3}{4}\right)^n+\cdots=\sum\limits_{k=1}^{\infty}\left(\frac{3}{4}\right)^{k-1}-1-\frac{3}{4}-\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^3-\left(\frac{3}{4}\right)^4.$

The first term has the form of a geometric series $\sum\limits_{k=1}^{\infty}ar^{k-1}$ with $a=1$ and $r=\frac{3}{4}$. Since $|r|<1$, this series will converge to a sum of $\frac{a}{1-r}=\frac{1}{1-\frac{3}{4}}=\frac{1}{1/4}=4$. The original series $\boxed{\textit{converges}}$ to a sum of $4-1-\frac{3}{4}-\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^3-\left(\frac{3}{4}\right)^4=3-\frac{3}{4}-\frac{9}{16}-\frac{27}{64}-\frac{81}{256}=\boxed{\frac{243}{256}.}$

39. The series can be written $\sum\limits_{k=0}^{\infty}\frac{1}{k+1}=\sum\limits_{k'=1}^{\infty}\frac{1}{k'}$, where $k'=k+1$. This is the harmonic series, which $\boxed{\textit{diverges.}}$

40. $\sum\limits_{k=4}^{\infty}k^{-1}=\sum\limits_{k=4}^{\infty}\frac{1}{k}=\sum\limits_{k=1}^{\infty}\frac{1}{k}-1-\frac{1}{2}-\frac{1}{3}$. The first term is the harmonic series. It $\boxed{\textit{diverges}}$, and so does the original series.

41. $\sum\limits_{k=1}^{\infty}\frac{1}{100^k}=\sum\limits_{k=1}^{\infty}\frac{1}{100}\cdot\left(\frac{1}{100}\right)^{k-1}$ has the form of a geometric series $\sum\limits_{k=1}^{\infty}ar^{k-1}$ with $a=\frac{1}{100}$ and $r=\frac{1}{100}$. Since $|r|<1$, this series $\boxed{\textit{converges}}$ to a sum of $\frac{a}{1-r}=\frac{\frac{1}{100}}{1-\frac{1}{100}}=\frac{1/100}{99/100}=\boxed{\frac{1}{99}.}$

42. $\sum\limits_{k=1}^{\infty}e^{-k}=\sum\limits_{k=1}^{\infty}\frac{1}{e^k}=\sum\limits_{k=1}^{\infty}\frac{1}{e}\cdot\left(\frac{1}{e}\right)^{k-1}$ has the form of a geometric series $\sum\limits_{k=1}^{\infty}ar^{k-1}$ with $a=\frac{1}{e}$ and $r=\frac{1}{e}$. Since $|r|<1$, this series $\boxed{\textit{converges}}$ to a sum of $\frac{a}{1-r}=\frac{\frac{1}{e}}{1-\frac{1}{e}}=\boxed{\frac{1}{e-1}.}$

43. $\sum\limits_{k=1}^{\infty}(-10k)$. The nth partial sum is $S_n=\sum\limits_{k=1}^{n}(-10k)=(-10)\sum\limits_{k=1}^{\infty}k=(-10)\cdot\frac{n(n+1)}{2}$, using the standard formula for the sum of the first n integers. We have

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}(-10)\frac{n(n+1)}{2}=-\infty.$$

Since the sequence of partial sums $\{S_n\}$ diverges, the series $\boxed{\textit{diverges}}$ as well.

44. $\sum\limits_{k=1}^{\infty} \frac{3k}{5}$. The nth partial sum is $S_n = \sum\limits_{k=1}^{n} \frac{3k}{5} = \frac{3}{5} \sum\limits_{k=1}^{\infty} k = \frac{3}{5} \cdot \frac{n(n+1)}{2}$, using the standard formula for the sum of the first n integers. We have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{5}\frac{n(n+1)}{2} = \infty.$$

Since the sequence of partial sums $\{S_n\}$ diverges, the series $\boxed{\textit{diverges}}$ as well.

45. $\sum\limits_{k=1}^{\infty} \cos^{k-1}\left(\frac{2\pi}{3}\right)$ has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \cos\frac{2\pi}{3} = -\frac{1}{2}$. Since $|r| < 1$, this series will $\boxed{\textit{converge}}$ to a sum of $\frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{\frac{3}{2}} = \boxed{\frac{2}{3}.}$

46. $\sum\limits_{k=1}^{\infty} \sin^{k-1}\left(\frac{\pi}{6}\right)$ has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = 1$ and $r = \sin\frac{\pi}{6} = \frac{1}{2}$. Since $|r| < 1$, this series will $\boxed{\textit{converge}}$ to a sum of $\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2.}$

47. $\sum\limits_{k=1}^{\infty} \frac{\tan^k\left(\frac{\pi}{4}\right)}{k} = \sum\limits_{k=1}^{\infty} \frac{1}{k}$, since $\tan\frac{\pi}{4} = 1$. This is a harmonic series and so it $\boxed{\textit{diverges.}}$

48. $\sum\limits_{k=1}^{\infty} \frac{\sin^k\left(\frac{\pi}{2}\right)}{k} = \sum\limits_{k=1}^{\infty} \frac{1}{k}$, since $\sin\frac{\pi}{2} = 1$. This is a harmonic series and so it $\boxed{\textit{diverges.}}$

49. $\sum\limits_{k=1}^{\infty} \cos(\pi k) = \cos\pi + \cos 2\pi + \cos 3\pi + \cdots + \cos n\pi + \cdots$ Now, $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, etc. In general, $\cos n\pi = (-1)^n$. So the series can be written $\sum\limits_{k=1}^{\infty} (-1)^k = \sum\limits_{k=1}^{\infty} (-1)(-1)^{k-1}$, which has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = -1$ and $r = -1$. Since $|r| = 1$, this series $\boxed{\textit{diverges.}}$

50. $\sum\limits_{k=1}^{\infty} \sin\left(\frac{\pi k}{2}\right) = \sin\frac{\pi}{2} + \sin\pi + \sin\frac{3\pi}{2} + \sin 2\pi + \cdots$. Now, $\sin\frac{\pi}{2} = 1$, $\sin\pi = 0$, $\sin\frac{3\pi}{2} = -1$, $\sin 2\pi = 0$, etc. We see that the partial sums $S_n = \sum\limits_{k=1}^{n} \sin\left(\frac{\pi k}{2}\right)$ will alternate between the values of $+1$ and 0, depending on the value of n. The sequence of partial sums is $\{S_n\} = \{1, 1, 0, 0, 1, 1, 0, 0, \cdots\}$. Since the limit of the sequence of partial sums does not exist, the series $\boxed{\textit{diverges.}}$

51. $\sum\limits_{k=1}^{\infty} 2^{-k}3^{k+1} = \sum\limits_{k=1}^{\infty} \frac{3^{k+1}}{2^k} = \sum\limits_{k=1}^{\infty} \frac{3^2\cdot 3^{k-1}}{2\cdot 2^{k-1}} = \sum\limits_{k=1}^{\infty} \frac{9}{2}\left(\frac{3}{2}\right)^{k-1}$ has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{9}{2}$ and $r = \frac{3}{2}$. Since $|r| > 1$, this series $\boxed{\textit{diverges.}}$

52. $\sum\limits_{k=1}^{\infty} 3^{1-k}2^{1+k} = \sum\limits_{k=1}^{\infty} \frac{2^{k+1}}{3^{k-1}} = \sum\limits_{k=1}^{\infty} 2^2 \cdot \left(\frac{2}{3}\right)^{k-1}$ has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = 2^2 = 4$ and $r = \frac{2}{3}$. Since $|r| < 1$, this series will $\boxed{\textit{converge}}$ to a sum of $\frac{a}{1-r} = \frac{4}{1-\frac{2}{3}} = \frac{4}{\frac{1}{3}} = \boxed{12.}$

53. $\sum\limits_{k=1}^{\infty}\left(-\frac{1}{3}\right)^k = \sum\limits_{k=1}^{\infty}\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)^{k-1}$ has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = -\frac{1}{3}$ and $r = -\frac{1}{3}$. Since $|r| < 1$, this series $\boxed{\textit{converges}}$ to a sum of $\frac{a}{1-r} = \frac{-\frac{1}{3}}{1-\left(-\frac{1}{3}\right)} = \frac{-1/3}{4/3} = \boxed{-\frac{1}{4}.}$

54. $\sum\limits_{k=1}^{\infty}\frac{\pi}{3^k} = \sum\limits_{k=1}^{\infty}\frac{\pi}{3}\cdot\left(\frac{1}{3}\right)^{k-1}$ has the form of a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{\pi}{3}$ and $r = \frac{1}{3}$. Since $|r| < 1$, this series $\boxed{\textit{converges}}$ to a sum of $\frac{a}{1-r} = \frac{\frac{\pi}{3}}{1-\frac{1}{3}} = \frac{\pi/3}{2/3} = \boxed{\frac{\pi}{2}.}$

55. $\sum\limits_{k=1}^{\infty}\ln\frac{k}{k+1} = \sum\limits_{k=1}^{\infty}\left(\ln k - \ln(k+1)\right)$. The nth partial sum is

$$\begin{aligned} S_n = \sum_{k=1}^{n}\left(\ln k - \ln(k+1)\right) &= (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots + (\ln(n-1) - \ln n) + (\ln n - \ln(n+1)) \\ &= \ln 1 - \ln(n+1) = -\ln(n+1). \end{aligned}$$

We have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[-\ln(n+1)\right] = -\infty.$$

Since the limit of the partial sums diverges, the sequence $\{S_n\}$ is a divergent sequence. This in turn means that the original series is $\boxed{\textit{divergent.}}$

56. $\sum\limits_{k=1}^{\infty}\left[e^{2k^{-1}} - e^{2(k+1)^{-1}}\right]$. The nth partial sum is

$$\begin{aligned} S_n = \sum_{k=1}^{n}\left(e^{2k^{-1}} - e^{2(k+1)^{-1}}\right) &= \left(e^{2/1} - e^{2/2}\right) + \left(e^{2/2} - e^{2/3}\right) + \cdots + \left(e^{2/(n-1)} - e^{2/n}\right) + \left(e^{2/n} - e^{2/(n+1)}\right) \\ &= e^2 - e^{2/(n+1)}. \end{aligned}$$

We have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\sum_{k=1}^{n}\left(e^{2k^{-1}} - e^{2(k+1)^{-1}}\right) = \lim_{n\to\infty} e^2 - \lim_{n\to\infty} e^{2/(n+1)} = e^2 - e^{\lim_{n\to\infty} 2/(n+1)} = e^2 - e^0 = e^2 - 1.$$

Since the limit of partial sums $\{S_n\}$ converges to a limit, the series $\boxed{\textit{converges}}$ to the same limit $\boxed{e^2 - 1.}$

57. $\sum\limits_{k=1}^{\infty}\left(\sin\frac{1}{k} - \sin\frac{1}{k+1}\right)$. The nth partial sum is

$$\begin{aligned} S_n &= \sum_{k=1}^{n}\left(\sin\frac{1}{k} - \sin\frac{1}{k+1}\right) \\ &= \left(\sin\frac{1}{1} - \sin\frac{1}{2}\right) + \left(\sin\frac{1}{2} - \sin\frac{1}{3}\right) + \cdots + \left(\sin\frac{1}{n-1} - \sin\frac{1}{n}\right) + \left(\sin\frac{1}{n} - \sin\frac{1}{n+1}\right) \\ &= \sin 1 - \sin\frac{1}{n+1}. \end{aligned}$$

We have

$$\begin{aligned} \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\sin 1 - \sin\frac{1}{n+1}\right) = \lim_{n\to\infty}\sin 1 - \lim_{n\to\infty}\sin\frac{1}{n+1} = \sin 1 - \sin\left(\lim_{n\to\infty}\frac{1}{n+1}\right) \\ = \sin 1 - \sin 0 = \sin 1. \end{aligned}$$

Since the limit of partial sums $\{S_n\}$ converges to a limit, the series $\boxed{\textit{converges}}$ to the same limit $\boxed{\sin 1.}$

58. $\sum\limits_{k=1}^{\infty}\left(\tan\frac{1}{k}-\tan\frac{1}{k+1}\right)$. The nth partial sum is

$$\begin{aligned}S_n &= \sum_{k=1}^{n}\left(\tan\frac{1}{k}-\tan\frac{1}{k+1}\right)\\ &= \left(\tan\frac{1}{1}-\tan\frac{1}{2}\right)+\left(\tan\frac{1}{2}-\tan\frac{1}{3}\right)+\cdots+\left(\tan\frac{1}{n-1}-\tan\frac{1}{n}\right)+\left(\tan\frac{1}{n}-\tan\frac{1}{n+1}\right)\\ &= \tan 1-\tan\frac{1}{n+1}.\end{aligned}$$

We have

$$\begin{aligned}\lim_{n\to\infty} S_n &= \lim_{n\to\infty}\left(\tan 1-\tan\frac{1}{n+1}\right)=\lim_{n\to\infty}\tan 1-\lim_{n\to\infty}\tan\frac{1}{n+1}=\tan 1-\tan\left(\lim_{n\to\infty}\frac{1}{n+1}\right)\\ &= \tan 1-\tan 0=\tan 1.\end{aligned}$$

Since the limit of partial sums $\{S_n\}$ converges to a limit, the series $\boxed{\textit{converges}}$ to the same limit $\boxed{\tan 1.}$

59. Write the given decimal as follows:

$$\begin{aligned}0.5555\cdots = 0.5+0.05+0.005+\cdots &= \frac{5}{10}+\frac{5}{100}+\frac{5}{1000}+\cdots\\ &= \frac{5}{10}\left(1+\frac{1}{10}+\frac{1}{100}+\cdots\right)\\ &= \sum_{k=1}^{\infty}\frac{5}{10}\left(\frac{1}{10}\right)^{k-1}.\end{aligned}$$

This is a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a=\frac{5}{10}$ and $r=\frac{1}{10}$. Since $|r|<1$, this series converges to a sum of $\frac{a}{1-r}=\frac{\frac{5}{10}}{1-\frac{1}{10}}=\frac{5/10}{9/10}=\frac{5}{9}$. So the decimal written as a rational number is $0.5555\cdots=\boxed{\frac{5}{9}.}$

60. Write the given decimal as follows:

$$\begin{aligned}0.727272\cdots = 0.72+0.0072+0.000072+\cdots &= \frac{72}{100}+\frac{72}{10000}+\frac{72}{1000000}+\cdots\\ &= \frac{72}{100}\left(1+\frac{1}{100}+\frac{1}{10000}+\cdots\right)\\ &= \sum_{k=1}^{\infty}\frac{72}{100}\left(\frac{1}{100}\right)^{k-1}.\end{aligned}$$

This is a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a=\frac{72}{100}$ and $r=\frac{1}{100}$. Since $|r|<1$, this series converges to a sum of $\frac{a}{1-r}=\frac{\frac{72}{100}}{1-\frac{1}{100}}=\frac{72/100}{99/100}=\frac{72}{99}=\frac{8}{11}$. So the decimal written as a rational number is $0.727272\cdots=\boxed{\frac{8}{11}.}$

61. Write the given decimal as follows:

$$\begin{aligned}4.28555\cdots &= 4.28 + 0.00555\cdots = 4.28 + 0.005 + 0.0005 + 0.00005 + \cdots\\ &= 4.28 + \frac{5}{1000} + \frac{5}{10000} + \frac{5}{100000} + \cdots = 4.28 + \frac{5}{1000}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right)\\ &= 4.28 + \sum_{k=1}^{\infty} \frac{5}{1000}\left(\frac{1}{10}\right)^{k-1}.\end{aligned}$$

The second term is a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{5}{1000}$ and $r = \frac{1}{10}$. Since $|r| < 1$, this series converges to a sum of $\frac{a}{1-r} = \frac{\frac{5}{1000}}{1-\frac{1}{10}} = \frac{5/1000}{9/10} = \frac{5}{900}$. So the original series sums to the value of the decimal in rational form, which is $4.28 + \frac{5}{900} = \frac{428}{100} + \frac{5}{900} = \boxed{\frac{3857}{900}}.$

62. Write the given decimal as follows:

$$\begin{aligned}7.162162162\cdots &= 7 + 0.162162162\cdots = 7 + 0.162 + 0.000162 + 0.000000162 + \cdots\\ &= 7 + \frac{162}{1,000} + \frac{162}{1,000,000} + \frac{162}{1,000,000,000} + \cdots\\ &= 7 + \frac{162}{1000}\left(1 + \frac{1}{1,000} + \frac{1}{1,000,000} + \cdots\right)\\ &= 7 + \sum_{k=1}^{\infty} \frac{162}{1000}\left(\frac{1}{1000}\right)^{k-1}.\end{aligned}$$

The second term is a geometric series $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{162}{1000}$ and $r = \frac{1}{1000}$. Since $|r| < 1$, this series converges to a sum of $\frac{a}{1-r} = \frac{\frac{162}{1000}}{1-\frac{1}{1000}} = \frac{162/1000}{999/1000} = \frac{162}{999}$. So the original series sums to the value of the decimal in rational form, which is $7 + \frac{162}{999} = \boxed{\frac{265}{37}}.$

Applications and Extensions

63. Let the original height from which the ball is dropped be h_0. The height attained after the first bounce $h_1 = h_0 \cdot r$, where $r =$ fraction of the original height. The height attained after the second bounce $h_2 = h_1 \cdot r = (h_0 r)r = h_0 r^2$. Proceeding in this fashion, we can say that the height attained after the nth bounce will be $h_n = h_0 r^n$.
The total vertical distance traveled by the ball, remembering that each bounce has an upward vertical travel followed by a downward vertical travel, is given by

$$\begin{aligned}H &= h_0 + 2h_1 + 2h_2 + 2h_3 + \cdots + 2h_n + \cdots\\ &= h_0 + 2(h_0 r + h_0 r^2 + h_0 r^3 + \cdots + h_0 r^n + \cdots)\\ &= h_0 + 2h_0 r(1 + r + r^2 + \cdots + r^{n-1} + \cdots)\\ &= h_0 + 2h_0 r \cdot \sum_{k=1}^{\infty} r^{k-1}.\end{aligned}$$

Since the process of bouncing is not an elastic collision for the ball in question, we have $|r| < 1$, so that the second term is a convergent geometric series of the form $\sum\limits_{k=1}^{\infty} ar^{k-1}$ which sums to

$\frac{a}{1-r}$, with $a = 1$, so the sum will be $\frac{1}{1-r}$. So the total vertical distance traversed by the bouncing ball is
$H = h_0 + 2h_0 r \cdot \frac{1}{1-r} = h_0 \left[1 + \frac{2r}{1-r}\right] = h_0 \left(\frac{1+r}{1-r}\right) = 18\text{ft} \left(\frac{1+\frac{2}{3}}{1-\frac{2}{3}}\right) = \frac{18\cdot(5/3)}{1/3}\text{ft} = \boxed{90\text{ft.}}$

64. (a) Let the initial dollar amount on January 1, 2020, be $A_1 = A_0$. The amount on January 2 will be $A_2 = rA_1 = rA_0$, where $r =$ the fraction of the original amount given on January 1. On January 3, the amount received will be $r(rA_2) = r^2 A_0$, and so forth. The total amount received $A = A_0 + rA_0 + r^2 A_0 + \cdots = A_0(1 + r + r^2 + \cdots) = \sum_{k=1}^{\infty} A_0 r^{k-1}$. If $|r| < 1$, this is a convergent geometric series with sum $\frac{A_0}{1-r}$. So the total amount received
$A = \frac{A_0}{1-r} = \frac{\$1000}{1-\frac{1}{9}} = \frac{\$1000}{1/10} = \boxed{\$10,000.}$

(b) We want the amount received on the nth day to be less than one cent = \$0.01. So $A_n = r^{n-1}A_0 < \$0.01$ gives $r^{n-1} < \frac{\$0.01}{\$1000} = 10^{-5}$. Taking the $\log_{10}$ of both sides, remembering that $\log_{10} r$ is an increasing function of r, we have

$$\begin{aligned}(n-1)\log_{10} r &< -5\log_{10} 10\\ n-1 &> -\frac{5}{\log_{10} r}\\ n-1 &> -\frac{5}{\log_{10}\frac{9}{10}}\\ &= \frac{-5}{-0.04576} \approx 109.27,\end{aligned}$$

(where the inequality has been reversed since $|r| < 1$ gives negative values for $\log_{10} r$), or $n - 1 = 110$ days, so $n = 111$ days. Now, 2020 is a leap year, so February 2020 has 29 days. January has 31 days, March has 31, for a total of $29 + 31 + 31 = 91$ days. So a total time period of 111 days after (and including) January 1 would fall on $\boxed{\text{April } 20, 2020.}$ On, and after this day, the amount paid out will be less than 1 cent.

65. (a) The population of rainbow trout in Mirror Lake in year n of the restocking program is $p_n = 3000r^n + h\sum_{k=1}^{n} r^{k-1}$. As $n \to \infty$, the manager can expect the steady rainbow trout population to be

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} 3000r^n + h \lim_{n\to\infty} \sum_{k=1}^{n} r^{k-1} = 3000 \lim_{n\to\infty} r^n + h\sum_{k=1}^{\infty} r^{k-1}.$$

Since $|r| < 1$. we have $\lim_{n\to\infty} r^n = 0$. Also the second term is a convergent geometric series with sum $\frac{h}{1-r}$. So the steady population of rainbow trout is $p = 3000 \cdot 0 + \frac{h}{1-r} = \boxed{\frac{h}{1-r}.}$

(b) We require $p = 4000$ with $r = 0.5$. This means $h = p(1 - r) = 4000(1 - 0.5) = \boxed{2000}$ is the number of fish to be added anually to obtain the steady population.

66. The multiplier for a marginal propensity to consume of 90% is
$\sum_{k=1}^{\infty} (0.90)^{k-1} = \frac{1}{1-0.9} = \frac{1}{0.1} = \boxed{10.}$

67. (a) The expression for stock pricing can be written as
$\text{Price} = P + P\frac{1+i}{1+r} + P\left(\frac{1+i}{1+r}\right)^2 + P\left(\frac{1+i}{1+r}\right)^3 + \cdots = \sum_{k=1}^{\infty} P\left(\frac{1+i}{1+r}\right)^{k-1} = \frac{P}{1-\left(\frac{1+i}{1+r}\right)} = \boxed{\frac{P(1+r)}{r-i}.}$

(Here, we have used the fact that in most normal circumstances, the annual rate of return $r\%$ will always be greater than the annual dividend increase $i\%$, which means $\left|\frac{1+i}{1+r}\right| < 1$, and so the geometric series will converge to the sum we found rather than diverge.)

(b) If $r = 9\%$ the annual rate of return on a stock, $P = \$4.00$ what the stock currently pays in annual dividends, and $i = 3\%$ the annual increase in dividend, then the highest stock price the investor should pay will be about $\frac{P(1+r)}{r-i} = \frac{\$4(1+9)}{9-3} = \frac{\$4(10)}{6} = \boxed{\$6.67.}$

68. Area of the Koch snowflake is

$$\begin{aligned}
A &= 1 + 3\left(\frac{1}{9}\right) + 12\left(\frac{1}{9}\right)^2 + 48\left(\frac{1}{9}\right)^3 + 192\left(\frac{1}{9}\right)^4 + \cdots \\
&= 1 + \frac{3}{9}\left[1 + 4\cdot\frac{1}{9} + 16\cdot\left(\frac{1}{9}\right)^2 + 64\cdot\left(\frac{1}{9}\right)^3 + \cdots\right] \\
&= 1 + \frac{1}{3}\left[1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \cdots\right] \\
&= 1 + \frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{4}{9}\right)^{k-1} \\
&= 1 + \frac{1}{3}\cdot\frac{1}{1-\frac{4}{9}} = 1 + \frac{1}{3}\cdot\frac{9}{9-4} = 1 + \frac{3}{5} \\
&= \boxed{\frac{8}{5}\text{ square units.}}
\end{aligned}$$

69. Let the length of the race track be L. Let the tortoise be given a lead of $d < L$. If Achilles' speed is v_A and the tortoise's speed is v_T, then we are given that $v_A - v_T = v$, a (positive) constant. If the tortoise starts off at $t = 0$, then the time elapsed before Achilles starts is $\Delta t = \frac{d}{v_T}$. The time taken by Achilles to cover half the distance separating them would be $t_1 = \frac{d}{2v_A}$. By this time, the tortoise has moved out further by a distance $= v_T t_1 = \frac{v_T}{2v_A}d$. The new remaining distance between them $= \frac{d}{2} + \frac{v_T}{2v_A}d = \frac{d}{2} + \frac{v_A - v}{v_A}\frac{d}{2} = d - \frac{v}{v_A}\frac{d}{2} = d_1$, say.
The time taken by Achilles to cover half this new distance is $t_2 = \frac{d_1}{2v_A}$. The tortoise will by then have moved out further by a distance $= \frac{v_T}{2v_A}d_1$. The new remaining distance separating them,

$$\begin{aligned}
d_2 &= d_1 - \frac{v}{v_A}\frac{d_1}{2} \\
&= d_1\left(1 - \frac{v}{2v_A}\right) \\
&= d\left(1 - \frac{v}{2v_A}\right)\left(1 - \frac{v}{2v_A}\right) \\
&= d\left(1 - \frac{v}{2v_A}\right)^2.
\end{aligned}$$

After n such halvings, the distance separating them $d_n = d\left(1 - \frac{v}{2v_A}\right)^n$. Since $\boxed{r = 1 - \frac{v}{2v_A}} < 1$, we have

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} dr^n = d\lim_{n\to\infty} r^n = 0.$$

This demonstrates that Achilles will eventually catch up with the tortoise, since the distance separating them tends to 0. In fact, we need to next show that this process takes a *finite* amount of time.

The total time for catching up after a time Δt has elapsed (this is the lead time given to the tortoise) is

$$\begin{aligned}
T &= t_1 + t_2 + \cdots + t_n + \cdots \\
&= \frac{d}{2v_A} + \frac{d_1}{2v_A} + \frac{d_2}{2v_A} + \cdots + \frac{d_{n-1}}{2v_A} + \cdots \\
&= \frac{d}{2v_A} + r\frac{d}{2v_A} + r^2\frac{d}{2v_A} + \cdots \\
&= \frac{d}{2v_A}(1 + r + r^2 + \cdots) \\
&= \frac{d}{2v_A}\sum_{k=1}^{\infty} r^{k-1} = \frac{d}{2v_A}\left(\frac{1}{1-r}\right) \\
&= \frac{d}{2v_A}\left(\frac{1}{1-\left(1-\frac{v}{2v_A}\right)}\right) = \frac{d}{2v_A}\left(\frac{1}{\frac{v}{2v_A}}\right) \\
&= \frac{d}{v}.
\end{aligned}$$

This is exactly as we would expect, the time taken by Achilles to overtake the tortoise is just the original distance separating them divided by the relative difference in speeds. So we see by using a series argument that Achilles will catch up with the tortoise in a finite time.

Next, we show using a series argument that the total distance covered by Achilles in catching up with the tortoise is also finite. The total distance is

$$\begin{aligned}
D &= \frac{d}{2} + \frac{d_1}{2} + \frac{d_2}{2} + \cdots \\
&= \frac{d}{2} + \frac{rd}{2} + \frac{r^2 d}{2} + \cdots \\
&= \frac{1}{2}d(1 + r + r^2 + \cdots) \\
&= \frac{1}{2}d\sum_{k=1}^{\infty} r^{k-1} = \frac{1}{2}d \cdot \frac{1}{1-r} = \frac{1}{2}d \cdot \left(\frac{1}{1-\left(1-\frac{v}{2v_A}\right)}\right) = \frac{v_A}{v}d \\
&= \frac{v_A}{v_A - v_T}d.
\end{aligned}$$

Since $v_T > 0$, this distance is bigger than d, as we would expect. If the speeds are such that $D < L$, then Achilles may even overtake the tortoise and win the race!

70. Probability of winning
$= \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots + \frac{1}{2^{2n-1}} + \cdots = \sum_{k=1}^{\infty} \frac{1}{2^{2k-1}} = \sum_{k=1}^{\infty} \frac{1}{2}\frac{1}{2^{2k-2}} = \sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{1}{4}\right)^{k-1}$.

This is a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{1}{2}$ and $r = \frac{1}{4}$, which, since $|r| < 1$, will converge to a sum of $\frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{1/2}{3/4} = \boxed{\frac{2}{3}.}$

71. **(a)** Amount of pig and cattle feces produced prior to cleaning on the first day $T(1) = p$. After the cleaning at the end of the first day, amount of feces left over = $(1-e)T(1) = (1-e)p$. Amount of feces produced on the second day, prior to cleaning, $T(2) = p + (1-e)p$. After the end of the second day's cleaning, amount of feces left over = $(1-e)T(2) = (1-e)[p + (1-e)p] = (1-e)p + (1-e)^2 p$. Then, amount of feces produced on the third day, prior to cleaning, $T(3) = p + (1-e)p + (1-e)^2 p$. Continuing this way, the total amount of accumulated fecal matter on the nth day prior to cleaning will be

$$T(n) = p + (1-e)p + (1-e)^2 p + \cdots + (1-e)^{n-1} p = \boxed{p \sum_{k=1}^{n} (1-e)^{k-1}.}$$

(b) $T = \lim_{n\to\infty} T(n) = \lim_{n\to\infty} p \sum_{k=1}^{n} (1-e)^{k-1} = \sum_{k=1}^{\infty} (1-e)^{k-1} = p \cdot \frac{1}{1-(1-e)} = \boxed{\frac{p}{e}}$, since the series is a converging geometric series of the type $\sum_{k=1}^{\infty} r^{k-1}$ with $|r| = |1-e| < 1$.

(c) From part (b) we see that $T = \frac{p}{e}$. If we require that $T \le L$, then we would have $T(n) \le L$ for all n, since $n < \infty$ and $T(n)$ is a monotonic increasing function of n. Now the condition $T \le L$ leads to $\frac{p}{e} \le L$, or $e \ge \frac{p}{L}$. This shows that the minimum efficiency of cleaning is $e_{\text{min}} = \boxed{\frac{p}{L}.}$

(d) We have $e_{\text{min}} = \frac{p}{L} = \frac{120 \text{ kg}}{180 \text{ kg}} = \boxed{\frac{2}{3}.}$ At $e = \frac{4}{5}$, we have $T(365) \approx T(\infty) = \frac{p}{e} = \frac{120 \text{ kg}}{4/5} = \boxed{150 \text{ kg.}}$

72. Write

$$\begin{aligned} 0.9999\cdots &= 0.9 + 0.09 + 0.009 + 0.0009 + \cdots \\ &= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \cdots \\ &= \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots\right) \\ &= \frac{9}{10} \sum_{k=1}^{\infty} \left(\frac{1}{10}\right)^{k-1} = \frac{9}{10} \cdot \frac{1}{1 - \frac{1}{10}} = \frac{9}{10} \cdot \frac{1}{\frac{9}{10}} = 1, \end{aligned}$$

that is $0.9999\cdots = 1$, as was needed to be demonstrated.

73. The right hand side is a geometric series: $\sum_{k=1}^{\infty} \frac{1}{x^{k-1}} = \sum_{k=1}^{\infty} \left(\frac{1}{x}\right)^{k-1}$. If $|x| > 1$, then $\left|\frac{1}{x}\right| < 1$, so this series will converge to a sum of $\frac{1}{1-\frac{1}{x}} = \frac{x}{x-1}$. So we have shown that $\frac{x}{x-1} = \sum_{k=1}^{\infty} \frac{1}{x^{k-1}}$ for $|x| > 1$.

74. The right hand side is a geometric series: $\sum_{k=0}^{\infty} (-1)^k x^k = \sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1} = \sum_{k=1}^{\infty} (-x)^{k-1}$. Since $|r| = |-x| = |x| < 1$, this geometric series will converge to a sum of $\frac{1}{1-r} = \frac{1}{1-(-x)} = \frac{1}{1+x}$. So we have shown that $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$ for $|x| < 1$.

75. Since there is no closed form expression for the partial sums of the harmonic series, we have to proceed by trial and error calculating the partial sums. The partial sum $S_n = \sum_{k=1}^{n} \frac{1}{k}$ for $n = 10$ is 2.929, and for $n = 11$ it is 3.020. So $\boxed{n = 11}$ is the smallest number for which $\sum_{k=1}^{n} \frac{1}{k} \ge 3$.

76. Since there is no closed form expression for the partial sums of the harmonic series, we have to proceed by trial and error calculating the partial sums. The partial sum $S_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ for $n = 30$ is 3.995, and for $n = 31$ it is 4.027. So $\boxed{n = 31}$ is the smallest number for which $\sum\limits_{k=1}^{n} \frac{1}{k} \geq 4$.

77. The given series can be rewritten as follows: $\sum\limits_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}} = \sum\limits_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right)$. Consider the nth partial sum of the series:

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) \\ &= \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \cdots + \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right) + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) \\ &= 1 - \frac{1}{\sqrt{n+1}}. \end{aligned}$$

We have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{\sqrt{n+1}}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 1 - 0 = 1.$$

Since the sequence of partial sums converges, the series $\boxed{\textit{converges}}$ as well, and has the sum $\boxed{1.}$

78. The given series can be rewritten as follows: $\sum\limits_{k=1}^{\infty} \frac{1}{k(k+2)} = \sum\limits_{k=1}^{\infty} \left(\frac{1}{2k} - \frac{1}{2(k+2)}\right)$, using partial fractions. Consider the nth partial sum of the series:

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \left(\frac{1}{2k} - \frac{1}{2(k+2)}\right) = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right) \\ &= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \\ &+ \frac{1}{2}\left(\frac{1}{n-2} - \frac{1}{n}\right) + \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) \\ &= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right). \end{aligned}$$

We have

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= \lim_{n\to\infty} \frac{3}{4} - \lim_{n\to\infty} \frac{1}{2(n+1)} - \lim_{n\to\infty} \frac{1}{2(n+2)} = \frac{3}{4} - 0 - 0 = \frac{3}{4}. \end{aligned}$$

Since the sequence of partial sums converges, the series converges as well, and has the sum $\boxed{\frac{3}{4}.}$

79. The given series can be rewritten using the techniques of partial fraction decomposition as follows:

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)(k+2)}=\sum_{k=1}^{\infty}\frac{1}{2k}-\frac{1}{k+1}+\frac{1}{2(k+2)}=\sum_{k=1}^{\infty}\frac{1}{2}\left(\frac{1}{k}-\frac{2}{k+1}+\frac{1}{k+2}\right).$$

Consider the nth partial sum of the series:

$$\begin{aligned}S_n&=\sum_{k=1}^{n}\frac{1}{2}\left(\frac{1}{k}-\frac{2}{k+1}+\frac{1}{k+2}\right)=\sum_{k=1}^{n}\frac{1}{2}\left(\frac{1}{k}-\frac{1}{k+1}\right)-\sum_{k=1}^{n}\frac{1}{2}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)\\&=\left\{\frac{1}{2}\left(\frac{1}{1}-\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\frac{1}{2}\left(\frac{1}{n-1}-\frac{1}{n}\right)+\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+1}\right)\right\}-\\&\quad\left\{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+1}\right)+\frac{1}{2}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)\right\}\\&=\left\{\frac{1}{2}\left(\frac{1}{1}\right)-\frac{1}{2}\left(\frac{1}{n+1}\right)\right\}-\left\{\frac{1}{2}\left(\frac{1}{2}\right)-\frac{1}{2}\left(\frac{1}{n+2}\right)\right\}\\&=\frac{1}{4}-\frac{1}{2(n+1)}+\frac{1}{2(n+2)}.\end{aligned}$$

We have

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(\frac{1}{4}-\frac{1}{2(n+1)}+\frac{1}{2(n+2)}\right)=\lim_{n\to\infty}\frac{1}{4}-\lim_{n\to\infty}\frac{1}{2(n+1)}+\lim_{n\to\infty}\frac{1}{2(n+2)}=\frac{1}{4}-0+0=\frac{1}{4}.$$

Since the sequence of partial sums converges, the series converges as well, and has the sum $\boxed{\frac{1}{4}.}$

80. The given series can be rewritten using the techniques of partial fraction decomposition as follows:

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)}&=\sum_{k=1}^{\infty}\frac{1}{6k}-\frac{1}{2(k+1)}+\frac{1}{2(k+2)}-\frac{1}{6(k+3)}\\&=\sum_{k=1}^{\infty}\frac{1}{6k}-\frac{3}{6(k+1)}+\frac{3}{6(k+2)}-\frac{1}{6(k+3)}.\end{aligned}$$

Consider the nth partial sum of the series:

$$\begin{aligned}S_n&=\sum_{k=1}^{n}\frac{1}{6}\left(\frac{1}{k}-\frac{1}{k+1}\right)-\frac{1}{3}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)+\frac{1}{6}\left(\frac{1}{k+2}-\frac{1}{k+3}\right)\\&=\left\{\frac{1}{6}\left(\frac{1}{1}-\frac{1}{2}\right)+\frac{1}{6}\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\frac{1}{6}\left(\frac{1}{n-1}-\frac{1}{n}\right)+\frac{1}{6}\left(\frac{1}{n}-\frac{1}{n+1}\right)\right\}\\&\quad-\left\{\frac{1}{3}\left(\frac{1}{2}-\frac{1}{3}\right)+\frac{1}{3}\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+1}\right)+\frac{1}{3}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)\right\}\\&\quad+\left\{\frac{1}{6}\left(\frac{1}{3}-\frac{1}{4}\right)+\frac{1}{6}\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\frac{1}{6}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\frac{1}{6}\left(\frac{1}{n+2}-\frac{1}{n+3}\right)\right\}\\&=\left\{\frac{1}{6}\left(\frac{1}{1}\right)-\frac{1}{6}\left(\frac{1}{n+1}\right)\right\}-\left\{\frac{1}{3}\left(\frac{1}{2}\right)-\frac{1}{3}\left(\frac{1}{n+2}\right)\right\}+\left\{\frac{1}{6}\left(\frac{1}{3}\right)-\frac{1}{6}\left(\frac{1}{n+3}\right)\right\}\\&=\frac{1}{6}-\frac{1}{6}+\frac{1}{18}-\frac{1}{6(n+1)}+\frac{1}{3(n+2)}-\frac{1}{6(n+3)}\\&=\frac{1}{18}-\frac{1}{6(n+1)}+\frac{1}{3(n+2)}-\frac{1}{6(n+3)}.\end{aligned}$$

We have

$$\begin{aligned}\lim_{n\to\infty} S_n &= \lim_{n\to\infty}\left(\frac{1}{18}-\frac{1}{6(n+1)}+\frac{1}{3(n+2)}-\frac{1}{6(n+3)}\right)\\ &= \lim_{n\to\infty}\frac{1}{18}-\lim_{n\to\infty}\frac{1}{6(n+1)}+\lim_{n\to\infty}\frac{1}{3(n+2)}-\lim_{n\to\infty}\frac{1}{6(n+3)}\\ &= \frac{1}{18}-0+0-0=\frac{1}{18}.\end{aligned}$$

Since the sequence of partial sums converges, the series converges as well, and has the sum $\boxed{\frac{1}{18}.}$

81. We use the following result from the theory of partial fractions:
$F(k)=\frac{1}{\prod\limits_{i=1}^{m}(k+a_i)}=\sum\limits_{i=1}^{m}\frac{A_i}{k+a_i}$, where $A_i=(k+a_i)F(k)|_{k=-a_i}$. To prove this, multiply both sides by $\prod\limits_{i=1}^{m}(k+a_i)$. Then we get

$$\begin{aligned}1 &= \sum_{i=1}^{m}\frac{A_i}{(k+a_i)}\prod_{i=1}^{m}(k+a_i)\\ &= A_1(k+a_2)(k+a_3)\cdots(k+a_m)+A_2(k+a_1)(k+a_3)\cdots(k+a_m)+\\ &\quad A_3(k+a_1)(k+a_2)(k+a_4)\cdots(k+a_m)+\\ &\quad \cdots+A_{m-1}(k+a_1)\cdots(k+a_{m-2})(k+a_m)+A_m(k+a_1)\cdots(k+a_{m-1}).\end{aligned}$$

Note that each set of factors multiplying the coefficient A_i is missing the term $(k+a_i)$. So for instance, when we set $k=-a_1$, all terms vanish *except* the term multiplying A_1:

$$1=A_1(-a_1+a_2)(-a_1+a_3)\cdots(-a_1+a_m)$$

or,

$$A_1=\frac{1}{(-a_1+a_2)(-a_1+a_3)\cdots(-a_1+a_m)}=(k+a_1)F(k)|_{k=-a_1}$$

where $F(k)=\frac{1}{\prod\limits_{i=1}^{m}(k+a_i)}$. Similarly, for $k=-a_2$, we obtain $A_2=(k+a_2)F(k)|_{k=-a_2}$, etc, and in general, for $k=-a_i$, we have $A_i=(k+a_i)F(k)|_{k=-a_i}$, which proves the stated result.
Let $F(k)=\frac{1}{k(k+1)(k+2)\cdots(k+a)}$. Since $a_1=0, a_2=1, a_3=2,\cdots,a_i=i-1,\cdots,a_m=a$, we get $A_1=\frac{1}{a!}$, $A_2=-\frac{a}{a!}$, $A_3=(-1)^2\frac{a(a-1)}{2a!}$, and so on. In general, for $1\le i\le m$,
$A_i=\frac{(-1)^{i-1}}{(i-1)!}\frac{a(a-1)\cdots(a-i+2)}{a!}$.
For example, in Problem 79, we had the case $a=2$. We have $A_1=\frac{1}{2!}=\frac{1}{2}$, $A_2=-\frac{2}{2!}=-1$, and $A_3=(-1)^2\frac{2(2-1)}{2\cdot 2!}=\frac{1}{2}$. The partial fraction decomposition and the partial sum were:

$$\begin{aligned}\frac{1}{k(k+1)(k+2)} &= \frac{A_1}{k}+\frac{A_2}{k+1}+\frac{A_3}{k+2}\\ &= \frac{1}{2}\cdot\frac{1}{k}-\frac{1}{k+1}+\frac{1}{2}\cdot\frac{1}{k+2}\\ &= \frac{1}{2!}\left(\frac{1}{k}-\frac{2}{k+1}+\frac{1}{k+2}\right)\\ &= \frac{1}{2!}\left(\frac{1}{k}-\frac{1}{k+1}\right)-\frac{1}{2!}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)\end{aligned}$$

$$S_n=\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}=\frac{1}{2\cdot 2!}-\frac{1}{2(n+1)}+\frac{1}{2(n+2)}.$$

In Problem 80, we had the case $a = 3$, so $A_1 = \frac{1}{3!} = \frac{1}{6}$, $A_2 = -\frac{3}{3!} = -\frac{1}{2}$, $A_3 = (-1)^2\frac{3(3-1)}{2\cdot 3!} = \frac{1}{2}$, and $A_4 = (-1)^3\frac{3(3-1)(3-2)}{3!\cdot 3!} = -\frac{1}{6}$. The partial fraction decomposition and the partial sum were

$$\begin{aligned}
\frac{1}{k(k+1)(k+2)(k+3)} &= \frac{A_1}{k} + \frac{A_2}{k+1} + \frac{A_3}{k+2} + \frac{A_4}{k+3} \\
&= \frac{1}{6}\cdot\frac{1}{k} - \frac{1}{2}\cdot\frac{1}{k+1} + \frac{1}{2}\cdot\frac{1}{k+2} - \frac{1}{6}\cdot\frac{1}{k+3} \\
&= \frac{1}{3!}\left(\frac{1}{k} - \frac{3}{k+1} + \frac{3}{k+2} - \frac{1}{k+3}\right) \\
&= \frac{1}{3!}\left(\frac{1}{k} - \frac{1}{k+1}\right) - \frac{1}{3}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) + \frac{1}{6}\left(\frac{1}{k+2} - \frac{1}{k+3}\right) \\
S_n = \sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)(k+3)} &= \frac{1}{3\cdot 3!} - \frac{1}{6(n+1)} + \frac{1}{3(n+2)} - \frac{1}{6(n+3)}.
\end{aligned}$$

For the general case of a, we get, as in the previous two problems, a series of terms multiplying the coefficients A_i. When they are added as telescoping series of the partial sum S_n, the result will have a term $\frac{1}{a\cdot a!}$ without any $n-$dependence, while all the surviving terms will have the form $\frac{\alpha_i A_i}{n+a_i}$ where α_i are whole numbers coming from the distribution of terms in the telescoping series. That is, we will obtain:

$$\begin{aligned}
\frac{1}{k(k+1)(k+2)\cdots(k+a)} &= \frac{A_1}{k} + \frac{A_2}{k+1} + \frac{A_3}{k+2} + \frac{A_4}{k+3} + \cdots + \frac{A_i}{k+i-1} + \cdots + \frac{A_m}{k+a} \\
&= \frac{1}{a!}\cdot\frac{1}{k} - \frac{a}{a!}\cdot\frac{1}{k+1} + (-1)^2\frac{a(a-1)}{2!a!}\cdot\frac{1}{k+2} + (-1)^3\frac{a(a-1)(a-2)}{3!a!}\cdot\frac{1}{k+3} \\
&\qquad + \cdots + (-1)^{i-1}\frac{a(a-1)\cdots(a-i)}{(i-1)!a!}\cdot\frac{1}{k+i-1} + \cdots + (-1)^{a-1}\frac{1}{a!}\cdot\frac{1}{k+a} \\
&= \frac{1}{a!}\left(\frac{1}{k} - \frac{a}{k+1} + \frac{a(a-1)}{2!(k+2)} - \frac{a(a-1)(a-2)}{3!(k+3)} + \cdots\right) \\
&\qquad + \frac{1}{a!}\left(\cdots + (-1)^{i-1}\frac{a(a-1)\cdots(a-i)}{(i-1)!(k+i-1)} + \cdots + (-1)^{a-1}\frac{1}{k+a}\right) \\
&= \frac{1}{a!}\left(\frac{1}{k} - \frac{1}{k+1}\right) - \frac{a-1}{a!}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) + \cdots + \frac{(-1)^{a-1}}{a!}\left(\frac{1}{k+a-1} - \frac{1}{k+a}\right) \\
S_n = \sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)\cdots(k+a)} &= \frac{1}{a\cdot a!} + \sum_{k=1}^{a}\frac{\alpha_i A_i}{n+k}.
\end{aligned}$$

In the limit, as $n\to\infty$, we get $\lim\limits_{n\to\infty} S_n = \boxed{\frac{1}{a\cdot a!}}$, which is also the sum of the series.

82. The right hand side of $\frac{x}{2+2x} = x + x^2 + x^3 + \cdots$ can be written as $\sum\limits_{k=1}^{\infty} x\cdot x^{k-1}$. This is a geometric series of the form $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = x$ and $r = x$. The series has a convergent sum, $\frac{a}{1-r} = \frac{x}{1-x}$ for $|r| = |x| < 1$. So the equation becomes:

$$\begin{aligned}
\frac{x}{2(1+x)} &= \frac{x}{1-x} \\
1 - x &= 2 + 2x \\
3x &= -1,
\end{aligned}$$

or $\boxed{x = -\frac{1}{3}.}$

83. A convergent geometric series of the form $\sum\limits_{k=1}^{\infty} ar^{k-1}$, $|r| < 1$, has the sum $\frac{a}{1-r}$. If a is rational, then $a = \frac{p}{q}$, where p and q are integers that have no common factors, and $q \neq 0$; similarly, if r is rational, then $r = \frac{s}{t} < 1$, where s and t are integers with no common factors, and $t \neq 0$. Then, we have

$$\text{Sum} = \frac{a}{1-r} = \frac{\frac{p}{q}}{1-\frac{s}{t}} = \frac{pt}{q(t-s)}.$$

Since pt and $q(t-s)$ are integers, the ratio is a rational number as well, with $q(t-s) \neq 0$, since $\frac{s}{t} < 1$ (possibly with common factors, which can be canceled out to simplify the result).

84. $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. When $r = 1$, $S_n = a + a + \cdots + a \quad (n \text{ times}) = na$. On the other hand, since for $r \neq 1$, $S_n = \frac{a(r^n-1)}{r-1}$, we have

$$\lim_{r\to 1} S_n = \lim_{r\to 1} \frac{a(r^n-1)}{r-1} = a \lim_{r\to 1} \frac{nr^{n-1}}{1} = an = na,$$

by using L'Hôpital's rule. So the results of using either the series directly for $r = 1$, or applying a limiting procedure to the partial sum formula, agree with each other.

Challenge Problems

85. Since $\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right)$ and the harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges very slowly, we would expect the limit above to converge very slowly as well. Using $S_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ gives $\gamma_n \approx S_n - \ln n$ as an approximation to γ. Using the given values of the partial sums, we have:

$$\begin{aligned}
n = 10 &: S_{10} = 2.92897; \gamma_{10} \approx S_{10} - \ln 10 \approx \boxed{0.62638.}\\
n = 20 &: S_{20} = 3.59744; \gamma_{20} \approx S_{20} - \ln 20 \approx \boxed{0.60171.}\\
n = 50 &: S_{50} = 4.49921; \gamma_{50} \approx S_{50} - \ln 50 \approx \boxed{0.58719.}\\
n = 100 &: S_{100} = 5.18738; \gamma_{100} \approx S_{100} - \ln 100 \approx \boxed{0.58221.}
\end{aligned}$$

As expected, the rate of convergence of the limit defining γ is very slow as well.

86. Using (see solution to Problem 85) $\gamma_n \approx \sum\limits_{k=1}^{n} \frac{1}{k} - \ln n$, we have $\gamma_{10^9} \approx \sum\limits_{k=1}^{10^9} \frac{1}{k} - \ln 10^9$, or $\sum\limits_{k=1}^{10^9} \frac{1}{k} \approx \gamma_{10^9} + \ln 10^9$. Using $\gamma_{10^9} \approx \gamma \approx 0.5772$, we have, to four decimal places, $\sum\limits_{k=1}^{10^9} \frac{1}{k} \approx \boxed{21.3005.}$

87. If a real number has a repeating decimal, the repeating part can be expressed as a geometric series. Such a real number may be represented as

$$R = x.yz\cdots abc\cdots abc\cdots abc\cdots$$

(x is the whole number part) where, if the repeated units $abc\cdots$ start in (say) the nth decimal place, and have a string length r, we can write:

$$\begin{aligned}R &= x.yz\cdots + abc\cdots \times 10^{-n+1-r} + abc\cdots \times 10^{-n+1-2r} + abc\cdots \times 10^{-n+1-3r} + \cdots\\ &= \frac{xyz\cdots}{10^{n-1}} + abc\cdots \times 10^{-n+1-r}\left(1 + 10^{-r} + 10^{-2r} + \cdots\right)\\ &= \frac{xyz\cdots}{10^{n-1}} + \frac{abc\cdots}{10^{n-1+r}}\sum_{k=1}^{\infty}\left(10^{-r}\right)^{k-1}\\ &= \frac{xyz\cdots}{10^{n-1}} + \frac{abc\cdots}{10^{n-1+r}}\left(\frac{1}{1-10^{-r}}\right)\\ &= \frac{xyz\cdots}{10^{n-1}} + \frac{abc\cdots}{10^{n-1}(10^{r}-1)},\end{aligned}$$

which, being a sum of rational numbers, is a rational number.
Conversely, any rational number is of the form $\frac{p}{q}$, where p and q are integers with $q \neq 0$ with no common factors. Then performing long division produces a remainder of with either 0 which will terminate the long division, or result a repeating block of numbers because there can only be at most $q-1$ different remainders. The moment you get a remainder that is non-zero, the rest of the long division will proceed as it had when the same remainder had earlier occured, causing a repeating pattern to emerge in the quotient. So we have shown that a real number has a repeating decimal if and only if it is a rational number.

88. (a) Since $s_n \geq 0$ for all $n \geq 1$, the sequence of partial sums $\{S_n\}$ is a nondecreasing sequence, and so if $\{S_n\}$ is bounded, then it converges, because a bounded monotonic sequence converges. This in turn means that the series $\sum_{k=1}^{\infty} s_k$ converges. Conversely, if the original series converges, then the partial sums also converge, so by the theorem on p.640, the sequence $\{S_n\}$ must also be bounded.
(b) Write the harmonic series as $1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$. Each of the bracketed terms is $> \frac{1}{2}$. The partial sums form an unbounded sequence since, for instance, $S_1 = 1$, $S_2 = S_{2^0} = 1 + \frac{1}{2}$; $S_4 = S_{2^2} > 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2}\cdot 2$;
$S_8 = S_{2^3} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2}\cdot 3$; $S_{16} = S_{2^4} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2}\cdot 4$; and in general, $S_{2^n} \geq 1 + \sum_{k=0}^{n}\frac{1}{2} = 1 + \frac{1}{2}\cdot n$. This shows that the partial sums S_{2^n} are unbounded, since $\lim_{n\to\infty} S_{2^n} \geq \lim_{n\to\infty}\left(1 + \frac{1}{2}\cdot n\right) = \infty$, which means the sequence $\{S_{2^n}\}$ will diverge, so the series will diverge as well.

AP® Practice Problems

1. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(k+1)(k+2)}$

   Use Partial Fraction Decomposition:

$$\begin{aligned}\frac{1}{(k+1)(k+2)} &= \frac{A}{k+1} + \frac{B}{k+2}\\ 1 &= A(k+2) + B(k+1)\end{aligned}$$

   Let $k = -2$

$$\begin{aligned}1 &= -B\\ B &= -1\end{aligned}$$

Let $k = -1$
$A = 1$

$$\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2}$$

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} \\
&= \sum_{k=1}^{n} \left(\frac{1}{k+1} - \frac{1}{k+2}\right) \\
&= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) \\
&\quad + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\
&= \frac{1}{2} + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) + \left(-\frac{1}{5} + \frac{1}{5}\right) \\
&\quad + \cdots + \left(-\frac{1}{n+1} + \frac{1}{n+1}\right) - \frac{1}{n+2} \\
&= \frac{1}{2} - \frac{1}{n+2}
\end{aligned}$$

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{n+2}\right) = \boxed{\frac{1}{2}}$$

CHOICE C

2. $$\sum_{k=1}^{\infty} \frac{7}{3^{k-1}} = \sum_{k=1}^{\infty} \frac{7}{3^k(3^{-1})} = 21\sum_{k=1}^{\infty} \frac{1}{3^k} = 21\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k$$
$$= 21 \cdot \frac{\frac{1}{3}}{1 - \frac{1}{3}} = 21 \cdot \frac{1}{2} = \boxed{\frac{21}{2}}$$

CHOICE C

3. $$\begin{aligned}
0.121212... &= 0.12 + 0.0012 + 0.000012 + \cdots \\
&= 12\left[0.01 + (0.01)^2 + (0.01)^3 + \cdots\right] \\
&= \frac{0.12}{1 - 0.01} = \frac{0.12}{0.99} = \frac{12}{99} = \boxed{\frac{4}{33}}
\end{aligned}$$

CHOICE A

4. Analyze each choice in turn:

I. $$\sum_{k=1}^{\infty} \left(\sqrt{2}\right)^{k-1} = \sum_{k=1}^{\infty} \frac{\left(\sqrt{2}\right)^k}{\sqrt{2}} = \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \left(\sqrt{2}\right)^k$$

This is a geometric series with $R = \sqrt{2} > 0$.

Therefore $\sum_{k=1}^{\infty} \left(\sqrt{2}\right)^{k-1}$ diverges.

II. $\displaystyle\sum_{k=1}^{\infty} -\left(\frac{3}{4^k}\right) = -3\sum_{k=1}^{\infty}\left(\frac{1}{4^k}\right) = -3\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k$

This is a Convergent Geometric Series since $0 < R = \dfrac{1}{4} < 1$.

III. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ is a p-series with $p = 1$, also known as the harmonic series, which diverges.

Therefore $\boxed{\text{I and III diverge}}$.

CHOICE B

5. $\displaystyle\sum_{k=1}^{\infty}\frac{7^{k-2}}{8^{k+1}} = \sum_{k=1}^{\infty}\frac{7^k(7^{-2})}{8^k(8)} = \frac{1}{392}\sum_{k=1}^{\infty}\left(\frac{7}{8}\right)^k$

This is a Geometric Series with $R = \dfrac{7}{8} < 1$.

$$\frac{1}{392}\sum_{k=1}^{\infty}\left(\frac{7}{8}\right)^k = \frac{1}{392}\cdot\frac{\frac{7}{8}}{1-\frac{7}{8}} = \frac{1}{392}\cdot 7 = \frac{1}{56}$$

Therefore $\boxed{\sum_{k=1}^{\infty}\frac{7^{k-2}}{8^{k+1}} \text{ converges to } \frac{1}{56}}$.

CHOICE B

6. (a) $100 + 90 + 90 + \dfrac{9}{10}\cdot 90 + \dfrac{9}{10}\cdot 90 + \dfrac{9}{10}\cdot\left(\dfrac{9}{10}\cdot 90\right) + \dfrac{9}{10}\cdot\left(\dfrac{9}{10}\cdot 90\right) + \ldots$

$$= 100 + 2\left[90\left(\frac{9}{10}\right)^0 + 90\left(\frac{9}{10}\right)^1 + 90\left(\frac{9}{10}\right)^2 + \ldots\right]$$

$$= \boxed{100 + 2\sum_{k=1}^{\infty} 90\left(\frac{9}{10}\right)^{k-1}}$$

(b) $\displaystyle 100 + 2\sum_{k=1}^{\infty}90\left(\frac{9}{10}\right)^{k-1} = 100 + 2\cdot\frac{90}{1-\frac{9}{10}} = 100 + 2\cdot 900 = \boxed{1900\text{ cm}}$

(c) 1900 cm (19 m) is the total distance that the object hanging from the spring moves up and down between being released and coming to rest.

8.3 Properties of Series; Series with Positive Terms; the Integral Test

Concepts and Vocabulary

1. $\boxed{\textbf{(a)}}$ If the series $\sum_{k=1}^{\infty} a_k$ converges then $\lim_{n\to\infty} a_n = 0$.

2. <u>False:</u> For example, $\lim_{n\to\infty}\frac{1}{n} = 0$ but $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\frac{1}{k}$ diverges (Harmonic series).

3. **True:** Since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^3 = \infty$. The series cannot converge.

4. **True:** Any finite number of terms in a series do not determine convergence.

5. **False:** For example, $\sum\limits_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)$ converges as it is a telescoping series (the sum is 1), whereas $\sum\limits_{k=1}^{\infty}\frac{1}{k}$ and $\sum\limits_{k=1}^{\infty}\frac{1}{k+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ both separately diverge, being the harmonic series and the harmonic series from the second term forward, respectively.

6. **True:** If a series is convergent to S then a scalar multiple of each term yields a convergent series of sum cS.

7. **True:** This is the basis of the integral test of convergence.

8. **False:** If the terms are all positive, then the sequence of partial sums will be an increasing sequence, and if this sequence is not bounded, the series will be divergent.

9. The $p-$series $\sum\limits_{k=1}^{\infty}\frac{1}{k^p}$ converges if $\underline{p>1}$ and diverges if $\underline{0<p\leq 1}$.

10. **Converges.** $\sum\limits_{k=1}^{\infty}\frac{1}{k^{3/2}}$ is a p-series $\sum\limits_{k=1}^{\infty}\frac{1}{k^p}$, $p>1$, which converges.

11. **Diverges.** $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty}\frac{1}{k^{-1/2}} = \sum\limits_{k=1}^{\infty} k^{1/2}$. We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^{1/2} = \infty \neq 0$. So by the Divergence Test, the series diverges.

12. **False:** The upper bound must be $1+\frac{1}{p-1}$; the lower bound is correct.

Skill Building

13. The series is $\sum\limits_{k=1}^{\infty} 16$. We have $\lim\limits_{n\to\infty} a_n = 16 \neq 0$. So by the Divergence Test, the series $\boxed{\textit{diverges.}}$

14. The series is $\sum\limits_{k=1}^{\infty}\frac{k+9}{k}$. We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(1+\frac{9}{n}\right) = 1 \neq 0$. So by the Divergence Test, the series $\boxed{\textit{diverges}}$.

15. The series is $\sum\limits_{k=1}^{\infty}\ln k$. We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\ln n = \infty \neq 0$. So by the Divergence Test, the series $\boxed{\textit{diverges.}}$

16. The series is $\sum\limits_{k=1}^{\infty} e^k$. We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} e^n = \infty \neq 0$. So by the Divergence Test, the series $\boxed{\textit{diverges.}}$

17. The series is $\sum\limits_{k=1}^{\infty}\frac{k^2}{k^2+4}$. We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{n^2}{n^2+4} = \lim\limits_{n\to\infty}\frac{1}{1+\frac{4}{n^2}} = \frac{1}{1+0} = 1 \neq 0$. So by the Divergence Test, the series $\boxed{\textit{diverges.}}$

18. The series is $\sum\limits_{k=1}^{\infty} \frac{k^2+3}{\sqrt{k}}$. We have
$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n^2+3}{\sqrt{n}} = \lim\limits_{n\to\infty} n^{3/2} + 3 \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = \lim\limits_{n\to\infty} n^{3/2} + 0 = \infty \neq 0$. So by the Divergence Test, the series $\boxed{\textit{diverges.}}$

19. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^{1.01}}$. The function $f(x) = \frac{1}{x^{1.01}}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 1$. Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find

$$\begin{aligned} I = \int_1^{\infty} f(x)\,dx &= \lim_{b\to\infty} \int_1^b \frac{dx}{x^{1.01}} = \lim_{b\to\infty} \left[\frac{x^{-1.01+1}}{-1.01+1}\right]_1^b \\ &= -\frac{1}{0.01} \lim_{b\to\infty} \left[b^{-0.01} - 1^{-0.01}\right] = -\frac{1}{0.01}[0-1] \\ &= 100. \end{aligned}$$

Since the improper integral I converges, by the integral test the series $\boxed{\textit{converges}}$ as well.

20. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^{0.9}}$. The function $f(x) = \frac{1}{x^{0.9}}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 1$. Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find

$$I = \int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{dx}{x^{0.9}} = \lim_{b\to\infty} \left[\frac{x^{-0.9+1}}{-0.9+1}\right]_1^b = \frac{1}{0.1} \lim_{b\to\infty} \left[b^{0.1} - 1^{0.1}\right] = \infty.$$

Since the improper integral I diverges, by the integral test the series $\boxed{\textit{diverges}}$ as well.

21. The function $f(x) = \frac{\sqrt{\ln x}}{x}$ is continuous, positive and decreasing on the interval $(e^{1/2}, \infty)$ because $f'(x) = \frac{\frac{1}{2\sqrt{\ln x}} - \sqrt{\ln x}}{x^2} = \frac{1 - 2\ln x}{2x^2\sqrt{\ln x}} < 0$ for $e^{1/2} > 0$.
Also, $a_k = f(k)$ for all positive integers k.
Thus, we can use the Integral Test:

$$\int_{e^{1/2}}^{\infty} \frac{\sqrt{\ln x}}{x}\,dx = \lim_{b\to\infty} \int_{e^{1/2}}^{b} \frac{\sqrt{\ln x}}{x}\,dx = \lim_{b\to\infty} \left[\frac{2}{3}(\ln x)^{3/2}\Big|_{e^{1/2}}^{b}\right] = \lim_{b\to\infty} \left[\frac{2}{3}(\ln b)^{3/2} - \frac{2}{3}\left(\frac{1}{2}\right)^{3/2}\right] = \infty$$

Therefore the series diverges on the interval $(e^{1/2}, \infty)$, so the infinite sum $\sum\limits_{k=1}^{\infty} \frac{\sqrt{\ln k}}{k}$ $\boxed{\textit{diverges}}$.

22. The series is $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}}$. The function $f(x) = \frac{1}{x\sqrt{\ln x}}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. To see $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x\sqrt{\ln x}$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[2, \infty)$. We have

$$g'(x) = 1 \cdot \sqrt{\ln x} + x \cdot \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \sqrt{\ln x} + \frac{1}{\sqrt{\ln x}} > 0 \text{ for } x \geq 2.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 2$. Using the integral test, we find $I = \int_2^\infty f(x)\,dx = \lim\limits_{b\to\infty} \int_2^b \frac{dx}{x\sqrt{\ln x}}$. Let $u = \ln x$. Then $du = \frac{dx}{x}$. We have

$$I = \int_2^\infty f(x)\,dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{\sqrt{u}} = \lim_{b\to\infty} \left[\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_{\ln 2}^{\ln b} = 2 \lim_{b\to\infty} \sqrt{u}\Big|_{\ln 2}^{\ln b} = 2 \lim_{b\to\infty} \left(\sqrt{\ln b} - \sqrt{\ln 2}\right) = \infty.$$

Since the improper integral I diverges, by the integral test the series $\boxed{\textit{diverges}}$ as well.

23. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} ke^{-k^2}$. The function $f(x) = xe^{-x^2}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 1$ since

$$f'(x) = 1 \cdot e^{-x^2} + xe^{-x^2}(-2x) = e^{-x^2}(1 - 2x^2) < 0 \text{ for } x \geq 1.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find $I = \int_1^\infty f(x)\,dx = \lim\limits_{b\to\infty} \int_1^b xe^{-x^2}\,dx$. Let $u = e^{-x^2}$. Then $du = -2xe^{-x^2}\,dx$. We have

$$I = \int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_{e^{-1}}^{e^{-b^2}} -\frac{1}{2}\,du = -\frac{1}{2} \lim_{b\to\infty} u\Big|_{e^{-1}}^{e^{-b^2}} = -\frac{1}{2} \lim_{b\to\infty} \left[e^{-b^2} - e^{-1}\right] = -\frac{1}{2}\left[0 - e^{-1}\right] = \frac{e^{-1}}{2}.$$

Since the improper integral I converges, by the integral test the series $\boxed{\textit{converges}}$ as well.

24. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} ke^{-k}$. The function $f(x) = xe^{-x}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 1$, since

$$f'(x) = 1 \cdot e^{x} + x(-e^{-x}) = e^{-x}(1 - x) \leq 0 \text{ for } x \geq 1.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find $I = \int_1^\infty f(x)\,dx = \lim\limits_{b\to\infty} \int_1^b xe^{-x}\,dx$. Let $u = x$, $dv = e^{-x}\,dx$. Then, $du = dx$, $v = -e^{-x}$. Integrating by parts, we have

$$\begin{aligned} I &= \lim_{b\to\infty} \left(-xe^{-x}\Big|_1^b + \int_1^b e^{-x}\,dx\right) = \lim_{b\to\infty} \left(-be^{-b} + e^{-1}\right) + \lim_{b\to\infty} \left[-e^{-x}\right]_1^b \\ &= -\lim_{b\to\infty} \frac{b}{e^b} + \frac{1}{e} - \lim_{b\to\infty} \left(e^{-b} - e^{-1}\right) = -\lim_{b\to\infty} \frac{1}{e^b} + \frac{1}{e} - 0 + \frac{1}{e} \\ &= 0 + \frac{1}{e} - 0 + \frac{1}{e} = \frac{2}{e}, \end{aligned}$$

using L'Hôpital's rule on the first limit in the second line. Since the improper integral I converges, by the integral test the series $\boxed{\textit{converges}}$ as well.

25. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^2+1}$. The function $f(x) = \frac{1}{x^2+1}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 1$, since $f'(x) = -\frac{1}{(x^2+1)^2} \cdot (2x) < 0$ for $x \geq 1$. Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find

$$I = \int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \tan^{-1} x\Big|_1^b = \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 1\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Since the improper integral I converges, by the integral test the series $\boxed{\textit{converges}}$ as well.

26. The series is $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{k\sqrt{k^2-1}}$. The function $f(x) = \frac{1}{x\sqrt{x^2-1}}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. To see that $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x\sqrt{x^2-1}$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[2, \infty)$. We have

$$g'(x) = 1 \cdot \sqrt{x^2-1} + x \cdot \frac{1}{2\sqrt{x^2-1}} \cdot (2x) = \frac{x^2-1+x^2}{\sqrt{x^2-1}} = \frac{2x^2-1}{\sqrt{x^2-1}} > 0 \text{ for } x \geq 2.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 2$. Using the integral test, we find $I = \int_2^{\infty} f(x)\,dx = \lim\limits_{b\to\infty} \int_2^b \frac{dx}{x\sqrt{x^2-1}}$. Let $x = \sec u$ Then $dx = \sec u \tan u\, du$. We have

$$I = \lim_{b\to\infty} \int_{\sec^{-1} 2}^{\sec^{-1} b} \frac{\sec u \tan u\, du}{\sec u\sqrt{\sec^2 u - 1}} = \lim_{b\to\infty} \int_{sec^{-1}2}^{\sec^{-1} b} \frac{\sec u \tan u\, du}{\sec u \tan u} = \lim_{b\to\infty} u\Big|_{\sec^{-1} 2}^{\sec^{-1} b} = \lim_{b\to\infty} \left(\sec^{-1} b - \sec^{-1} 2\right)$$
$$= \frac{\pi}{2} - \sec^{-1} 2.$$

Since the improper integral I converges, by the integral test the series $\boxed{\textit{converges}}$ as well.

27. The series is $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{k\ln k}$. The function $f(x) = \frac{1}{x\ln x}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. To see that $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x\ln x$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[2, \infty)$. We have

$$g'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1 > 0 \text{ for } x \geq 2.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 2$. Using the integral test, we find $I = \int_2^{\infty} f(x)\,dx = \lim\limits_{b\to\infty} \int_2^b \frac{dx}{x\ln x}$. Let $u = \ln x$. Then $du = \frac{dx}{x}$. We have

$$I = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u} = \lim_{b\to\infty} [\ln u]\Big|_{\ln 2}^{\ln b} = \lim_{b\to\infty} [\ln(\ln b) - \ln(\ln 2)] = \infty.$$

Since the improper integral I diverges, by the integral test the series $\boxed{\textit{diverges}}$ as well.

28. The series is $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$. The function $f(x) = \frac{1}{x(\ln x)^3}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. To see that $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x(\ln x)^3$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[2, \infty)$. We have

$$g'(x) = 1 \cdot (\ln x)^3 + x \cdot 3(\ln x)^2 \cdot \frac{1}{x} = (\ln x)^3 + 3(\ln x)^2 > 0 \text{ for } x \geq 2.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 2$. Using the integral test, we find $I = \int_2^{\infty} f(x)\,dx = \lim\limits_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^3}$. Let $u = \ln x$. Then $du = \frac{dx}{x}$. We have

$$I = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u^3} = \lim_{b\to\infty} \left[\frac{u^{-3+1}}{-3+1}\right]_{\ln 2}^{\ln b} = -\frac{1}{2} \lim_{b\to\infty} \left[\frac{1}{(\ln b)^2} - \frac{1}{(\ln 2)^2}\right] = \frac{1}{2}\frac{1}{(\ln 2)^2}.$$

Since the improper integral I converges, by the integral test the series $\boxed{\textit{converges}}$ as well.

29. The series $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ is a p-series with $p = 2 > 1$. So it $\boxed{\textit{converges.}}$

30. The series $\sum\limits_{k=1}^{\infty} \frac{1}{k^4}$ is a p-series with $p = 4 > 1$. So it $\boxed{\textit{converges.}}$

31. The series $\sum\limits_{k=1}^{\infty} \frac{1}{k^{1/3}}$ is a p-series with $0 < p = \frac{1}{3} < 1$. So it $\boxed{\textit{diverges.}}$

32. The series $\sum\limits_{k=1}^{\infty} \frac{1}{k^{2/3}}$ is a p-series with $0 < p = \frac{2}{3} < 1$. So it $\boxed{\textit{diverges.}}$

33. The series $\sum\limits_{k=1}^{\infty} \frac{1}{k^e}$ is a p-series with $p = e > 1$. So it $\boxed{\textit{converges.}}$

34. The series $\sum\limits_{k=1}^{\infty} \frac{1}{k^{\sqrt{2}}}$ is a p-series with $p = \sqrt{2} > 1$, so it $\boxed{\textit{converges.}}$

35. $1 + \frac{1}{2\sqrt{3}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots = \sum\limits_{k=1}^{\infty} \frac{1}{k\sqrt{k}} = \sum\limits_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is a p-series with $p = \frac{3}{2} > 1$. So it $\boxed{\textit{converges.}}$

36. $1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots = \sum\limits_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}} = \sum\limits_{k=1}^{\infty} \frac{1}{k^{1/3}}$ is a p-series with $0 < p = \frac{1}{3} < 1$ and so $\boxed{\textit{diverges.}}$

37. $1 + \frac{1}{4\sqrt{2}} + \frac{1}{9\sqrt{3}} + \frac{1}{16\sqrt{4}} + \cdots = \sum\limits_{k=1}^{\infty} \frac{1}{k^2\sqrt{k}} = \sum\limits_{k=1}^{\infty} \frac{1}{k^{5/2}}$ is a p-series with $p = \frac{5}{2} > 1$ and so $\boxed{\textit{converges.}}$

38. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots = \sum\limits_{k=1}^{\infty} \frac{1}{k^3}$ is a p-series with $p = 3 > 1$ and so $\boxed{\textit{converges.}}$

39. The series is $\sum\limits_{k=1}^{\infty} \frac{10}{k}$. Since the harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges, the series $\sum\limits_{k=1}^{\infty} \frac{10}{k}$ $\boxed{\textit{diverges}}$ as well since it is a nonzero constant multiple of a divergent series.

40. The series is $\sum\limits_{k=1}^{\infty} \frac{2}{1+k}$. Since $\sum\limits_{k=1}^{\infty} \frac{1}{1+k}$ is just the harmonic series $\sum\limits_{k=2}^{\infty} \frac{1}{k}$, which diverges, the series $\sum\limits_{k=1}^{\infty} \frac{2}{1+k}$ which is a nonzero constant multiple of a divergent series also $\boxed{\textit{diverges}}$.

41. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k^2+1}{4k+1}$. Since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n^2+1}{4n+1} = \lim\limits_{n\to\infty} \frac{n+\frac{1}{n}}{4+\frac{1}{n}} = \infty \neq 0$, by the Divergence Test the series $\boxed{\textit{diverges.}}$

42. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k^3}{k^3+3}$. Since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n^3}{n^3+3} = \lim\limits_{n\to\infty} \frac{1}{1+\frac{3}{n^3}} = 1 \neq 0$, by the Divergence Test, the series $\boxed{\textit{diverges.}}$

43. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \left(k + \frac{1}{k}\right)$. Since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(n + \frac{1}{n}\right) = \lim\limits_{n\to\infty} n + 0 = \infty \neq 0$, by the Divergence Test, the series $\boxed{\textit{diverges.}}$

44. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \left(k - \frac{10}{k}\right)$. We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(n - \frac{10}{n}\right) = \lim\limits_{n\to\infty} n - 0 = \infty \neq 0$. So by the Divergence Test the series $\boxed{\textit{diverges.}}$

45. The series is $\sum\limits_{k=1}^{\infty} \left(\frac{1}{3k} - \frac{1}{4k}\right) = \sum\limits_{k=1}^{\infty} \frac{1}{12k}$ is a constant multiple of the harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ which diverges. So the given series also $\boxed{\textit{diverges.}}$

46. The series is $\sum\limits_{k=1}^{\infty} \left(\frac{1}{3^k} - \frac{1}{4^k}\right) = \sum\limits_{k=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{k-1} - \sum\limits_{k=1}^{\infty} \frac{1}{4}\left(\frac{1}{4}\right)^{k-1} = \frac{\frac{1}{3}}{1-\frac{1}{3}} - \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1/3}{2/3} - \frac{1/4}{3/4} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. Here, we used the fact that each term was a convergent geometric series, and the sum and difference property of convergent series to simplify the result. So, the series $\boxed{\textit{converges.}}$

47. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \sin\left(\frac{\pi}{2}k\right)$. Since $\sin\left(\frac{\pi}{2}k\right)$ is $+1$ for $k = 1, 5, 9, \cdots$; 0 for $k = 2, 4, 6, \cdots$; and -1 for $k = 3, 7, 11, \cdots$, the partial sums $S_n = \sum\limits_{k=1}^{n} a_k$ will oscillate between 0 and 1 for all n. The sequence of partial sums $\{S_n\} = \{1, 1, 0, 0, 1, 1, 0, 0, \cdots\}$ does not converge to a limit, so the series will $\boxed{\textit{diverge.}}$

48. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \sec \pi k$. Since $\sec \pi k$ is -1 for $k = 1, 3, 5, \cdots$, and $+1$ for $k = 2, 4, 6, \cdots$, the partial sums $S_n = \sum\limits_{k=1}^{n} a_k$ will oscillate between -1 and 0. The series will $\boxed{\textit{diverge}}$ because the sequence of partial sums $\{S_n\} = \{-1, 0, -1, 0, \cdots\}$ does not converge to a limit.

49. The series is $\sum\limits_{k=3}^{\infty} a_k = \sum\limits_{k=3}^{\infty} \frac{k+1}{k-2}$. We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n+1}{n-2} = \lim\limits_{n\to\infty} \frac{1+\frac{1}{n}}{1-\frac{2}{n}} = \frac{1+0}{1-0} = 1 \neq 0$. So the series $\boxed{\textit{diverges.}}$

50. The series is $\sum\limits_{k=5}^{\infty} a_k = \sum\limits_{k=5}^{\infty} \frac{2k^5+3}{k^5-4k^4}$. We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{2n^5+3}{n^5-4n^4} = \lim\limits_{n\to\infty} \frac{2+\frac{3}{n^5}}{1-\frac{4}{n}} = \frac{2+0}{1-0} = 2 \neq 0$. So the series $\boxed{\textit{diverges.}}$

51. The series is $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{k(\ln k)^{1/2}}$. The function $f(x) = \frac{1}{x(\ln x)^{1/2}} = \frac{1}{x\sqrt{\ln x}}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. To see that $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x\sqrt{\ln x}$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[2, \infty)$. We have

$$g'(x) = 1 \cdot \sqrt{\ln x} + x \cdot \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}} > 0 \text{ for } x \geq 2.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 2$. Using the integral test, we find $I = \int_2^\infty f(x)\,dx = \lim\limits_{b\to\infty} \int_2^b \frac{dx}{x\sqrt{\ln x}}$. Let $u = \ln x$. Then $du = \frac{dx}{x}$. We have

$$I = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{\sqrt{u}} = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u^{-\frac{1}{2}}\,du = \lim_{b\to\infty} \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_{\ln 2}^{\ln b} = 2\lim_{b\to\infty}\left(\sqrt{\ln b} - \sqrt{\ln 2}\right) = \infty.$$

Since the improper integral I diverges, by the integral test the series $\boxed{\textit{diverges}}$ as well.

52. The series is $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$. The function $f(x) = \frac{1}{x(\ln x)^2}$ is defined on $[2,\infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. To see that $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x(\ln x)^2$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[2,\infty)$. We have

$$g'(x) = 1\cdot(\ln x)^2 + x\cdot(2\ln x)\cdot\frac{1}{x} = (\ln x)^2 + 2\ln x > 0 \text{ for } x \geq 2.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 2$. Using the integral test, we find $I = \int_2^\infty f(x)\,dx = \lim\limits_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^2}$. Let $u = \ln x$. Then $du = \frac{dx}{x}$. We have

$$I = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u^2} = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u^{-2}\,du = \lim_{b\to\infty}\left[\frac{u^{-1}}{-1}\right]_{\ln 2}^{\ln b} = -\lim_{b\to\infty}\left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right) = -\left(0 - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

Since the improper integral I converges, by the integral test the series $\boxed{\textit{converges}}$ as well.

53. The series is $\sum\limits_{k=3}^{\infty} a_k = \sum\limits_{k=3}^{\infty} \frac{2k}{k^2-4}$. The function $f(x) = \frac{2x}{x^2-4}$ is defined on $[3,\infty)$ and is continuous, positive, and decreasing for all $x \geq 3$, since

$$f'(x) = \frac{(x^2-4)\cdot 2 - 2x\cdot(2x)}{(x^2-1)^2} = -\frac{3x^2+8}{(x^2-1)^2} < 0 \text{ for } x \geq 3.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 3$. Using the integral test, we find $I = \int_3^\infty f(x)\,dx = \lim\limits_{b\to\infty} \int_3^b \frac{2x\,dx}{x^2-4}$. Let $u = x^2 - 4$. Then $du = 2x\,dx$. We have

$$I = \lim_{b\to\infty} \int_5^{b^2-4} \frac{du}{u} = \lim_{b\to\infty} [\ln u]\Big|_5^{b^2-4} = \lim_{b\to\infty}\left(\ln(b^2-4) - \ln 5\right) = \infty.$$

Since the improper integral I diverges, by the integral test the series $\boxed{\textit{diverges.}}$

54. The series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{(2k-1)(2k)}$. The function $f(x) = \frac{1}{(2x-1)(2x)} = \frac{1}{2x-1} - \frac{1}{2x}$ is defined on $[1,\infty)$ and is continuous, positive, and decreasing for all $x \geq 1$. Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find

$$\begin{aligned} I &= \int_1^\infty f(x)\,dx = \lim_{b\to\infty}\left(\int_1^b \frac{dx}{2x-1} - \frac{dx}{2x}\right) = \lim_{b\to\infty}\left[\frac{1}{2}\ln|2x-1| - \frac{1}{2}\ln|x|\right]_1^b \\ &= \lim_{b\to\infty}\left[\frac{1}{2}\ln|2b-1| - \frac{1}{2}\ln|b|\right] = \frac{1}{2}\lim_{b\to\infty}\ln\left|\frac{2b-1}{b}\right| = \frac{1}{2}\lim_{b\to\infty}\ln\left|\frac{2-\frac{1}{b}}{1}\right| \\ &= \frac{1}{2}\ln\lim_{b\to\infty}\left|2 - \frac{1}{b}\right| = \frac{1}{2}\ln 2. \end{aligned}$$

Since the improper integral I converges, by the integral test the series $\boxed{\textit{converges}}$ as well.

55. The bounds for the sum of a Convergent p-series are, for $p > 1$, $\frac{1}{p-1} < \sum\limits_{k=1}^{\infty} \frac{1}{k^p} < 1 + \frac{1}{p-1}$.

Here, for $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$, the bounds for the sum of the Convergent series are

$$\frac{1}{2-1} < \sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \frac{1}{2-1}$$

$$\boxed{1 < \sum_{k=1}^{\infty} \frac{1}{k^2} < 2}$$

56. The bounds for the sum of a Convergent p-series are, for $p > 1$, $\frac{1}{p-1} < \sum\limits_{k=1}^{\infty} \frac{1}{k^p} < 1 + \frac{1}{p-1}$.

Here, for $\sum\limits_{k=1}^{\infty} \frac{1}{k^4}$, the bounds for the sum of the Convergent series are

$$\frac{1}{4-1} < \sum_{k=1}^{\infty} \frac{1}{k^4} < 1 + \frac{1}{4-1}$$

$$= \boxed{\frac{1}{3} < \sum_{k=1}^{\infty} \frac{1}{k^4} < \frac{4}{3}}$$

57. The bounds for the sum of a Convergent p-series are, for $p > 1$, $\frac{1}{p-1} < \sum\limits_{k=1}^{\infty} \frac{1}{k^p} < 1 + \frac{1}{p-1}$.

Here, for $\sum\limits_{k=1}^{\infty} \frac{1}{k^e}$, the bounds for the sum of the Convergent series are

$$\frac{1}{e-1} < \sum_{k=1}^{\infty} \frac{1}{k^e} < 1 + \frac{1}{e-1}$$

$$\boxed{\frac{1}{e-1} < \sum_{k=1}^{\infty} \frac{1}{k^e} < \frac{e}{e-1}}$$

58. The bounds for the sum of a Convergent p-series are, for $p > 1$, $\frac{1}{p-1} < \sum\limits_{k=1}^{\infty} \frac{1}{k^p} < 1 + \frac{1}{p-1}$.

Here, for $\sum\limits_{k=1}^{\infty} \frac{1}{k^{\sqrt{2}}}$, the bounds for the sum of the Convergent series are

$$\frac{1}{\sqrt{2}-1} < \sum_{k=1}^{\infty} \frac{1}{k^{\sqrt{2}}} < 1 + \frac{1}{\sqrt{2}-1}$$

$$\frac{1}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} < \sum_{k=1}^{\infty} \frac{1}{k^{\sqrt{2}}} < 1 + \frac{1}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$\frac{1+\sqrt{2}}{2-1} < \sum_{k=1}^{\infty} \frac{1}{k^{\sqrt{2}}} < 1 + \frac{1+\sqrt{2}}{2-1}$$

$$\boxed{1+\sqrt{2} < \sum_{k=1}^{\infty} \frac{1}{k^{\sqrt{2}}} < 2+\sqrt{2}}$$

59. $1+\dfrac{1}{2\sqrt{2}}+\dfrac{1}{3\sqrt{3}}+\dfrac{1}{4\sqrt{4}}+\ldots=\dfrac{1}{1^2\sqrt{1}}+\dfrac{1}{2^2\sqrt{2}}+\dfrac{1}{3^2\sqrt{3}}+\dfrac{1}{4^2\sqrt{4}}+\ldots$

The general term is $\frac{1}{k\sqrt{k}}$ and thus the series is $\sum\limits_{k=1}^{\infty}\frac{1}{k\sqrt{k}}=\sum\limits_{k=1}^{\infty}\left(\frac{1}{k^{3/2}}\right)$

The bounds for the sum of a Convergent p-series are, for $p>1$, $\frac{1}{p-1}<\sum\limits_{k=1}^{\infty}\frac{1}{k^p}<1+\frac{1}{p-1}$.

Here, for $\sum\limits_{k=1}^{\infty}\frac{1}{k^{3/2}}$, the bounds for the sum of the Convergent series are

$$\frac{1}{\frac{3}{2}-1}<\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}<1+\frac{1}{\frac{3}{2}-1}$$
$$\boxed{2<\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}<3}$$

60. $1+\dfrac{1}{4\sqrt{2}}+\dfrac{1}{9\sqrt{3}}+\dfrac{1}{16\sqrt{4}}+\ldots=\dfrac{1}{1^2\sqrt{1}}+\dfrac{1}{2^2\sqrt{2}}+\dfrac{1}{3^2\sqrt{3}}+\dfrac{1}{4^2\sqrt{4}}+\ldots+\dfrac{1}{k^2\sqrt{k}}=\sum\limits_{k=1}^{\infty}\left(\dfrac{1}{k^{5/2}}\right)$

The bounds for the sum of a Convergent p-series are, for $p>1$, $\frac{1}{p-1}<\sum\limits_{k=1}^{\infty}\frac{1}{k^p}<1+\frac{1}{p-1}$.

Here, for $\sum\limits_{k=1}^{\infty}\frac{1}{k^{5/2}}$, the bounds for the sum of the Convergent series are

$$\frac{1}{\frac{5}{2}-1}<\sum_{k=1}^{\infty}\frac{1}{k^{5/2}}<1+\frac{1}{\frac{5}{2}-1}$$
$$\frac{1}{\frac{3}{2}}<\sum_{k=1}^{\infty}\frac{1}{k^{5/2}}<1+\frac{1}{\frac{3}{2}}$$
$$=\boxed{\frac{2}{3}<\sum_{k=1}^{\infty}\frac{1}{k^{5/2}}<\frac{5}{3}}$$

61. Consider $f(x)=xe^{-x^2}$

$$\begin{aligned}f'(x)&=1\cdot e^{-x^2}+xe^{-x^2}(-2x)\\&=e^{-x^2}\left(1-2x^2\right)\end{aligned}$$

$f'(x)$ is negative on the interval $[1,\infty)$. Therefore f is decreasing on the interval $[1,\infty)$. Observe that $f(x)$ is positive on the interval $[1,\infty)$. If f is differentiable at $x=c$, then f is continuous at $x=c$. $f(x)=xe^{-x^2}$ is differentiable for all x on the interval $[1,\infty)$.
Since f is continuous, positive, and decreasing on $[1,\infty)$ and since $a_k=f(k)$ for all positive integers k, therefore if the improper integral $\int_1^{\infty}f(x)\,dx$ converges then the sum of the series is bounded by

$$\int_1^{\infty}f(x)\,dx<\sum_{k=1}^{\infty}a_k<a_1+\int_1^{\infty}f(x)\,dx.$$

Here,

$$\begin{aligned}\int_1^\infty xe^{-x^2}\,dx &= \lim_{b\to\infty}\int_1^b xe^{-x^2}\,dx = \lim_{b\to\infty}\left[\frac{e^{-x^2}}{-2}\right]_1^b\\ &= \lim_{b\to\infty}\left[\frac{e^{-b^2}}{-2}-\frac{e^{-1}}{-2}\right] = \lim_{b\to\infty}\left[\frac{1}{-2e^{b^2}}+\frac{1}{2e}\right]\\ &= \frac{1}{2e}.\end{aligned}$$

Therefore,

$$\int_1^\infty x^2e^{-3x^3}\,dx < \sum_{k=1}^\infty k^2e^{-3k^3} < a_1 + \int_1^\infty x^2e^{-3x^3}\,dx$$

$$\frac{1}{2e} < \sum_{k=1}^\infty k^2e^{-3k^3} < e^{-1}+\frac{1}{2e}$$

$$\frac{1}{2e} < \sum_{k=1}^\infty k^2e^{-3k^3} < \frac{1}{e}+\frac{1}{2e}$$

$$\boxed{\frac{1}{2e} < \sum_{k=1}^\infty ke^{-k^2} < \frac{3}{2e}}$$

62. Consider $f(x) = x^2e^{-3x^3}$

$$\begin{aligned}f'(x) &= 2xe^{-3x^3} + x^2e^{-3x^3}\left(-9x^2\right)\\ &= xe^{-3x^3}\left(2-9x^3\right)\end{aligned}$$

$f'(x)$ is negative on the interval $[1,\infty)$. Therefore $f(x)$ is decreasing on the interval $[1,\infty)$. Observe that $f(x)$ is positive on the interval $[1,\infty)$. If f is differentiable at $x=c$, then f is continuous at $x=c$. $f(x)=x^2e^{-3x^3}$ is differentiable for all x on the interval $[1,\infty)$.
Since f is continuous, positive, and decreasing on $[1,\infty)$ and since $a_k=f(k)$ for all positive integers k, therefore if the improper integral $\int_1^\infty f(x)\,dx$ converges then the sum of the series is bounded by

$$\int_1^\infty f(x)\,dx < \sum_{k=1}^\infty a_k < a_1 + \int_1^\infty f(x)\,dx.$$

Here,

$$\begin{aligned}\int_1^\infty x^2e^{-3x^3}\,dx &= \lim_{b\to\infty}\int_1^b x^2e^{-3x^3}\,dx = \lim_{b\to\infty}\left[\frac{e^{-3x^3}}{-9}\right]_1^b\\ &= \lim_{b\to\infty}\left[\frac{e^{-3b^3}}{-9}-\frac{e^{-3}}{-9}\right] = \lim_{b\to\infty}\left[\frac{1}{-9e^{3b^3}}-\frac{1}{-9(e^3)}\right]\\ &= \frac{1}{9e^3}.\end{aligned}$$

Therefore,

$$\int_1^\infty x^2e^{-3x^3}\,dx < \sum_{k=1}^\infty k^2e^{-3k^3} < a_1 + \int_1^\infty x^2e^{-3x^3}\,dx$$

$$\frac{1}{9e^3} < \sum_{k=1}^\infty k^2e^{-3k^3} < e^{-3} + \frac{1}{9e^3}$$

$$\frac{1}{9e^3} < \sum_{k=1}^\infty k^2e^{-3k^3} < \frac{1}{e^3} + \frac{1}{9e^3}$$

$$\boxed{\frac{1}{9e^3} < \sum_{k=1}^\infty k^2e^{-3k^3} < \frac{10}{9e^3}}$$

63. Consider $f(x) = \frac{1}{x^2+9} = \left(x^2+9\right)^{-1}$

$$f'(x) = -1\left(x^2+9\right)^{-2}(2x) = \frac{-2x}{\left(x^2+9\right)^2}$$

$f'(x)$ is negative on the interval $[1,\infty)$. Therefore f is decreasing on the interval $[1,\infty)$. Observe that $f(x)$ is positive on the interval $[1,\infty)$. If f is differentiable at $x = c$, then f is continuous at $x = c$. $f(x) = \frac{1}{4x^2+9}$ is differentiable for all x on the interval $[1,\infty)$.
Since f is continuous, positive, and decreasing on $[1,\infty)$ and since $a_k = f(k)$ for all positive integers k, therefore if the improper integral $\int_1^\infty f(x)\,dx$ converges then the sum of the series is bounded by

$$\int_1^\infty f(x)\,dx < \sum_{k=1}^\infty a_k < a_1 + \int_1^\infty f(x)\,dx.$$

Here,

$$\int_1^\infty \frac{1}{x^2+9}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2+9}\,dx = \lim_{b\to\infty}\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_1^b$$
$$= \frac{1}{3}\lim_{b\to\infty}\left(\tan^{-1}\frac{b}{3} - \tan^{-1}\frac{1}{3}\right) = \frac{1}{3}\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{3}\right) = \frac{\pi}{6} - \frac{1}{3}\tan^{-1}\frac{1}{3}$$

Therefore,

$$\int_1^\infty \frac{1}{x^2+9}\,dx < \sum_{k=1}^\infty \frac{1}{k^2+9} < \frac{1}{10} + \int_1^\infty \frac{1}{x^2+9}\,dx$$

$$\boxed{\frac{\pi}{6} - \frac{1}{3}\tan^{-1}\frac{1}{3} < \sum_{k=1}^\infty \frac{1}{4k^2+1} < \frac{1}{10} + \frac{\pi}{6} - \frac{1}{3}\tan^{-1}\frac{1}{3}}$$

64. Consider $f(x) = \frac{1}{4x^2+1} = \left(4x^2+1\right)^{-1}$

$$f'(x) = -1\left(4x^2+1\right)^{-2}(8x) = \frac{-8x}{\left(4x^2+1\right)^2}$$

$f'(x)$ is negative on the interval $[1,\infty)$. Therefore $f(x)$ is decreasing on the interval $[1,\infty)$. Observe that $f(x)$ is positive on the interval $[1,\infty)$. If f is differentiable at $x = c$, then f is continuous at $x = c$. $f(x) = \frac{1}{4x^2+1}$ is differentiable for all x on the interval $[1,\infty)$.

Since f is continuous, positive, and decreasing on $[1, \infty)$ and since $a_k = f(k)$ for all positive integers k, therefore if the improper integral $\int_1^\infty f(x)\,dx$ converges then the sum of the series is bounded by

$$\int_1^\infty f(x)\,dx < \sum_{k=1}^\infty a_k < a_1 + \int_1^\infty f(x)\,dx.$$

Here,

$$\int_1^\infty \frac{1}{4x^2+1}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{4x^2+1}\,dx = \lim_{b\to\infty}\left[\frac{1}{2}\tan^{-1}(2x)\right]_1^b$$
$$= \frac{1}{2}\lim_{b\to\infty}\left[\tan^{-1}(2b) - \tan^{-1} 2\right] = \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1} 2\right) = \frac{\pi}{4} - \frac{\tan^{-1} 2}{4}$$

Therefore,

$$\int_1^\infty \frac{1}{4x^2+1}\,dx < \sum_{k=1}^\infty \frac{1}{4k^2+1} < \frac{1}{5} + \int_1^\infty \frac{1}{4x^2+1}\,dx$$

$$\boxed{\frac{\pi}{4} - \frac{\tan^{-1} 2}{2} < \sum_{k=1}^\infty \frac{1}{4k^2+1} < \frac{1}{5} + \frac{\pi}{4} - \frac{\tan^{-1} 2}{2}}$$

65. The series is $\sum\limits_{k=2}^\infty a_k = \sum\limits_{k=2}^\infty \frac{1}{k(\ln k)^p}$. The function $f(x) = \frac{1}{x(\ln x)^p}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. To check whether $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x(\ln x)^p$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[2, \infty)$. We have

$$g'(x) = 1 \cdot (\ln x)^p + x \cdot p(\ln x)^{p-1} \cdot \frac{1}{x} = (\ln x)^p + p(\ln x)^{p-1} > 0.$$

This final condition is satisfied for all $p > 0$ and for x in the interval $[2, \infty)$, so we conclude that the function $f(x)$ must be decreasing on $[2, \infty)$. Also we have $a_k = f(k)$ for all positive integers $k \geq 2$. To use the Integral Test evaluate $I = \int_2^\infty f(x)\,dx = \lim\limits_{b\to\infty}\int_2^b \frac{dx}{x(\ln x)^p}$. Let $u = \ln x$. Then $du = \frac{dx}{x}$. We have, for $p \neq 1$,

$$I = \lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{du}{u^p} = \lim_{b\to\infty}\int_{\ln 2}^{\ln b} u^{-p}\,du = \lim_{b\to\infty}\left[\frac{u^{-p+1}}{-p+1}\right]_{\ln 2}^{\ln b} = \frac{1}{1-p}\lim_{b\to\infty}\left(\frac{1}{(\ln b)^{p-1}} - \frac{1}{(\ln 2)^{p-1}}\right).$$

This improper integral converges if and only if $p > 1$, and diverges for $p < 1$, so the series converges, by the Integral Test, for $p > 1$, and diverges for $p < 1$.
Next, consider the case $p = 1$. We have

$$I = \int_2^\infty f(x)\,dx \lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{du}{u} = \lim_{b\to\infty}\left[\ln u\right]\Big|_{\ln 2}^{\ln b} = \lim_{b\to\infty}\left[\ln(\ln b) - \ln(\ln 2)\right] = \infty.$$

Since the improper integral I diverges, by the Integral Test, the series diverges for $p = 1$.
In conclusion, using the integral test, we have shown that the series $\sum\limits_{k=2}^\infty a_k = \sum\limits_{k=2}^\infty \frac{1}{k(\ln k)^p}$ converges if and only if $p > 1$.

66. The series is $\sum_{k=3}^{\infty} a_k = \sum_{k=3}^{\infty} \frac{1}{k(\ln k)[\ln(\ln k)]^p}$. The function $f(x) = \frac{1}{x(\ln x)[\ln(\ln x)]^p}$ is defined on $[3, \infty)$ and is continuous, positive, and decreasing for all $x \geq 3$. To check whether $f(x)$ is decreasing, consider $g(x) = \frac{1}{f(x)} = x(\ln x)[\ln(\ln x)]^p$. It suffices to show that $g(x)$ is increasing and has a nonzero derivative on $[3, \infty)$. We have

$$\begin{aligned} g'(x) &= \ln x[\ln(\ln x)]^p + x \cdot \frac{1}{x}[\ln(\ln x)]^p + x \ln x \cdot p[\ln(\ln x)]^{p-1} \cdot \frac{1}{\ln x} \cdot \frac{1}{x} \\ &= \ln x[\ln(\ln x)]^p + [\ln(\ln x)]^p + p[\ln(\ln x)]^{p-1} \\ &= [\ln(\ln x)]^{p-1}[(\ln x + 1)(\ln(\ln x) + p)] \\ &> 0. \end{aligned}$$

This final condition is satisfied for all $p > 0$ and for x in the interval $[3, \infty)$, so we can conclude that the function $f(x)$ must be decreasing on $[3, \infty)$. Also we have $a_k = f(k)$ for all positive integers $k \geq 3$. Using the integral test, we find $I = \int_3^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_3^b \frac{dx}{x(\ln x)[\ln(\ln x)]^p}$. Let $u = \ln(\ln x)$. Then $du = \frac{dx}{x \ln x}$. We have for $p \neq 1$,

$$\begin{aligned} I = \lim_{b \to \infty} \int_{\ln(\ln 3)}^{\ln(\ln b)} \frac{du}{u^p} = \lim_{b \to \infty} \int_{\ln(\ln 3)}^{\ln(\ln b)} u^{-p}\,du &= \lim_{b \to \infty} \left[\frac{u^{-p+1}}{-p+1} \right]_{\ln(\ln 3)}^{\ln(\ln b)} \\ &= \frac{1}{1-p} \lim_{b \to \infty} \left(\frac{1}{[\ln(\ln b)]^{p-1}} - \frac{1}{\ln[(\ln 3)]^{p-1}} \right). \end{aligned}$$

This improper integral converges if and only if $p > 1$, and diverges if $p < 1$ so the series converges, by the Integral Test, for $p > 1$ and diverges for $p < 1$.
Next, consider the case $p = 1$. We have

$$I = \lim_{b \to \infty} \int_{\ln(\ln 3)}^{\ln(\ln b)} \frac{du}{u} = \lim_{b \to \infty} [\ln u] \Big|_{\ln(\ln 3)}^{\ln(\ln b)} = \lim_{b \to \infty} [\ln(\ln(\ln b)) - \ln(\ln(\ln 3))] = \infty.$$

Since the improper integral I diverges, by the Integral Test, the series diverges for $p = 1$. In conclusion, using the Integral Test, we have shown that the series $\sum_{k=3}^{\infty} a_k = \sum_{k=3}^{\infty} \frac{1}{k(\ln k)[\ln(\ln k)]^p}$ converges if and only if $p > 1$.

67. $S = 1 + 2 + 4 + 8 + \cdots$ is a divergent geometric series, because the sequence of partial sums $S_n = \sum_{k=1}^{n} 2^{k-1}$ diverges since $|r| = 2 > 1$. Multiplying the series S by a constant nonzero multiple such as 2 again gives a divergent series. The equation $2S = -1 + S$ can be solved for S by the usual operations of arithmetic only if S and $2S$ are finite, which they are not. (Note that since both S and $2S$ are infinite, the equation $2S = -1 + S$ is satisfied identically, that is in the sense of $\infty = \infty$.) This is the reason for the paradox.

68. Let $a_k = 1$ and $b_k = -1$ for all k. Then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ diverge, but $\sum_{k=1}^{\infty} (a_k + b_k)$ converges. Let $a_k = b_k = 1$ for all k. Then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ diverge, but $\sum_{k=1}^{\infty} (a_k - b_k)$ converges.
For another example, let $a_k = \frac{1}{k}$ and $b_k = \frac{1}{k+1}$ for all k. Then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ being harmonic series diverge, but $\sum_{k=1}^{\infty} (a_k - b_k)$ will be a telescoping series and converge. Let $a_k = \frac{1}{k}$ and $b_k = -\frac{1}{k+1}$ for all k. Then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ being harmonic series diverge, but $\sum_{k=1}^{\infty} (a_k + b_k)$ will be a telescoping series and converge.

69. Using a CAS, we find that $\pi^2 \approx 6\sum_1^{100} \frac{1}{k^2} \approx 6(1.63498) \approx \boxed{9.8099.}$

70. (a) Since $\frac{1}{p-1} < \sum\limits_{k=1}^{\infty} \frac{1}{k^p} < 1 + \frac{1}{p-1}$, an interval of width 1 for the case of $p = 3$ is $\left(\frac{1}{p-1}, 1 + \frac{1}{p-1}\right) = \left(\frac{1}{3-1}, 1 + \frac{1}{3-1}\right) = \boxed{\left(\frac{1}{2}, \frac{3}{2}\right).}$

(b) Using a CAS, $\sum\limits_{k=1}^{100} \frac{1}{k^3} \approx \boxed{1.2020.}$

71. The series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} k^6 e^{-k}$. The function $f(x) = x^6 e^{-x}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 6$, since

$$f'(x) = 6x^5 e^{-x} - x^6 e^{-x} = x^5(6 - x)e^{-x} \leq 0 \text{ for } x \geq 6.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 6$. Using the integral test, we find

$$I = \int_6^{\infty} f(x)\, dx = \int_6^{\infty} x^6 e^{-x}\, dx = \frac{176112}{e},$$

using a CAS. Since the improper integral I converges, by the integral test the series $\sum\limits_{k=6}^{\infty} a_k = \sum\limits_{k=6}^{\infty} k^6 e^{-k}$ converges as well. The original series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} k^6 e^{-k}$ differs from the convergent series $\sum\limits_{k=6}^{\infty} a_k = \sum\limits_{k=6}^{\infty} k^6 e^{-k}$ by the five terms: $\sum\limits_{k=1}^{5} a_k = \sum\limits_{k=1}^{5} k^6 e^{-k} \approx 225.625$ using a CAS, so since the two series differ by a finite value, it means that the original series $\boxed{\textit{converges}}$ as well.

72. The series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k+3}{k^2+6k+7}$. The function $f(x) = \frac{x+3}{x^2+6x+7}$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $x \geq 1$, since

$$f'(x) = \frac{(x^2 + 6x + 7) \cdot 1 - (x + 3) \cdot (2x + 6)}{(x^2 + 6x + 7)^2} = -\frac{x^2 + 6x + 11}{(x^2 + 6x + 7)^2} < 0 \text{ for } x \geq 1.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the integral test, we find

$$I = \int_1^{\infty} f(x)\, dx = \lim_{b\to\infty} \int_1^b \frac{x+3}{x^2 + 6x + 7}\, dx = \frac{1}{2} \lim_{b\to\infty} \big(\ln(b^2 + 6b + 7) - \ln 14\big) = \infty,$$

using a CAS. Since the improper integral I diverges, by the integral test the series $\boxed{\textit{diverges}}$ as well.

73. The series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{5k+6}{k^3-1}$. The function $f(x) = \frac{5x+6}{x^3-1}$ is defined on $[2, \infty)$ and is continuous, positive, and decreasing for all $x \geq 2$, since

$$f'(x) = \frac{(x^3 - 1) \cdot 5 - (5x + 6) \cdot (3x^2)}{(x^3 - 1)^2} = -\frac{10x^3 + 18x^2 + 5}{(x^3 - 1)^2} < 0 \text{ for } x \geq 2.$$

Also, $a_k = f(k)$ for all positive integers $k \geq 2$. Using the integral test, we find using a CAS,

$$I = \int_2^{\infty} f(x)\, dx = \int_2^{\infty} \frac{5x + 6}{x^3 - 1}\, dx = \frac{1}{6}\left(-\sqrt{3}\pi + 11\ln 7 + 2\sqrt{3}\tan^{-1}\left(\frac{5}{\sqrt{3}}\right)\right) \approx 3.375.$$

Since the improper integral I converges, by the integral test the series $\boxed{\textit{converges}}$ as well.

74. **(a)** For the series $\sum\limits_{k=1}^{\infty} \frac{1}{k^{0.99}}$: $\boxed{S_{10} \approx\ 2.9561}$ $\boxed{S_{1000} \approx 7.7290}$ $\boxed{S_{100,000} \approx 12.7783}$.

For the series $\sum\limits_{k=1}^{\infty} \frac{1}{k^{1.01}}$: $\boxed{S_{10} \approx\ 2.9023}$ $\boxed{S_{1000} \approx 7.2530}$ $\boxed{S_{100,000} \approx 11.4529}$.

(b) The partial sums of the convergent series $\sum\limits_{k=1}^{\infty} \frac{1}{k^{1.01}}$ grow slower compared to the partial sums of the divergent series $\sum\limits_{k=1}^{\infty} \frac{1}{k^{0.99}}$, as one might have expected.

75. Since $\sum\limits_{k=1}^{\infty} a_k$ is convergent, let the sequence of partial sums $\{S_n\}$ for $\sum\limits_{k=1}^{\infty} a_k$ converge to L, that is, $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_k = L$. Similarly, since $\sum\limits_{k=1}^{\infty} b_k$ is convergent, let the sequence of partial sums $\{S'_n\}$ for $\sum\limits_{k=1}^{\infty} b_k$ converge to M, which means $\lim\limits_{n\to\infty} S'_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} b_k = M$. The sequence of partial sums of $\sum\limits_{k=1}^{\infty} (a_k + b_k)$ is $\{S_n + S'_n\}$, and we have

$$\lim_{n\to\infty} (S_n + S'_n) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} S'_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k + \lim_{n\to\infty} \sum_{k=1}^{n} b_k = L + M,$$

using the additive property of limits, showing that the series $\sum\limits_{k=1}^{\infty} (a_k + b_k)$ converges.

76. Let $S = \sum\limits_{k=1}^{\infty} a_k$ be a convergent series which converges to S; this means that the sequence $\{S_n\}$ of partial sums $S_n = \sum\limits_{k=1}^{n} a_k$ converges to S, or $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_k = S$. Consider the series $\sum\limits_{k=1}^{\infty} ca_k$. The sequence of partial sums $\{S'_n\}$ of this series is $S'_n = \sum\limits_{k=1}^{n} ca_k$. We have

$$\lim_{n\to\infty} S'_n = \lim_{n\to\infty} \sum_{k=1}^{n} ca_k = c \lim_{n\to\infty} \sum_{k=1}^{n} a_k = cS,$$

using the scalar multiple property of limits. Since the sequence of partial sums $\{S'_n\}$ converges to cS, this means that the sum of the series $\sum\limits_{k=1}^{\infty} ca_k = cS$.

77. We have $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{N} a_k + \sum\limits_{k=N+1}^{\infty} a_k = K + S = S + K$. This proves that $\sum\limits_{k=1}^{\infty} a_k$ converges with sum $S + K$.

78. Let the series $\sum\limits_{k=1}^{\infty} a_k$ converge to a sum L. This means the limit of the sequence of partial sums $\{S_n\}$ is L, that is $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_k = L$. If the series $\sum\limits_{k=1}^{\infty} b_k$ diverges, then the limit of the sequence of partial sums $\{S'_n\}$ is either nonexistent or is infinite, in other words, $\lim\limits_{n\to\infty} S'_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} b_k$ does not exist or is infinite. The sequence of partial sums $\{S''_n\}$ of the series $\sum\limits_{k=1}^{\infty} (a_k + b_k)$ is given by

$$S''_n = \sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = S_n + S'_n.$$

Taking the limits on both sides, we have

$$\lim_{n\to\infty} S''_n = \lim_{n\to\infty} S_n + \lim_{n\to\infty} S'_n = L + \lim_{n\to\infty} S'_n,$$

which does not exist or is infinite because $\lim\limits_{n\to\infty} S'_n$ does not exist or is infinite. This shows that the sequence of partial sums $\{S''_n\}$ of the series $\sum\limits_{k=1}^{\infty}(a_k+b_k)$ diverges, so the series itself diverges.

79. Since the series $\sum\limits_{k=1}^{\infty} a_k$ with $a_k > 0$ for all k converges, the limit of the sequence $\{S_n\}$ of partial sums with nth term given by $S_n = \sum\limits_{k=1}^{n} a_k$ is finite, say L. That is, $L = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n} a_k$. The nth term of the sequence $\{S'_n\}$ of partial sums for the series $\sum\limits_{k=1}^{\infty}\frac{a_k}{1+a_k}$ is given by $S'_n = \sum\limits_{k=1}^{n}\frac{a_k}{a_k+1}$. Since the terms $a_k > 0$ for all k, we have

$$0 < S'_n = \sum_{k=1}^{n}\frac{a_k}{a_k+1} < \sum_{k=1}^{n} a_k < L,$$

which means that the partial sum is bounded from below by 0 and above by L. Also, the terms of the sequence of partial sums is an increasing sequence which can be seen using the Algebraic Difference Test:

$$S'_{n+1} - S'_n = \sum_{k=1}^{n+1}\frac{a_k}{a_k+1} - \sum_{k=1}^{n}\frac{a_k}{a_k+1} = \frac{a_{n+1}}{a_{n+1}+1} > 0,$$

since $a_{n+1} > 0$ for all n. Since the sequence $\{S'_n\}$ of partial sums is bounded from above and increasing, it converges.

Challenge Problems

80. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty}\frac{1}{k\ln\left(1+\frac{1}{k}\right)}$. We have

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\frac{1}{n\ln\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{\frac{1}{n}}{\ln\left(1+\frac{1}{n}\right)}\\ &= \lim_{x\to\infty}\frac{\frac{1}{x}}{\ln\left(1+\frac{1}{x}\right)} = \lim_{x\to\infty}\frac{-\frac{1}{x^2}}{\frac{1}{1+\frac{1}{x}}\cdot -\frac{1}{x^2}} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right) = 1 \neq 0,\end{aligned}$$

using L'Hôpital's rule on a related function of the sequence $\{a_k\}$. So the series $\boxed{\textit{diverges.}}$

81. The series is $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty}\frac{1}{(\ln k)^p}$. The function $f(x) = \frac{1}{(\ln x)^p}$ is defined on $[2,\infty)$ and is continuous, positive, and decreasing for all $x \geq 2$. Also, $a_k = f(k)$ for all positive integers k. Using the integral test, we find $I = \int_2^{\infty} f(x)\,dx = \lim\limits_{b\to\infty}\int_2^b \frac{dx}{(\ln x)^p}$. Let $u = \frac{1}{(\ln x)^p}$, $dv = dx$. Then, $du = -p(\ln x)^{-p-1}\cdot\frac{1}{x}\,dx$, and $v = x$. Integrating by parts, we have

$$I = \lim_{b\to\infty}\left(\left[\frac{x}{(\ln x)^p}\right]_2^b + \int_2^b \frac{dx}{(\ln x)^{p+1}}\right).$$

The first term yields

$$\lim_{b\to\infty}\frac{b}{(\ln b)^p}-\frac{2}{(\ln 2)^p}=\lim_{b\to\infty}\frac{1}{p(\ln b)^{p-1}\cdot\frac{1}{b}}-\frac{2}{(\ln 2)^p}=\lim_{b\to\infty}\frac{b}{p(\ln b)^{p-1}}-\frac{2}{(\ln 2)^p}$$
$$=\cdots=\lim_{b\to\infty}\frac{b}{p!\ln b}-\frac{2}{(\ln 2)^p}=\frac{1}{p!}\lim_{b\to\infty}\frac{1}{\frac{1}{b}}-\frac{2}{(\ln 2)^p}=\infty,$$

via a repeated application of L'Hôpital's rule to the first term. This means that the improper integral I diverges, so by the integral test the series $\boxed{\textit{diverges}}$ as well.

82. The series is $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{(\ln k)^q}{x^p}$. We assume in what follows that $p>0$ and $q>0$ are integers. The function $f(x)=\frac{(\ln x)^q}{x^p}$ is defined on $[2,\infty)$ and is continuous, positive, and decreasing for all $x\geq 2$. Also, $a_k=f(k)$ for all positive integers k. To use the integral test, we evaluate

$$I(p,q)=\int_2^{\infty} f(x)\,dx=\lim_{b\to\infty}\int_2^b x^{-p}(\ln x)^q\,dx.$$

Let $u=(\ln x)^q$, $dv=x^{-p}\,dx$. Then $du=\frac{(\ln x)^{q-1}}{x}\,dx$, $v=\frac{x^{-p+1}}{-p+1}$, assuming $p\neq 1$ (we will consider the case $p=1$ separately below). Integrating by parts, we have

$$I(p,q)=\lim_{b\to\infty}\left[\frac{(\ln x)^q\cdot x^{-p+1}}{-p+1}\Big|_2^b-\int_2^b\frac{x^{-p+1}}{-p+1}\cdot\frac{(\ln x)^{q-1}}{x}\,dx\right]$$
$$=\frac{1}{1-p}\lim_{b\to\infty}\left[(\ln x)^q x^{-p+1}\Big|_2^b-\int_2^b\frac{(\ln x)^{q-1}}{x^p}\,dx\right]$$
$$=\frac{1}{1-p}\lim_{b\to\infty}\left[\frac{(\ln b)^q}{b^{p-1}}-\frac{(\ln 2)^q}{2^{p-1}}-I(p,q-1)\right]$$

The first term limit is of an indeterminate form, and can be found by repeated application of L'Hôpital's rule,

$$\lim_{b\to\infty}\frac{(\ln b)^q}{b^{p-1}}=\lim_{b\to\infty}\frac{q(\ln b)^{q-1}\cdot\frac{1}{b}}{(p-1)b^{p-2}}=\lim_{b\to\infty}\frac{q(\ln b)^{q-1}}{(p-1)b^{p-1}}=\cdots=\lim_{b\to\infty}\frac{q!}{(p-1)!b^{p-1}}.$$

This limit exists if $p>1$. The other term obtained while integrating $I(p,q)$ by parts, namely $I(p,q-1)$ can again be integrated by parts, and will produce similar limits that are defined only if $p>1$—to see this, just replace q by $q-k$ everywhere in the above calculation, to see that

$$I(p,q-k)=\frac{1}{1-p}\lim_{b\to\infty}\left[\frac{(\ln b)^{q-k}}{b^{p-1}}-\frac{(\ln 2)^{q-k}}{2^{p-1}}-I(p,q-k-1)\right],$$

where $k=0,1,2,\cdots q-1$. In the final step, for $k=q$, we would obtain

$$I(p,0)=\int_2^b\frac{dx}{x^p}=\frac{1}{1-p}\frac{1}{b^{p-1}},$$

and once again, this term has convergent limit as $b\to\infty$ only if $p>1$. We have shown that the series converges for any $q>0$ if $p>1$.

If $p = 1$, then we have $dv = \frac{dx}{x}$ so $v = \ln x$. Proceeding with the integration of parts as before, we have

$$\begin{aligned} I(p=1,q) &= \lim_{b\to\infty}\left[(\ln x)^q\cdot(\ln x)^1\Big|_2^b - \int_2^b (\ln x)^1\cdot\frac{(\ln x)^{q-1}}{x}\,dx\right] \\ I(1,q) &= \lim_{b\to\infty}\left[(\ln x)^{q+1}\Big|_2^b\right] - \lim_{b\to\infty}\int_2^b \frac{(\ln x)^q}{x}\,dx \\ &= \lim_{b\to\infty}\left[(\ln b)^{q+1} - (\ln 2)^{q+1}\right] - I(1,q) \\ \text{or,}\qquad 2I(1,q) &= \lim_{b\to\infty}\left[(\ln b)^{q+1} - (\ln 2)^{q+1}\right] \\ I(1,q) &= \frac{1}{2}\lim_{b\to\infty}\left[(\ln b)^{q+1} - (\ln 2)^{q+1}\right], \end{aligned}$$

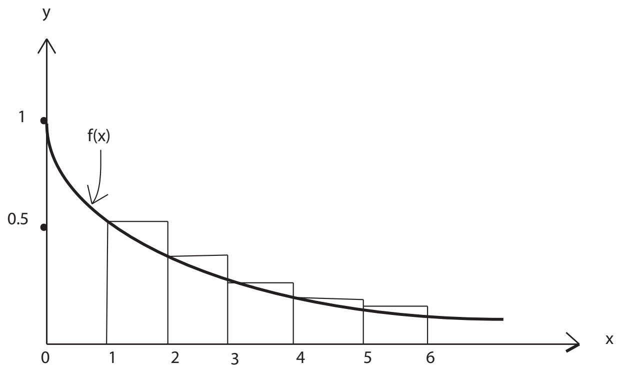
which exists only if $q + 1 \leq 0$, or $q \leq -1$. Since we assumed that $q > 0$, we conclude that the series does not converge for $p = 1$. So we have shown using the Integral Test that the original series converges for the integers $\boxed{p > 1 \text{ for any } q > 0.}$

83. The series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} k^x$. The function $f(y) = y^x$ is defined on $[1, \infty)$ and is continuous, positive, and decreasing for all $y \geq 1$, provided $x < 0$, since $f'(x) = xy^{x-1} < 0$ if $x < 0$ for $y \geq 1$. Also, $a_k = f(k)$ for all positive integers $k \geq 1$. Using the Integral Test, we find

$$I = \int_1^{\infty} f(y)\,dy = \lim_{b\to\infty}\int_1^b y^x\,dy = \lim_{b\to\infty}\frac{y^{x+1}}{x+1}\Big|_1^b = \lim_{b\to\infty}\frac{1}{(x+1)b^{-1-x}} - \frac{1}{(x+1)} = -\frac{1}{x+1}$$

that is, the limit exists and is finite only if $-1 - x > 0$, or $x < -1$. So for $x \leq 0$ the range of x values that lead to a convergent series is $\boxed{(-\infty, -1).}$ For $x > 0$, the function $f(y) = y^x$ is an increasing function of x, and the Integral Test is not applicable. But for this case, we have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^x = \infty \neq 0$, so the series diverges by the Divergence Test.

84. (a) Consider the finite sum $S_n = \sum\limits_{k=1}^{n} \frac{1}{1+k^2}$. The function $f(x) = \frac{1}{1+x^2}$ is defined on $[0, \infty)$ and is continuous, positive, and decreasing for all $x \geq 0$. Also, $a_k = f(k)$ for all positive integers k. So we have $\int_1^n \frac{1}{1+x^2}\,dx \leq \sum\limits_{k=1}^{n} \frac{1}{1+k^2} \leq \int_0^n \frac{dx}{1+x^2}$ or $\tan^{-1} n - \frac{\pi}{4} \leq S_n \leq \tan^{-1} n$. So, $S_n \leq \tan^{-1} n$. See figure below for the case $n = 6$:



(b) Since $S_n < \tan^{-1} n < \frac{\pi}{2}$, the sequence of partial sums $\{S_n\}$ is bounded from above. The terms of the sum S_n are positive, so the sequence $\{S_n\}$ is also increasing, as seen by the Algebraic Difference Test:

$$S_{n+1} - S_n = \sum_{k=1}^{n+1} \frac{1}{1+k^2} - \sum_{k=1}^{n} \frac{1}{1+k^2} = \frac{1}{1+(n+1)^2} > 0.$$

Since the sequence of partial sums $\{S_n\}$ is bounded from above and is increasing, it converges to a finite limit, which means that the series $\sum\limits_{k=1}^{\infty} \frac{1}{1+k^2}$ also converges.

(c) From part (a), we have since $\tan^{-1} n - \frac{\pi}{4} \le S_n \le \tan^{-1} n$. Taking limits on every term of the inequality, we get $\lim\limits_{n\to\infty} \left(\tan^{-1} n - \frac{\pi}{4}\right) \le \lim\limits_{n\to\infty} \left(\sum\limits_{k=1}^{n} \frac{1}{1+k^2}\right) \le \lim\limits_{n\to\infty} \left(\tan^{-1} n\right)$. So, $\frac{\pi}{2} - \frac{\pi}{4} \le \sum\limits_{k=1}^{\infty} \frac{1}{1+k^2} \le \frac{\pi}{2}$, or $\frac{\pi}{4} \le \sum\limits_{k=1}^{\infty} \frac{1}{1+k^2} \le \frac{\pi}{2}$, as was required to be proved.

85. (a) $\sum\limits_{k=1}^{10} \frac{1}{k^3} \approx \boxed{1.1975.}$

(b) We have since $\frac{1}{p-1} < \sum\limits_{k=1}^{\infty} \frac{1}{k^p} < 1 + \frac{1}{p-1}$, for the case of $p = 3$, $\frac{1}{3-1} < \sum\limits_{k=1}^{\infty} \frac{1}{k^3} < 1 + \frac{1}{3-1}$, or

$$\boxed{\frac{1}{2} < \sum_{k=1}^{\infty} \frac{1}{k^3} < \frac{3}{2}.}$$

(c) We conclude that $\zeta(3)$ converges to a value between $\frac{1}{2}$ and $\frac{3}{2}$.

86. (a) For the rectangles under the curve $y = f(x)$, as shown in the left figure of the problem, the height of the tallest rectangle of unit width is $f(n+1)$, while the height of the smallest rectangle of unit width is $f(m)$. The area under the curve is larger than the areas of the rectangles of unit width under the curve, so we have $f(n+1) + \cdots + f(m) \le \int_n^m f(x)\,dx$. For the rectangles over the curve $y = f(x)$, as shown in the right figure of the problem, the height of the tallest rectangle of unit width is $f(n)$ while the height of the smallest rectangle of unit width is $f(m-1)$. The area under the curve is now smaller than the areas of all the rectangles of unit width over the curve, so we have $\int_n^m f(x)\,dx \le f(n) + \cdots + f(m-1)$. Putting these two geometric results together gives us the desired inequalities,

$$f(n+1) + \cdots + f(m) \le \int_n^m f(x)\,dx \le f(n) + \cdots + f(m-1).$$

(b) We have, since $\sum\limits_{k=1}^{\infty} f(k)$ converges, then $\sum\limits_{k=n+1}^{\infty} f(k)$ and $\sum\limits_{k=n}^{\infty} f(k)$ also converge, since either of these series differs from the convergent series $\sum\limits_{k=1}^{\infty} f(k)$ only by a finite sum of positive terms. Since from part (a), we had $f(n+1) + \cdots + f(m) \le \int_n^m f(x)\,dx \le f(n) + \cdots + f(m-1)$, taking the limits on every term of the inequality as $m \to \infty$ gives us

$$\lim_{m\to\infty} \sum_{k=n+1}^{m} f(k) \le \lim_{m\to\infty} \int_n^m f(x)\,dx \le \lim_{m\to\infty} \sum_{k=n}^{m} f(k)$$

or $\sum\limits_{k=n+1}^{\infty} f(k) \le \int_n^{\infty} f(x)\,dx \le \sum\limits_{k=n+1}^{\infty} f(k)$, as needed to be shown, since the improper integral exists by the Squeeze Theorem.

(c) From the left hand inequality of part (b), we have $\sum\limits_{k=n+1}^{\infty} \frac{1}{k^2} \le \int_n^{\infty} \frac{1}{x^2}\,dx$, or

$$\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} \le \left[-\frac{1}{x}\right]_n^{\infty}.$$

Now, we use $\zeta(2) = \sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ (see Problem 75). The Error, which refers to the difference between the full value of the convergent series and its nth partial sum, is given by

$$\text{Error} = \left|\frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2}\right| \le \frac{1}{n}.$$

So for an error $\frac{1}{n} < \left(\frac{1}{2}\right) 10^{-2}$, we have $\boxed{n > 200.}$ For an error $\frac{1}{n} < \left(\frac{1}{2}\right) 10^{-10}$, we have $\boxed{n > 2 \times 10^{10}.}$

AP® Practice Problems

1. The series $\sum\limits_{k=1}^{\infty} \frac{1}{k^{p/3}}$ is a p-series which converges for $\frac{p}{3} > 1$, $\boxed{p > 3}$.

   CHOICE C

2. Determine which of the following series diverge by examining each one:

   I. $$\sum_{k=1}^{\infty} \frac{e^{k-1}}{3^{k-1}} = \sum_{k=1}^{\infty} \frac{(e^k)e^{-1}}{(3^k)3^{-1}} = \sum_{k=1}^{\infty} \frac{(e^k)3}{(3^k)e} = \frac{3}{e}\sum_{k=1}^{\infty}\left(\frac{e}{3}\right)^k$$

   This is a $\boxed{\text{convergent}}$ geometric series since $R = \frac{e}{3} < 1$.

   II. $$\begin{aligned}\sum_{k=1}^{\infty} \cos\left(\pi + \frac{1}{k}\right) &= \sum_{k=1}^{\infty}\left(\cos\pi\cos\frac{1}{k} - \sin\pi\sin\frac{1}{k}\right)\\ &= \sum_{k=1}^{\infty}\left(-1\cdot\cos\frac{1}{k} - 0\cdot\sin\frac{1}{k}\right)\\ &= \sum_{k=1}^{\infty}\left(-\cos\frac{1}{k}\right)\\ &= -\sum_{k=1}^{\infty}\cos\frac{1}{k}\end{aligned}$$

   By the nth term test, $\lim\limits_{k\to\infty}\left(\cos\frac{1}{k}\right) = 1 \ne 0$.

   Therefore, by the nth term test, $\sum\limits_{k=1}^{\infty}\cos\left(\pi + \frac{1}{k}\right)$ $\boxed{\text{diverges}}$.

   III. $$\sum_{k=1}^{\infty}\left(\frac{10}{k}\right) = 10\sum_{k=1}^{\infty}\frac{1}{k} \text{ is the } \boxed{\text{divergent}} \text{ harmonic series.}$$

   This is also a divergent p-series since $p = 1$.
   A p-series only converges for $p > 1$.

   $\boxed{\text{II and III}}$ are the only series that diverge.

   CHOICE C

3. $\sum\limits_{k=1}^{\infty} k^p = \sum\limits_{k=1}^{\infty} \frac{1}{k^{-p}}$ is a convergent p-series if $-p > 1$, $\boxed{p < -1}$

CHOICE B

4. $$\sum_{k=1}^{\infty} \frac{5^{k-1} - 3^{k-1}}{8^{k-1}} = \sum_{k=1}^{\infty} \left(\frac{5^{k-1}}{8^{k-1}} - \frac{3^{k-1}}{8^{k-1}} \right) = \sum_{k=1}^{\infty} \left(\frac{5}{8} \right)^{k-1} - \sum_{k=1}^{\infty} \left(\frac{3}{8} \right)^{k-1}$$
$$= \sum_{k=1}^{\infty} \left(\frac{5}{8} \right)^{k} \left(\frac{5}{8} \right)^{-1} - \sum_{k=1}^{\infty} \left(\frac{3}{8} \right)^{k} \left(\frac{3}{8} \right)^{-1}$$
$$= \sum_{k=1}^{\infty} \left(\frac{5}{8} \right)^{k} \left(\frac{8}{5} \right) - \sum_{k=1}^{\infty} \left(\frac{3}{8} \right)^{k} \left(\frac{8}{3} \right)$$
$$= \frac{8}{5} \sum_{k=1}^{\infty} \left(\frac{5}{8} \right)^{k} - \frac{8}{3} \sum_{k=1}^{\infty} \left(\frac{3}{8} \right)^{k}$$

Each of the individual Series is a convergent Geometric Series for which $\lim\limits_{n\to\infty} t_n = \frac{t_1}{1-r}$, so the sum

$$= \frac{8}{5}\left(\frac{\frac{5}{8}}{1 - \frac{5}{8}} \right) - \frac{8}{3}\left(\frac{\frac{3}{8}}{1 - \frac{3}{8}} \right)$$
$$= \frac{8}{5}\left(\frac{5}{3} \right) - \frac{8}{3}\left(\frac{3}{5} \right)$$
$$= \frac{8}{3} - \frac{8}{5}$$
$$= \boxed{\frac{16}{15}}$$

CHOICE B

5. $\sum\limits_{k=1}^{\infty} \frac{2}{k^{3/2}}$ is a convergent p-series since $p = \frac{3}{2} > 1$.

For a convergent p-series, the bounds are: $\frac{1}{p-1} < \sum\limits_{k=1}^{\infty} \frac{1}{k^p} < 1 + \frac{1}{p-1}$

Here,

$$\frac{1}{\frac{3}{2} - 1} < \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} < 1 + \frac{1}{\frac{3}{2} - 1}$$
$$2 < \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} < 3$$
$$\boxed{4 < \sum_{k=1}^{\infty} \frac{2}{k^{3/2}} < 6}$$

6. (a) A related function is $\boxed{f(x) = \frac{\ln x}{x}}$.

   (b) $f(x) = \frac{\ln x}{x}$ is positive for all $x > 0$ since $\ln x > 0$ and $x > 0$.

To determine if $f(x)$ is continuous and decreasing on $[3, \infty)$ calculate $f'(x)$ as follows:

$$f(x) = \frac{\ln x}{x}$$
$$f'(x) = \frac{\frac{1}{x}(x) - 1 \cdot \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

$f'(x) < 0$ for $x > e$ so $f(x)$ is decreasing for all $x > e$ and therefore for all x on the interval $[3, \infty)$.

$f(x)$ is continuous where $f'(x)$ is differentiable. Here, $f'(x)$ is differentiable for all $x > 0$ and therefore on the interval $[3, \infty)$.

(c) Since the related function, $f(x)$, is positive, continuous and differentiable on $[3, \infty)$ the integral test is applicable to determine whether $\sum\limits_{k=3}^{\infty} \frac{\ln k}{k}$ converges or diverges.

Investigate the improper Integral, $\int_3^\infty \frac{\ln x}{x}\, dx$:

$$\int_3^\infty \frac{\ln x}{x}\, dx = \lim_{b\to\infty}\left(\int_3^b \frac{\ln x}{x}\, dx\right)$$

Let $u = \ln x \quad du = \frac{dx}{x}$

$$\lim_{b\to\infty}\left(\int_3^b \frac{\ln x}{x}\, dx\right) = \lim_{\ln b\to\infty}\left(\int_{\ln 3}^{\ln b} u\, du\right) = \lim_{\ln b\to\infty}\left[\frac{u^2}{2}\right]_{\ln 3}^{\ln b}$$
$$= \lim_{\ln b\to\infty}\left[\frac{(\ln b)^2}{2} - \frac{(\ln 3)^2}{2}\right] = \infty$$

Since the improper Integral diverges, by the Integral Test the series $\sum\limits_{k=3}^{\infty} \frac{\ln k}{k}$ also $\boxed{\text{diverges}}$.

7. (a) For $\sum\limits_{k=1}^{\infty} \frac{1}{1+9k^2}$, a related function to $\frac{1}{1+9k^2}$ is $f(x) = \frac{1}{1+9x^2}$.

To determine if $\sum\limits_{k=1}^{\infty} \frac{1}{1+9k^2}$ converges apply the Integral Test as follows:

$f(x)$ must be positive, continuous, and decreasing for $x \geq 1$.

Observe that $f(x) = \frac{1}{1+9x^2} > 0$ for all x including $x \geq 1$.

To determine if $f(x)$ is continuous and decreasing, calculate $f'(x)$.

$$f'(x) = -1(1 + 9x^2)^{-2}(18x) = \frac{-18x}{(1 + 9x^2)^2}$$

$f(x)$ is differentiable for $x > 1$ since $f'(x)$ is defined for $x > 1$, so $f(x)$ is continuous for $x > 1$.

$f'(x)$ is negative for $x > 1$, so $f(x)$ is decreasing for $x > 1$.

With these preconditions met, the Integral Test is applicable to determine if $\sum\limits_{k=1}^{\infty} \frac{1}{1+9k^2}$ converges.

$$\int_1^\infty \frac{1}{1 + 9x^2}\, dx = \lim_{b\to\infty}\left(\int_1^b \frac{1}{1 + 9x^2}\, dx\right) = \lim_{b\to\infty}\left[\frac{1}{3}\tan^{-1}(3x)\right]_1^b$$
$$= \frac{1}{3}\lim_{b\to\infty}\left[\tan^{-1}(3b) - \tan^{-1}(3)\right] = \frac{1}{3}\left[\frac{\pi}{2} - \tan^{-1} 3\right]$$

Since the Improper Integral converges the series $\sum\limits_{k=1}^{\infty} \frac{1}{1+9k^2}$ also $\boxed{\text{converges}}$.

(b) In general, if f is a continuous, positive, decreasing function on the interval $[1,\infty)$ and $a_k = f(k)$ for all positive integers k and the improper integral $\int_1^\infty f(x)\,dx$ converges, then the sum of the series is bounded by

$$\int_1^\infty f(x)\,dx < \sum_{k=1}^\infty a_k < a_1 + \int_1^\infty f(x)\,dx$$

Here, $\boxed{\frac{\pi}{6} - \frac{1}{3}\tan^{-1} 3 < \sum_{k=1}^\infty \frac{1}{1+9k^2} < \frac{1}{10} + \frac{\pi}{6} - \frac{1}{3}\tan^{-1} 3}$

8.4 Comparison Tests

Concepts and Vocabulary

1. $\boxed{\textbf{(b)}}$ The series $\sum\limits_{k=1}^\infty a_k$ is divergent by the Comparison Test.

2. **False:** We require $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L \neq 0$ in order to be able to conclude that $\sum\limits_{k=1}^\infty a_k$ and $\sum\limits_{k=1}^\infty b_k$ are both convergent or both divergent.

3. **False:** If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L > 0$, then all that can be concluded is either $\sum\limits_{k=1}^\infty a_k$ and $\sum\limits_{k=1}^\infty b_k$ both converge or they both diverge.

4. **False:** The limit comparison test merely decides if the unknown series converges or diverges by comparing it with a known convergent or divergent series. It does not tell us what the sum of the series is.

Skill Building

5. Comparing the nth term of $\sum\limits_{k=1}^\infty \frac{1}{k(k+1)}$ with the nth term of $\sum\limits_{k=1}^\infty \frac{1}{k^2}$ we have

$$\frac{1}{n(n+1)} = \frac{1}{n^2+n} < \frac{1}{n^2}$$

whenever $n \geq 1$. Since the p-series $\sum\limits_{k=1}^\infty \frac{1}{k^p} = \sum\limits_{k=1}^\infty \frac{1}{k^2}$ converges for $p = 2 > 1$, by the Comparison Test, $\sum\limits_{k=1}^\infty \frac{1}{k(k+1)}$ also $\boxed{\textit{converges.}}$

6. Comparing the nth term of $\sum\limits_{k=1}^\infty \frac{1}{(k+1)^2}$ with the nth term of $\sum\limits_{k=1}^\infty \frac{1}{k^2}$ we have

$$\frac{1}{(n+1)^2} = \frac{1}{n^2+2n+1} < \frac{1}{n^2}$$

for $n \geq 1$. By the Comparison Test, since the p-series $\sum_{k=1}^{\infty} \frac{1}{k^p} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges for $p = 2 > 1$, the series $\sum_{k=1}^{\infty} \frac{1}{(k+1)^2}$ also $\boxed{\textit{converges.}}$

7. Comparing the nth term of $\sum_{k=2}^{\infty} \frac{4^k}{7^k+1}$ with the nth term of $\sum_{k=2}^{\infty} \left(\frac{4}{7}\right)^k$, we have

$$\frac{4^n}{7^n+1} < \frac{4^n}{7^n}$$

for $n \geq 1$. Since the geometric series $\sum_{k=2}^{\infty} \left(\frac{4}{7}\right)^k$ converges since $|r| = \frac{4}{7} < 1$, by the Comparison Test, the series $\sum_{k=2}^{\infty} \frac{4^k}{7^k+1}$ also $\boxed{\textit{converges.}}$

8. Comparing the nth term of $\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2^k)}$ with the nth term of $\sum_{k=1}^{\infty} \frac{1}{2^k}$ we wish to have

$$\begin{aligned} \frac{1}{(2n-1)(2^n)} &< \frac{1}{(2^n)} \\ \text{or,} \qquad \frac{1}{2n-1} &< 1 \\ \text{or,} \qquad 2n-1 &> 1, \end{aligned}$$

or $n > 1$. So for $n > 1$, by the Comparison test, since $\sum_{k=1}^{\infty} \frac{1}{2^k}$ is a convergent geometric series, since $|r| = \frac{1}{2} < 1$, the series $\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2^k)}$ also $\boxed{\textit{converges.}}$

9. Comparing the nth term of $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k(k-1)}}$ with the nth term of $\sum_{k=2}^{\infty} \frac{1}{k}$ we have

$$\frac{1}{\sqrt{n(n-1)}} = \frac{1}{n\sqrt{1-\frac{1}{n}}} > \frac{1}{n}$$

for $n > 1$. Since $\sum_{k=2}^{\infty} \frac{1}{k}$ is the divergent harmonic series, by the Comparison Test, the series $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k(k-1)}}$ also $\boxed{\textit{diverges.}}$

10. Comparing the nth term of $\sum_{k=2}^{\infty} \frac{\sqrt{k}}{k-1}$ with the nth term of $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}}$, we wish to have

$$\begin{aligned} \frac{\sqrt{n}}{n-1} &> \frac{1}{\sqrt{n}} \\ \text{or, cross-multiplying,} \qquad n &> n-1, \end{aligned}$$

which is true for all $n \geq 1$. Since the p-series $\sum_{k=2}^{\infty} \frac{1}{k^p} = \sum_{k=2}^{\infty} \frac{1}{k^{1/2}}$ diverges since $0 < p < 1$, by the Comparison Test, the series $\sum_{k=2}^{\infty} \frac{\sqrt{k}}{k-1}$ also $\boxed{\textit{diverges.}}$

11. Comparing the nth term of $\sum\limits_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)}$ with the nth term of the convergent p-series $\sum\limits_{k=1}^{\infty} \frac{1}{k^p} = \sum\limits_{k=1}^{\infty} \frac{1}{k^3}$ (convergent since $p = 3 > 1$) we have

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{n^3 + 3n^2 + 2n} < \frac{1}{n^3}$$

for $n \geq 1$. By the Comparison Test, the series $\sum\limits_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)}$ also $\boxed{\textit{converges.}}$

12. We compare the nth term of $\sum\limits_{k=1}^{\infty} \frac{6}{5k-2}$ with the nth term of the divergent harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$. Since

$$\begin{aligned} \frac{6}{5n-2} &> \frac{1}{n} \\ \text{or} \qquad 6n &> 5n - 2, \end{aligned}$$

for $n > -2$, by the Comparison Test, the series $\sum\limits_{k=1}^{\infty} \frac{6}{5k-2}$ also $\boxed{\textit{diverges.}}$

13. We compare $\sum\limits_{k=1}^{\infty} \frac{\sin^2 k}{k^{\pi}}$ with the convergent p-series $\sum\limits_{k=1}^{\infty} \frac{1}{k^{\pi}}$, which converges since $p = \pi > 1$. We have

$$\frac{\sin^2 n}{n^{\pi}} \leq \frac{1}{n^{\pi}}$$

since $\sin^2 n \leq 1$ for all $n \geq 1$, so by the Comparison Test, the series $\sum\limits_{k=1}^{\infty} \frac{\sin^2 k}{k^{\pi}}$ also $\boxed{\textit{converges.}}$

14. We compare $\sum\limits_{k=1}^{\infty} \frac{\cos^2 k}{k^2+1}$ with the convergent p-series $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$, which converges since $p = 2 > 1$. We have

$$\frac{\cos^2 n}{n^2+1} \leq \frac{1}{n^2+1} < \frac{1}{n^2}$$

since $\cos^2 n \leq 1$ for all $n \geq 1$, so by the Comparison Test, the series $\sum\limits_{k=1}^{\infty} \frac{\cos^2 k}{k^2+1}$ also $\boxed{\textit{converges.}}$

15. The nth term of the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$ behaves like

$$a_n = \frac{1}{(n+1)(n+2)} = \frac{1}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} \approx \frac{1}{n^2}$$

for large n. So we compare with the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$, which is a convergent p-series since $p = 2 > 1$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)(n+2)} = \lim_{n\to\infty} \frac{n^2}{n^2+3n+2} = \lim_{n\to\infty} \frac{1}{1+\frac{3}{n}+\frac{2}{n^2}} = 1.$$

Since the limit is a positive number, and the series $\sum\limits_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$ also $\boxed{\textit{converges.}}$

16. The nth term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2+1}$ behaves like

$$a_n = \frac{1}{n^2+1} = \frac{1}{n^2\left(1+\frac{1}{n^2}\right)} \approx \frac{1}{n^2}$$

for large n. So we compare with the convergent p-series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which converges since $p = 2 > 1$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1.$$

Since the limit is a positive number, and the series $\sum_{k=1}^{\infty} b_k$ converges, then by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2+1}$ also $\boxed{\textit{converges.}}$

17. The nth term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$ behaves like

$$a_n = \frac{1}{\sqrt{n^2+1}} = \frac{1}{n\sqrt{1+\frac{1}{n^2}}} \approx \frac{1}{n}$$

for large n. So we compare with the divergent harmonic series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1.$$

Since the limit is a positive number, and the series $\sum_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$ also $\boxed{\textit{diverges.}}$

18. The nth term of the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k+4}$ behaves like

$$a_n = \frac{\sqrt{n}}{n+4} = \frac{\sqrt{n}}{n\left(1+\frac{4}{n}\right)} = \frac{1}{\sqrt{n}\left(1+\frac{4}{n}\right)} \approx \frac{1}{\sqrt{n}}$$

for large n. So we compare with the p-series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ which is divergent since $0 < p < 1$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n+4}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n}{n+4} = \lim_{n\to\infty} \frac{1}{1+\frac{4}{n}} = 1.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k+4}$ also $\boxed{\textit{diverges.}}$

19. The nth term of the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{3\sqrt{k}+2}{2k^2+5}$ behaves like

$$a_n = \frac{3\sqrt{n}+2}{2n^2+5} = \frac{\sqrt{n}\left(3+\frac{2}{\sqrt{n}}\right)}{n^2\left(2+\frac{5}{n^2}\right)} \approx \frac{1}{n^{3/2}}$$

for large n. So we compare with the p-series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^{3/2}}$ which is convergent since $p = \frac{3}{2} > 1$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3\sqrt{n}+2}{2n^2+5}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n\sqrt{n}(3\sqrt{n}+2)}{2n^2+5} = \lim_{n\to\infty} \frac{3n^2+2n\sqrt{n}}{2n^2+5} = \lim_{n\to\infty} \frac{3+\frac{2}{\sqrt{n}}}{2+\frac{5}{n^2}} = \frac{3}{2}.$$

Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{3\sqrt{k}+2}{2k^2+5}$ also $\boxed{\textit{converges.}}$

20. The nth term of the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{3\sqrt{k}+2}{2k-3}$ behaves like

$$a_n = \frac{3\sqrt{n}+2}{2n-3} = \frac{\sqrt{n}\left(3+\frac{2}{\sqrt{n}}\right)}{n\left(2-\frac{3}{n}\right)} \approx \frac{1}{\sqrt{n}}$$

for large n. So we compare with the p-series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k}}$, which is divergent since $0 < p = \frac{1}{2} < 1$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3\sqrt{n}+2}{2n-3}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{3n+2\sqrt{n}}{2n-3} = \lim_{n\to\infty} \frac{3+\frac{2}{\sqrt{n}}}{2-\frac{3}{n}} = \frac{3}{2}.$$

Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{3\sqrt{k}+2}{2k-3}$ also $\boxed{\textit{diverges.}}$

21. The nth term of the series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{k\sqrt{k^2-1}}$ behaves like

$$a_n = \frac{1}{n\sqrt{n^2-1}} = \frac{1}{n^2\sqrt{1-\frac{1}{n^2}}} \approx \frac{1}{n^2}$$

for large n. So we compare with the p-series $\sum\limits_{k=2}^{\infty} b_k = \sum\limits_{k=2}^{\infty} \frac{1}{k^2}$, which converges since $p = 2 > 1$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{1}{n^2}}} = 1.$$

Since the limit is a positive number and the series $\sum\limits_{k=2}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{k\sqrt{k^2-1}}$ also $\boxed{\textit{converges.}}$

22. The nth term of the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k}{(2k-1)^2}$ behaves like

$$a_n = \frac{n}{(2n-1)^2} = \frac{n}{n^2\left(2-\frac{1}{n}\right)^2} \approx \frac{1}{n}$$

for large n. So we compare with the divergent harmonic series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k}$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{(2n-1)^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{(2n-1)^2} = \lim_{n\to\infty} \frac{1}{\left(2-\frac{1}{n}\right)^2} = \frac{1}{4}.$$

Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k}{(2k-1)^2}$ also $\boxed{\textit{diverges.}}$

23. Use the Limit Comparison Test and compare the series $\sum\limits_{k=1}^{\infty} \frac{3^k}{2^k+5}$ to the Geometric Series $\sum\limits_{k=1}^{\infty} \left(\frac{3}{2}\right)^k$. Then with $a_n = \frac{3^n}{2^n+5}$ and $b_n = \left(\frac{3}{2}\right)^n$ we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3^n}{2^n+5}}{\left(\frac{3}{2}\right)^n} = \lim_{n\to\infty} \left[\frac{3^n}{2^n+5}\left(\frac{2}{3}\right)^n\right] = \lim_{n\to\infty} \frac{2^n}{2^n+5} = \lim_{n\to\infty} \frac{1}{1+\frac{5}{2^n}} = 1.$$

Since the limit is a positive number and $\sum\limits_{k=1}^{\infty} \left(\frac{3}{2}\right)^k$ diverges, by the Limit Comparison Test, $\sum\limits_{k=1}^{\infty} \frac{3^k}{2^k+5}$ also $\boxed{\text{diverges}}$.

24. Use the Limit Comparison Test and compare the series $\sum\limits_{k=1}^{\infty} \frac{8^k}{5+3^k}$ to the Geometric Series $\sum\limits_{k=1}^{\infty} \left(\frac{8}{3}\right)^k$. Then with $a_n = \frac{8^n}{5+3^n}$ and $b_n = \left(\frac{8}{3}\right)^n$ we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{8^n}{5+3^n}}{\left(\frac{8}{3}\right)^n} \lim_{n\to\infty} \left[\frac{8^n}{5+3^n}\left(\frac{3}{8}\right)^n\right] = \lim_{n\to\infty} \frac{3^n}{5+3^n} = \lim_{n\to\infty} \frac{1}{1+\frac{5}{3^n}} = 1.$$

Since the limit is a positive number and $\sum\limits_{k=1}^{\infty} \left(\frac{8}{3}\right)^k$ diverges, by the Limit Comparison Test, $\sum\limits_{k=1}^{\infty} \frac{8^k}{5+3^k}$ also $\boxed{\text{diverges}}$.

25. Use the Limit Comparison Test and compare the series $\sum\limits_{k=1}^{\infty} \frac{3^k+4}{5^k+3}$ to the Geometric Series $\sum\limits_{k=1}^{\infty} \left(\frac{3}{5}\right)^k$. Then with $a_n = \frac{3^n+4}{5^n+3}$ and $b_n = \left(\frac{3}{5}\right)^n$ we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3^n+4}{5^n+3}}{\left(\frac{3}{5}\right)^n} \lim_{n\to\infty} \left[\left(\frac{3^n+4}{5^n+3}\right)\left(\frac{5}{3}\right)^n\right] = \lim_{n\to\infty} \frac{15^n+4\cdot 5^n}{15^n+3\cdot 3^n} = \lim_{n\to\infty} \frac{1+4\left(\frac{1}{3}\right)^n}{1+3\left(\frac{1}{5}\right)^n} = 1.$$

Since the limit is a positive number and $\sum\limits_{k=1}^{\infty}\left(\frac{3}{5}\right)^k$ converges, by the Limit Comparison Test $\sum\limits_{k=1}^{\infty}\frac{3^k+4}{5^k+3}$ also $\boxed{\text{converges}}$.

26. Use the Limit Comparison Test and compare the series $\sum\limits_{k=1}^{\infty}\frac{2^k+1}{7^k+4}$ to the Geometric Series $\sum\limits_{k=1}^{\infty}\left(\frac{2}{7}\right)^k$. Then with $a_n=\frac{2^n+1}{7^n+4}$ and $b_n=\left(\frac{2}{7}\right)^n$ we have

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{2^n+1}{7^n+4}}{\left(\frac{2}{7}\right)^n}\lim_{n\to\infty}\left[\left(\frac{2^n+1}{7^n+4}\right)\left(\frac{7}{2}\right)^n\right]=\lim_{n\to\infty}\frac{14^n+1\cdot 7^n}{14^n+4\cdot 2^n}=\lim_{n\to\infty}\frac{1+\left(\frac{1}{2}\right)^n}{1+4\left(\frac{1}{7}\right)^n}=1.$$

Since the limit is a positive number and $\sum\limits_{k=1}^{\infty}\left(\frac{2}{7}\right)^k$ converges, by the Limit Comparison Test $\sum\limits_{k=1}^{\infty}\frac{2^k+1}{7^k+4}$ also $\boxed{\text{converges}}$.

27. Use the Limit Comparison Test and compare the series $\sum\limits_{k=1}^{\infty}\frac{2\cdot 3^k+5}{4\cdot 8^k}$ to the Geometric Series $\sum\limits_{k=1}^{\infty}\left(\frac{3}{8}\right)^k$. Then with $a_n=\frac{2\cdot 3^n+5}{4\cdot 8^n}$ and $b_n=\left(\frac{3}{8}\right)^n$ we have

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{2\cdot 3^n+5}{4\cdot 8^n}}{\left(\frac{3}{8}\right)^n}=\lim_{n\to\infty}\left[\frac{2\cdot 3^n+5}{4\cdot 8^n}\left(\frac{8}{3}\right)^n\right]=\lim_{n\to\infty}\frac{2\cdot 24^n+5\cdot 8^n}{4\cdot 24^n}=\lim_{n\to\infty}\frac{2+5\left(\frac{1}{3}\right)^n}{4}=\frac{1}{2}.$$

Since the limit is a positive number and $\sum\limits_{k=1}^{\infty}\left(\frac{3}{8}\right)^k$ converges, by the Limit Comparison Test $\sum\limits_{k=1}^{\infty}\frac{2\cdot 3^k+5}{4\cdot 8^k}$ also $\boxed{\text{converges}}$.

28. The Limit Comparison Test is not necessary for this series: $\sum\limits_{k=1}^{\infty}\frac{-4\cdot 5^k}{2\cdot 6^k}=-\frac{4}{2}\sum\limits_{k=1}^{\infty}\frac{5^k}{6^k}=-2\sum\limits_{k=1}^{\infty}\left(\frac{5}{6}\right)^k$, and $\sum\limits_{k=1}^{\infty}\left(\frac{5}{6}\right)^k$ converges. Therefore $\sum\limits_{k=1}^{\infty}\frac{-4\cdot 5^k}{2\cdot 6^k}$ $\boxed{\text{converges}}$.
The Limit Comparison Test does also give the same result, of course:
Compare the series $\sum\limits_{k=1}^{\infty}\frac{-4\cdot 5^k}{2\cdot 6^k}$ to the Geometric Series $\sum\limits_{k=1}^{\infty}-\left(\frac{5}{6}\right)^k$.
Then with $a_n=\frac{-4\cdot 5^n}{2\cdot 6^n}$ and $b_n=-\left(\frac{5}{6}\right)^n$ we have

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{-4\cdot 5^n}{2\cdot 6^n}}{-\left(\frac{5}{6}\right)^n}=\lim_{n\to\infty}\left[\frac{4\cdot 5^n}{2\cdot 6^n}\cdot\left(\frac{6}{5}\right)^n\right]=2.$$

Since the limit is a positive number and $\sum\limits_{k=1}^{\infty}-\left(\frac{5}{6}\right)^k$ converges, by the Limit Comparison Test, $\sum\limits_{k=1}^{\infty}\frac{-4(5^k)}{2(6^k)}$ also $\boxed{\text{converges}}$.

29. Use the Limit Comparison Test and compare the series $\sum\limits_{k=1}^{\infty} \frac{8\cdot 3^k+4\cdot 5^k}{8^{k-1}}$ to the Geometric Series $\sum\limits_{k=1}^{\infty} \left(\frac{5}{8}\right)^k$. Then with $a_n = \frac{8\cdot 3^n+4\cdot 5^n}{8^{n-1}}$ and $b_n = \left(\frac{5}{8}\right)^n$ we have

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{8\cdot 3^n+4\cdot 5^n}{8^{n-1}}}{\left(\frac{5}{8}\right)^n} = \lim_{n\to\infty} \left[\frac{8\cdot 3^n+4\cdot 5^n}{8^{n-1}}\left(\frac{8}{5}\right)^n\right] = \lim_{n\to\infty}\left(8\cdot\frac{8\cdot 24^n+4\cdot 40^n}{40^n}\right)\\ &= \lim_{n\to\infty}\left\{8\left[8\cdot\left(\frac{3}{5}\right)^n+4\right]\right\} = 32.\end{aligned}$$

Since the limit is a positive number and $\sum\limits_{k=1}^{\infty} \left(\frac{5}{8}\right)^k$ converges, by the Limit Comparison Test, $\sum\limits_{k=1}^{\infty} \frac{8\cdot 3^k+4\cdot 5^k}{8^{k-1}}$ also $\boxed{\text{converges}}$.

30. Use the Limit Comparison Test and compare the series $\sum\limits_{k=1}^{\infty} \frac{2\cdot 5^k+3^k}{2^{k-1}}$ to the Geometric Series $\sum\limits_{k=1}^{\infty} \left(\frac{5}{2}\right)^k$. Then with $a_n = \frac{2\cdot 5^n+3^n}{2^{n-1}}$ and $b_n = \left(\frac{5}{2}\right)^n$ we have

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{2\cdot 5^n+3^n}{2^{n-1}}}{\left(\frac{5}{2}\right)^n} \lim_{n\to\infty} \left[\frac{2\cdot 5^n+3^n}{2^{n-1}}\left(\frac{2}{5}\right)^n\right] = \lim_{n\to\infty}\left(2\cdot\frac{2\cdot 10^n+6^n}{10^n}\right)\\ &= \lim_{n\to\infty}\left\{2\left[2+\left(\frac{3}{5}\right)^n\right]\right\} = 4.\end{aligned}$$

Since the limit is a positive number and $\sum\limits_{k=1}^{\infty} \left(\frac{5}{2}\right)^k$ diverges, by the Limit Comparison Test, $\sum\limits_{k=1}^{\infty} \frac{2\cdot 5^k+3^k}{2^{k-1}}$ also $\boxed{\text{diverges}}$.

31. The nth term of the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{3k+4}{k2^k}$ behaves like

$$a_n = \frac{3n+4}{n2^n} = \left(3+\frac{4}{n}\right)\frac{1}{2^n} \approx \frac{1}{2^n}$$

for large n. So we compare with the convergent geometric series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{2^k}$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{3n+4}{n2^n}}{\frac{1}{2^n}} = \lim_{n\to\infty}\left(3+\frac{4}{n}\right) = 3.$$

Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{3k+4}{k2^k}$ also $\boxed{\textit{converges.}}$

32. The nth term of the series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{k-1}{k2^k}$ behaves like

$$a_n = \frac{n-1}{n2^n} = \frac{1}{2^n}\left(1-\frac{1}{n}\right) \approx \frac{1}{2^n}$$

for large n. So we compare with the convergent geometric series $\sum\limits_{k=2}^{\infty} b_k = \sum\limits_{k=2}^{\infty} \frac{1}{2^k}$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n-1}{n2^n}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1.$$

Since the limit is a positive number and the series $\sum\limits_{k=2}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{k-1}{k2^k}$ also $\boxed{\textit{converges.}}$

33. The nth term of the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{2^k+1}$ behaves like $a_n = \frac{1}{2^n+1} \approx \frac{1}{2^n}$ for large n So we compare with the convergent geometric series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{2^n}$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n+1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{2^n}} = 1.$$

Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{2^k+1}$ also $\boxed{\textit{converges.}}$

34. The nth term of the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{5}{3^k+2}$ behaves like $a_n = \frac{5}{3^n+2} \approx \frac{5}{3^n}$ for large n So we compare with the convergent geometric series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{3^n}$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{5}{3^n+2}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{5\cdot 3^n}{3^n+2} = \lim_{n\to\infty} \frac{5}{1+\frac{2}{3^n}} = 5.$$

Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{5}{3^k+2}$ also $\boxed{\textit{converges.}}$

35. The nth term of $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k+5}{k^{k+1}}$ behaves like

$$a_n = \frac{n+5}{n^{n+1}} = \frac{n\left(1+\frac{5}{n}\right)}{n^{n+1}} \approx \frac{1}{n^n}$$

for large n. So we compare with the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^k}$ which was shown (see Example 1, p. 672 of text) to be convergent. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+5}{n^{n+1}}}{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{(n+5)n^n}{n^{n+1}} = \lim_{n\to\infty} \frac{n+5}{n} = \lim_{n\to\infty} \left(1+\frac{5}{n}\right) = 1.$$

Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k+5}{k^{k+1}}$ also $\boxed{\textit{converges.}}$

36. The nth term of the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{5}{k^k+1}$ behaves like $a_n = \frac{5}{n^n+1} \approx \frac{5}{n^n}$ for large n. So we compare with the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^k}$ which was shown (see Example 1, p. 672 of text) to be convergent. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{5}{n^n+1}}{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{5n^n}{n^n+1} = \lim_{n\to\infty} \frac{5}{1+\frac{1}{n^n}} = \frac{5}{1+0} = 5.$$

Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{5}{k^k+1}$ also $\boxed{\textit{converges.}}$

37. Comparing the nth term of the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{6k}{5k^2+2}$ with the nth term of the harmonic series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k}$ gives $\frac{6n}{5n^2+2} > \frac{1}{n}$ since $6n^2 > 5n^2+2$ for $n > \sqrt{2}$. This means for $n \geq 2$, the nth terms satisfy $a_n > b_n$. Since the harmonic series $\sum\limits_{k=1}^{\infty} b_k$ diverges, by the Comparison test, the given series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{6k}{5k^2+2}$ also $\boxed{\textit{diverges.}}$

38. The nth term of the series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{6k+3}{2k^3-2}$ behaves like

$$a_n = \frac{6n+3}{2n^3-2} = \frac{n\left(6+\frac{3}{n}\right)}{n^3\left(2-\frac{2}{n^3}\right)} \approx \frac{1}{n^2}$$

for large n. So we compare with the series $\sum\limits_{k=2}^{\infty} b_k = \sum\limits_{k=2}^{\infty} \frac{1}{k^2}$ which is a convergent p-series (since $p = 2 > 1$). We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{6n+3}{2n^3-2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{6n^3+3n^2}{2n^3-2} = \lim_{n\to\infty} \frac{6+\frac{3}{n}}{2-\frac{2}{n}} = 3.$$

Since the limit is a positive number and the series $\sum\limits_{k=2}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{6k+3}{2k^3-2}$ also $\boxed{\textit{converges.}}$

39. The nth term of the series $\sum\limits_{k=1}^{\infty} \frac{7+k}{(1+k^2)^4}$ behaves like

$$a_n = \frac{7+n}{(1+n^2)^4} = \frac{n\left(\frac{7}{n}+1\right)}{n^8\left(\frac{1}{n^2}+1\right)^4} \approx \frac{1}{n^7}$$

for large n. So we compare with the series $\sum\limits_{k=1}^{\infty} b_k = \frac{1}{k^7}$ which is a convergent p-series (since $p = 7 > 1$). We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{7+n}{(1+n^2)^4}}{\frac{1}{n^7}} = \lim_{n\to\infty} \frac{7n^7+n^8}{(1+n^2)^4} = \lim_{n\to\infty} \frac{\frac{7}{n}+1}{\left(\frac{1}{n^2}+1\right)^4} = 1.$$

Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{7+k}{(1+k^2)^4}$ also $\boxed{\textit{converges.}}$

40. The nth term of the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \left(\frac{7+k}{1+k^2}\right)^4$ behaves like

$$a_n = \left(\frac{7+n}{1+n^2}\right)^4 = \frac{n^4}{n^8}\left(\frac{\frac{7}{n}+1}{\frac{1}{n^2}+1}\right)^4 \approx \frac{1}{n^4}$$

for large n. So we compare with the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^4}$ which is a convergent p-series, since $p = 4 > 1$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{7+n}{1+n^2}\right)^4}{\frac{1}{n^4}} = \lim_{n\to\infty} \left(\frac{7n+n^2}{1+n^2}\right)^4 = \lim_{n\to\infty} \left(\frac{\frac{7}{n}+1}{\frac{1}{n^2}+1}\right)^4 = 1.$$

Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \left(\frac{7+k}{1+k^2}\right)^4$ also $\boxed{\textit{converges.}}$

41. We compare the series $\sum\limits_{k=1}^{\infty} \frac{e^{1/k}}{k}$ with the divergent harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$. Comparing the nth term of each series we have $\frac{e^{1/n}}{n} > \frac{1}{n}$ since $e^{1/n} > 1$ for any $n \geq 1$ (since the nth root of a number bigger than 1 is also bigger than 1; here $e > 1$). By the Comparison Test, the original series also $\boxed{\textit{diverges}}$.

42. We compare the series $\sum\limits_{k=1}^{\infty} \frac{1}{1+e^k}$ with the convergent geometric series $\sum\limits_{k=1}^{\infty} \left(\frac{1}{e}\right)^k$, which converges since $|r| = \frac{1}{e} < 1$. Comparing the nth term of each series we have $\frac{1}{1+e^n} < \frac{1}{e^n}$ for all $n \geq 1$, so by the Comparison Test, the original series also $\boxed{\textit{converges.}}$

43. The nth term of the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^2}{e^k}$ behaves like $\frac{\left(1+\frac{1}{n}\right)^2}{e^n} \approx \frac{1}{e^n}$ for large n. So we compare with the convergent geometric series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \left(\frac{1}{e}\right)^k$ which converges since $|r| = \frac{1}{e} < 1$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\left(1+\frac{1}{n}\right)^2}{e^n}}{\frac{1}{e^n}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 = 1.$$

Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^2}{e^k}$ also $\boxed{\textit{converges.}}$

44. We compare the series $\sum\limits_{k=1}^{\infty} \frac{1}{k2^k}$ with the convergent geometric series $\sum\limits_{k=1}^{\infty} \frac{1}{2^k}$ which converges since $|r| = \frac{1}{2} < 1$. Comparing the nth term of each series we have $\frac{1}{n2^n} < \frac{1}{2^n}$ for $n > 1$. By the Comparison Test, we see that the original series also $\boxed{\textit{converges.}}$

45. We compare the series $\sum\limits_{k=1}^{\infty} \frac{1+\sqrt{k}}{k}$ with the divergent harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$. Since the nth terms of the series satisfy $\frac{1+\sqrt{n}}{n} > \frac{1}{n}$ for $n \geq 1$, we see by the Comparison Test that the original series $\boxed{\textit{diverges.}}$

46. The nth term of the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1+3\sqrt{k}}{k^2}$ behaves like

$$a_n = \frac{1+3\sqrt{n}}{n^2} = \frac{\sqrt{n}\left(\frac{1}{\sqrt{n}}+3\right)}{n^2} \approx \frac{1}{n^{3/2}}$$

for large n. So we compare with the p-series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^{3/2}}$, which converges since $p = \frac{3}{2} > 1$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1+3\sqrt{n}}{n^2}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{3/2}+3n^2}{n^2} = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}}+3\right) = 3.$$

Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1+3\sqrt{k}}{k^2}$ also $\boxed{\textit{converges.}}$

47. We compare the series $\sum\limits_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \sin^2 k$ with the convergent geometric series $\sum\limits_{k=1}^{\infty} \frac{1}{2^k}$ which converges since $|r| = \frac{1}{2} < 1$. Comparing the nth term of each series we have $\left(\frac{1}{2}\right)^n \sin^2 n \leq \frac{1}{2^n}$ since $\sin^2 n \leq 1$ for all $n \geq 1$. By the Comparison test, we see that the original series also $\boxed{\textit{converges.}}$

48. We compare the series $\sum\limits_{k=1}^{\infty} \frac{\tan^{-1} k}{k^3}$ with the p-series $\sum\limits_{k=1}^{\infty} \frac{\pi}{2k^3}$, convergent since $p = 3 > 1$. Since the nth terms of the series satisfy $\frac{\tan^{-1} n}{n^3} < \frac{\pi}{2n^3}$ because $\tan^{-1} n < \frac{\pi}{2}$ for any finite and positive $n \geq 1$, we see that, by the Comparison Test, the original series also $\boxed{\textit{converges.}}$

Applications and Extensions

49. We compare the series $\sum\limits_{k=2}^{\infty} \frac{2}{k^3 \ln k}$ with the series $\sum\limits_{k=2}^{\infty} \frac{2}{k^3}$, which is a convergent p-series since $p = 3 > 1$. Since the nth terms of the series satisfy $\frac{2}{n^3 \ln n} < \frac{2}{n^3}$ since $\ln n > 1$ when $n \geq 3$ (recall $\ln e = 1$) we see that by the Comparison Test, the original series also $\boxed{\textit{converges.}}$

50. (Using the result of Problem 59 below) we compare the series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{\sqrt{k}(\ln k)^4}$ with the divergent harmonic series $\sum\limits_{k=2}^{\infty} d_k = \sum\limits_{k=2}^{\infty} \frac{1}{k}$. Applying L'Hôpital's rule repeatedly ($\star$) and arranging the fractions after each application, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_n}{d_n} &= \lim_{n\to\infty}\left[\frac{\frac{1}{\sqrt{n}(\ln n)^4}}{\frac{1}{n}}\right] = \lim_{n\to\infty}\frac{\sqrt{n}}{(\ln n)^4} \\
&\overset{\star}{=} \lim_{n\to\infty}\left[\frac{\frac{1}{2\sqrt{n}}}{\frac{4(\ln n)^3}{n}}\right] = \lim_{n\to\infty}\frac{\sqrt{n}}{8(\ln n)^3} \\
&\overset{\star}{=} \lim_{n\to\infty}\left[\frac{\frac{1}{2\sqrt{n}}}{\frac{24(\ln n)^2}{n}}\right] = \lim_{n\to\infty}\frac{\sqrt{n}}{48(\ln n)^2} \\
&\overset{\star}{=} \lim_{n\to\infty}\left[\frac{\frac{1}{2\sqrt{n}}}{\frac{96\ln n}{n}}\right] = \lim_{n\to\infty}\frac{\sqrt{n}}{192\ln n} \\
&\overset{\star}{=} \lim_{n\to\infty}\left[\frac{\frac{1}{2\sqrt{n}}}{\frac{192}{n}}\right] = \lim_{n\to\infty}\frac{\sqrt{n}}{384} \\
&= \infty.
\end{aligned}$$

So, by the result of Problem 59, since the series $\sum\limits_{k=2}^{\infty} d_k = \sum\limits_{k=2}^{\infty}\frac{1}{k}$ diverges, the series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty}\frac{1}{\sqrt{k}(\ln k)^4}$ also $\boxed{\textit{diverges.}}$

51. We compare the series $\sum\limits_{k=2}^{\infty}\frac{\ln k}{k+3}$ with the series $\sum\limits_{k=2}^{\infty}\frac{1}{k+3}$, which is a divergent harmonic series. The nth terms of the series satisfy $\frac{\ln n}{n+3} > \frac{1}{n+3}$ since $\ln n > 1$ if $n \geq 3$ (recall that $\ln e = 1$). By the Comparison test, we see that the original series also $\boxed{\textit{diverges.}}$

52. We compare the series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty}\frac{(\ln k)^2}{k^{5/2}}$ with the series $\sum\limits_{k=2}^{\infty} b_k = \sum\limits_{k=2}^{\infty}\frac{1}{k^{3/2}}$, which is a convergent p-series since $p = \frac{3}{2} > 1$. The nth term satisfies

$$a_n = \frac{(\ln n)^2}{n^{5/2}} < \frac{1}{n^{3/2}} = b_n$$

if $\ln n < n^{1/2}$. To find under what conditions $\ln n < n^{1/2}$, let $f(x) = \ln x - x^{1/2}$. We wish to find when this function is decreasing. Since $f'(x) = \frac{1}{x} - \frac{1}{2}x^{-1/2} < 0$ whenever $\sqrt{x} > 2$ or $x > 4$, and $f(4) = \ln 4 - \sqrt{2} < 0$, and we have shown that $f(x)$ is decreasing for $x > 4$, and is negative at $x = 4$, this proves that $f(x) < 0$ for $x > 4$. That is, $\ln n < n^{1/2}$ for $n \geq 4$. Note that the Comparison Test does not care about the relative behavior of series for any initial finite number of terms, but only about the *large n* behavior; due to this, by the Comparison Test, we see that the original series also $\boxed{\textit{converges.}}$

53. We compare the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty}\sin\frac{1}{k}$ with the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty}\frac{1}{k}$ which is a divergent harmonic series. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{y\to 0}\frac{\sin y}{y} = 1,$$

where the substitution $y = \frac{1}{n}$ was made and the standard limit result $\lim\limits_{y\to 0}\frac{\sin y}{y} = 1$ was used. Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k$ also $\boxed{\textit{diverges.}}$

54. We compare the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \tan \frac{1}{k}$ with the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k}$ which is a divergent harmonic series. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan \frac{1}{n}}{\frac{1}{n}} = \lim_{y\to 0} \frac{\tan y}{y} = \lim_{y\to 0} \frac{\sec^2 y}{1} = 1,$$

where the substitution $y = \frac{1}{n}$ was made and L'Hôpital's rule was used. Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \tan \frac{1}{k}$ also $\boxed{\textit{diverges.}}$

55. Compare the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k!}$ with the convergent geometric series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{2^k}$. Since

$$n! = n(n-1)(n-2)\cdots 3\cdot 2 \geq 2\cdot 2\cdot 2\cdot \cdots 2 \quad \text{(n factors of 2)},$$

we have $n! \geq 2^n$ or

$$a_n = \frac{1}{n!} \leq \frac{1}{2^n} = b_n$$

for all $n \geq 2$. So by the Comparison Test that the original series also $\boxed{\textit{converges.}}$

56. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k!}{k^k}$. We have

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \leq n\cdot n\cdot n\cdots n\cdot 2\cdot 1 = 2n^{n-2} = 2\frac{n^n}{n^2}$$

so that

$$a_n = \frac{n!}{n^n} \leq \frac{2}{n^2} = b_n.$$

Comparing with the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{2}{k^2}$, which is a constant multiple of a convergent p-series (since $p = 2 > 1$), we conclude based on the Comparison Test that the original series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k!}{k^k}$ $\boxed{\textit{converges.}}$

57. **(a)** The nth term of $\sum\limits_{k=1}^{\infty} \frac{1}{k^2+1}$ satisfies $\frac{1}{n^2+1} < \frac{1}{n^2}$ for all $n \geq 1$. So by the Comparison Test, since the series $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ converges (being a p-series for $p = 2 > 1$), the original series also $\boxed{\textit{converges.}}$

(b) We have the nth term of $\sum\limits_{k=2}^{\infty} \frac{1}{k^2-1}$ satisfy $\frac{1}{n^2-1} > \frac{1}{n^2}$. This however provides no information about the convergence or divergence of the original series, since the nth term of the original series is *greater* and not less than the nth term of a convergent p-series.

(c) Let $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{k^2-1}$ and $\sum\limits_{k=2}^{\infty} b_k = \sum\limits_{k=2}^{\infty} \frac{1}{k^2}$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{n^2}} = 1.$$

Since the limit is a positive number and the series $\sum\limits_{k=2}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{k^2-1}$ also $\boxed{\textit{converges.}}$ So, $\boxed{\textit{yes}}$, the Limit Comparison Test can be used to decide the convergence of the series.

58. Compare the series $\sum\limits_{k=1}^{\infty} \frac{d}{10^k}$ with the geometric series $\sum\limits_{k=1}^{\infty} \frac{1}{10^{k-1}}$, which is convergent since $|r| = \frac{1}{10} < 1$. Since the nth terms of the series satisfy $\frac{d_n}{10^n} < \frac{1}{10^{n-1}}$, since $d_n < 10$, by the Comparison Test, the original series also $\boxed{\textit{converges.}}$

59. If $\lim\limits_{n\to\infty} \frac{a_n}{d_n} = \infty$ it follows that we can choose an $M > 0$ such that there exists an $n > N$, such that for $n > N$, we have $\frac{a_n}{d_n} > M$, that is, $a_n > Md_n$. By the Comparison Test, since $\sum\limits_{k=1}^{\infty} d_k$ diverges, so does $\sum\limits_{k=1}^{\infty} a_k$.

60. If $\lim\limits_{n\to\infty} \frac{a_n}{d_n} = 0$ it follows that we can choose an $\epsilon > 0$ such that there exists $n > N$, such that for $n > N$, we have $\left|\frac{a_n}{d_n} - 0\right| < \epsilon$, or $a_n < \epsilon\, d_n$. By the Comparison Test, since $\sum\limits_{k=1}^{\infty} d_k$ converges, so does $\sum\limits_{k=1}^{\infty} a_k$.

61. Let $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{\ln k}$ and $\sum\limits_{k=2}^{\infty} d_k = \sum\limits_{k=2}^{\infty} \frac{1}{k}$. Then

$$\lim_{n\to\infty} \frac{a_n}{d_n} = \lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x = \infty,$$

by using L'Hôpital's rule on a related function of the ratio of the nth terms. So by the result of Problem 59, since the harmonic series $\sum\limits_{k=2}^{\infty} d_k$ diverges, the original series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{\ln k}$ also $\boxed{\textit{diverges.}}$

62. Let $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \left(\frac{1}{\ln k}\right)^2$ and $\sum\limits_{k=2}^{\infty} d_k = \sum\limits_{k=2}^{\infty} \frac{1}{k}$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{d_n} &= \lim_{n\to\infty} \frac{\left(\frac{1}{\ln n}\right)^2}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{(\ln n)^2} = \lim_{x\to\infty} \frac{x}{(\ln x)^2}\\ &= \lim_{x\to\infty} \frac{1}{2\ln x \cdot \frac{1}{x}} = \lim_{x\to\infty} \frac{x}{2\ln x} = \frac{1}{2}\lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \frac{1}{2}\lim_{x\to\infty} x = \infty,\end{aligned}$$

by using L'Hôpital's rule on a related function of the ratio of the nth terms. So by the result of Problem 59, since the harmonic series $\sum\limits_{k=2}^{\infty} d_k$ diverges, the original series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \left(\frac{1}{\ln k}\right)^2$ also $\boxed{\textit{diverges.}}$

63. Let $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\ln k}{k^2}$ and $\sum\limits_{k=1}^{\infty} d_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^{3/2}}$. Then

$$\lim_{n\to\infty} \frac{a_n}{d_n} = \lim_{n\to\infty} \frac{\frac{\ln n}{n^2}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = 2\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0,$$

by using L'Hôpital's rule on a related function of the ratio of the nth terms. So by the result of Problem 60, since the p-series $\sum\limits_{k=2}^{\infty} d_k$ converges (since $p = \frac{3}{2} > 1$), the original series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\ln k}{k^2}$ also $\boxed{\textit{converges.}}$

64. Let $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{(k\ln k)^2}$ and $\sum\limits_{k=2}^{\infty} d_k = \frac{1}{k^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{d_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2(\ln n)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{(\ln n)^2} = 0.$$

So by the result of Problem 60, since the p-series $\sum\limits_{k=2}^{\infty} d_k$ converges (since $p = 2 > 1$), the original series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{(k\ln k)^2}$ also $\boxed{\textit{converges.}}$

65. If $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{\ln k}$ and $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{e^k}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{e^n}} = \lim_{n\to\infty} \frac{e^n}{\ln n} = \lim_{x\to\infty} \frac{e^x}{\ln x} = \lim_{x\to\infty} \frac{e^x}{\frac{1}{x}} = \lim_{x\to\infty} xe^x = \infty,$$

by using L'Hôpital's rule on a related function of the ratio of the nth terms. Because the limit in question is not a finite positive number, but infinity, the Limit Comparison test is inconclusive in regards to whether the series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{\ln k}$ converges or diverges, even though the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{e^k}$ converges, as it is a geometric series with $|r| = \frac{1}{e} < 1$.

66. (a) Let $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\ln k}{k^p}$ and $\sum\limits_{k=1}^{\infty} d_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^q}$, where $1 < q < 2$ and $p > 1$. Then we have

$$\lim_{n\to\infty} \frac{a_n}{d_n} = \lim_{n\to\infty} \frac{\frac{\ln n}{n^p}}{\frac{1}{n^q}} = \lim_{n\to\infty} \frac{\ln n}{n^{p-q}} = \lim_{x\to\infty} \frac{\ln x}{x^{p-q}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{(p-q)x^{p-q-1}} = \frac{1}{p-q} \lim_{x\to\infty} \frac{1}{x^{p-q}} = 0,$$

by L'Hôpital's rule, and since from $p > 1$, and $1 < q < 2$, we have $p - q > 0$, since $p > 1$ implies $p \geq 2 > q$. So by the result of Problem 60, since the p-series $\sum\limits_{k=1}^{\infty} d_k$ converges (for $q > 1$), the original series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\ln k}{k^p}$ also $\boxed{\textit{converges.}}$

(b) Let $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(\ln k)^r}{k^p}$ and $\sum\limits_{k=1}^{\infty} d_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^q}$, where $1 < q < 2$, $p > 1$, and $r > 0$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{d_n} &= \lim_{n\to\infty} \frac{\frac{(\ln n)^r}{n^p}}{\frac{1}{n^q}} = \lim_{n\to\infty} \frac{(\ln n)^r}{n^{p-q}} = \lim_{x\to\infty} \frac{(\ln x)^r}{x^{p-q}} \\
&= \lim_{x\to\infty} \frac{r(\ln x)^{r-1}\cdot\frac{1}{x}}{(p-q)x^{p-q-1}} = \frac{r}{p-q} \lim_{x\to\infty} \frac{(\ln x)^{r-1}}{x^{p-q}} = \cdots \\
&= \frac{r!}{(p-q)^r} \lim_{x\to\infty} \frac{1}{x^{p-q}} = 0
\end{aligned}$$

by repeatedly applying L'Hôpital's rule, and since from $p > 1$, and $1 < q < 2$, we have $p - q > 0$, since $p > 1$ implies $p \geq 2 > q$. So by the result of Problem 60, since the p-series $\sum\limits_{k=1}^{\infty} d_k$ converges (for $q > 1$), the original series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(\ln k)^r}{k^p}$ also $\boxed{\textit{converges.}}$

67. (a) Let $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\ln k}{k^p}$ and $\sum\limits_{k=1}^{\infty} d_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^p}$, where $0 < p \leq 1$. We have

$$\lim_{n\to\infty} \frac{a_n}{d_n} = \lim_{n\to\infty} \frac{\frac{\ln n}{n^p}}{\frac{1}{n^p}} = \lim_{n\to\infty} \ln n = \infty.$$

By the results of exercise 59, since $\sum\limits_{k=1}^{\infty} d_k$ is a divergent p-series (since $0 < p \leq 1$), it follows that $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\ln k}{k^p}$ also diverges, as was needed to be shown.

(b) Let $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(\ln k)^r}{k^p}$ and $\sum\limits_{k=1}^{\infty} d_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^p}$, where $0 < p \leq 1$ and $r > 0$. We have

$$\lim_{n\to\infty} \frac{a_n}{d_n} = \lim_{n\to\infty} \frac{\frac{(\ln n)^r}{n^p}}{\frac{1}{n^p}} = \lim_{n\to\infty} (\ln n)^r = \infty,$$

since r is given to be a positive real number. By the results of exercise 59, since $\sum\limits_{k=1}^{\infty} d_k$ is a divergent p-series (since $0 < p \leq 1$), it follows that $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(\ln k)^r}{k^p}$ also diverges, as was needed to be shown.

68. The series $\sum\limits_{k=1}^{\infty} \frac{\ln k}{k}$ $\boxed{\textit{diverges}}$ since it is of the form $\sum\limits_{k=1}^{\infty} \frac{\ln k}{k^p}$, $p = 1$, which diverges by the results of Problem 67, part (a).

69. The series $\sum\limits_{k=1}^{\infty} \frac{\sqrt{\ln k}}{\sqrt{k}}$ $\boxed{\textit{diverges}}$ since it is of the form $\sum\limits_{k=1}^{\infty} \frac{(\ln k)^r}{k^p}$, with $0 < p = \frac{1}{2} \leq 1$, and $r = \frac{1}{2} > 0$, which converges by the results of Problem 67, part (b).

70. The series $\sum\limits_{k=2}^{\infty} \frac{\ln k}{k^3}$ $\boxed{\textit{converges}}$ since it is of the form $\sum\limits_{k=2}^{\infty} \frac{\ln k}{k^p}$, where $p = 3 > 1$, which converges by the results of Problem 66, part (a).

71. Since we can write $\sum\limits_{k=1}^{\infty} \frac{(\ln k)^2}{\sqrt{k^3}} = \sum\limits_{k=1}^{\infty} \frac{(\ln k)^2}{k^{3/2}}$, then the series is in the form $\displaystyle\sum_{k=1}^{\infty} \frac{(\ln k)^r}{k^p}$ with $r = 2 > 0$ and $p = \frac{3}{2} > 1$. By Exercise 66(b), the series $\boxed{\text{converges}}$.

72. (a) We first show that the p-series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p = 1$ using the Integral Test. Let $f(x) = \frac{1}{x^p}$ be a related function of the nth term of the series. $f(x)$ is continuous and decreasing on $[1, \infty)$, and $f(k) = a_k$ for all $k \geq 1$. To apply the Integral Test, we evaluate

$$\begin{aligned} I = \int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx &= \lim_{b\to\infty} \left[\frac{x^{-p-1}}{-p-1}\right]_1^b \\ &= -\frac{1}{p+1} \lim_{b\to\infty} \left[b^{-p-1} - 1^{-p-1}\right] = -\frac{1}{p+1}[0-1] = \frac{1}{p+1}. \end{aligned}$$

Since the limit is a finite real number, the improper integral converges, so by the Integral Test, the series converges for $p > 1$. At $p = 1$, the integral in question evaluates to

$$I = \int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} [\ln x]_1^b = \lim_{b\to\infty} [\ln b - \ln 1] = \infty,$$

so by the Integral Test, since the improper integral diverges, so does the series for $p = 1$.
(b) For $0 < p < 1$, $n^p < n$ for $n \geq 1$, or

$$a_n = \frac{1}{n^p} < \frac{1}{n} = b_n$$

for $n \geq 1$. Since we have shown in part (a) that the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges, by the Comparison Test, so does the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^p}$ for $0 < p < 1$.
If $p = 0$, the series becomes $\sum_{k=1}^{\infty} 1$. Since $\lim_{n\to\infty} a_n = 1 \neq 0$, the series diverges by the Divergence Test.
Finally, let $p < 0$. Then $p = -q$, with $q > 0$. Since

$$a_n = \frac{1}{n^p} = n^q \geq 1$$

for $n \geq 1$, and the series $\sum_{k=1}^{\infty} 1$ diverges (see above), by the Comparison Test, the series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^p}$ is divergent for $p < 0$.

73. If the series $\sum_{k=1}^{\infty} a_k$ converges, then by the Comparison Test the series $\sum_{k=1}^{\infty} \frac{a_k}{k}$ also converges since the nth terms satisfy $\frac{a_n}{n} < a_n$ for $n > 1$, provided that a_n are all positive (and they are given to be positive).

74. The series $\sum_{k=1}^{\infty} \frac{1}{1+2^k}$ is convergent as can be seen by comparing it with the (convergent) geometric series $\sum_{k=1}^{\infty} \frac{1}{2^k}$. The nth terms satisfy $\frac{1}{1+2^n} < \frac{1}{2^n}$ for all $n \geq 1$, so by the Comparison Test, the original series also converges.

75. We are given that the harmonic series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges. Choose $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ which is a convergent p-series (since $p = 2 > 1$). Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

On the other hand, choose $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k\ln(k+1)}$. Since $\frac{1}{n\ln(n+1)} > \frac{1}{(n+1)\ln(n+1)}$, it suffices to show that $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{1}{(k+1)\ln(k+1)}$ is divergent, for then the divergence of $\sum_{k=1}^{\infty} a_k$ would follow from the Comparison Test. That $\sum_{k=1}^{\infty} c_k$ is a divergent series is seen by using Integral Test. In fact, this was done in Problem 27, of Section 8.3, where it is shown that $\sum\limits_{k=2}^{\infty} \frac{1}{k \ln k} = \sum\limits_{k=1}^{\infty} \frac{1}{(k+1)\ln(k+1)}$ is divergent. Now we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n\ln(n+1)}}{\frac{1}{n}} = \frac{1}{\ln(n+1)} = 0.$$

This shows that the Limit Comparison test gives no guidance as to the convergence or divergence of a given $\sum\limits_{k=1}^{\infty} a_k$ series if $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$.

Challenge Problems

76. The nth term of the series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{\ln(2k+1)}{\sqrt{k^2 - 2\sqrt{k^3 - 2k - 3}}}$ behaves like

$$a_n = \frac{\ln(2n+1)}{\sqrt{n^2 - 2\sqrt{n^3 - 2n - 3}}} = \frac{\ln(2n+1)}{n \cdot n^{3/2}\sqrt{1 - \frac{2}{n^2}}\sqrt{1 - \frac{2}{n^2} - \frac{3}{n^3}}} \approx \frac{\ln(2n+1)}{n^{5/2}}$$

for large n. We compare with the series $\sum\limits_{k=2}^{\infty} b_k = \sum\limits_{k=2}^{\infty} \frac{\ln(2k+1)}{k^{5/2}}$. To see that this series converges, examine the nth term

$$b_n = \frac{\ln(2n+1)}{n^{5/2}} \le \frac{\ln(2n+n)}{n^{5/2}} = \frac{\ln(3n)}{n^{5/2}} = \frac{\ln 3 + \ln n}{n^{5/2}} = \frac{\ln 3}{n^{5/2}} + \frac{\ln n}{n^{5/2}} = c_n + d_n$$

for $n \ge 2$. The series $\sum\limits_{k=2}^{\infty} c_k = \sum\limits_{k=2}^{\infty} \frac{\ln 3}{k^{5/2}}$ is a constant multiple of a convergent p-series (since $p = \frac{5}{2} > 1$), while the series $\sum\limits_{k=2}^{\infty} d_k = \sum\limits_{k=2}^{\infty} \frac{\ln k}{k^{5/2}}$ converges by the result of Problem 66. (In the terminology of that problem, $p = \frac{5}{2} > 1$.) So we have shown that the series $\sum\limits_{k=2}^{\infty} b_k$ converges. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\ln(2n+1)}{\sqrt{n^2-2\sqrt{n^3-2n-3}}}}{\frac{\ln(2n+1)}{n^{5/2}}} = \lim_{n\to\infty} \frac{n^{5/2}}{\sqrt{n^2 - 2\sqrt{n^3 - 2n - 3}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1 - \frac{2}{n^2}}\sqrt{1 - \frac{2}{n^2} - \frac{3}{n^3}}} = 1.$$

Since the limit is a positive number and the series $\sum\limits_{k=2}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{\ln(2k+1)}{\sqrt{k^2 - 2\sqrt{k^3 - 2k - 3}}}$ also $\boxed{\textit{converges.}}$

77. The nth term of the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{(k^3 - k + 1)\ln(2k+1)}}$ behaves like

$$a_n = \frac{\sqrt{n}}{\sqrt{(n^3 - n + 1)\ln(2n+1)}} = \frac{\sqrt{n}}{n^{3/2}\sqrt{1 - \frac{1}{n^2} + \frac{1}{n^3}}} \cdot \frac{1}{\sqrt{\ln(2n+1)}} \approx \frac{1}{n} \cdot \frac{1}{\sqrt{\ln(2n+1)}}$$

for large n. We compare with $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k\sqrt{\ln(2k+1)}}$. This series diverges, and this is seen as follows: Let $\sum\limits_{k=1}^{\infty} d_k = \sum\limits_{k=1}^{\infty} \frac{1}{k\ln(2k+1)}$. This series diverges by comparing with the series

$\sum\limits_{k=2}^{\infty} c_k = \sum\limits_{k=2}^{\infty} \frac{1}{k \ln k}$ (which itself diverges by the Integral Test since $\lim\limits_{b\to\infty} \int_2^b \frac{dx}{x \ln x} = \lim\limits_{b\to\infty} \ln(\ln b) - \ln(\ln 2) = \infty$) and applying the Limit Comparison Test: $\lim\limits_{n\to\infty} \frac{d_n}{c_n} = \lim\limits_{n\to\infty} \frac{n \ln n}{n \ln(2n+1)} = \lim\limits_{x\to\infty} \frac{\ln x}{\ln(2x+1)} = \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2x+1}} = \lim\limits_{x\to\infty} \left(2 + \frac{1}{x}\right) = 2$. Compare now the nth terms of the series $\sum\limits_{k=1}^{\infty} b_k$ and $\sum\limits_{k=1}^{\infty} d_k$:

$$\lim_{n\to\infty} \frac{b_n}{d_n} = \lim_{n\to\infty} \frac{\frac{1}{n} \frac{1}{\sqrt{\ln(2n+1)}}}{\frac{1}{n} \frac{1}{\ln(2n+1)}} = \lim_{n\to\infty} \sqrt{\ln(2n+1)} = \infty.$$

So, by the result of Problem 59, the series $\sum\limits_{k=1}^{\infty} b_k$ indeed diverges. Now, finally, we compare the series $\sum\limits_{k=1}^{\infty} a_k$ with the series $\sum\limits_{k=1}^{\infty} b_k$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n\sqrt{1-\frac{1}{n^2}+\frac{1}{n^3}}} \cdot \frac{1}{\sqrt{\ln(2n+1)}}}{\frac{1}{n} \cdot \frac{1}{\sqrt{\ln(2n+1)}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1 - \frac{1}{n^2} + \frac{1}{n^3}}} = 1.$$

Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{(k^3-k+1)\ln(2k+1)}}$ also $\boxed{\textit{diverges.}}$

78. The nth term of the series $\sum\limits_{k=1}^{\infty} \frac{1+\sin k}{4^k}$ satisfies $0 < \frac{1+\sin n}{4^n} \le \frac{2}{4^n} < \frac{3}{4^n}$, since $\sin n > 0$ for $n \ge 1$. So, comparing with the convergent geometric series $\sum\limits_{k=1}^{\infty} \frac{3}{4}\left(\frac{1}{4}\right)^{k-1}$ we see by the Comparison Test that the given series also $\boxed{\textit{converges.}}$

79. We compare the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^{1+1/k}}$ with the divergent harmonic series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k}$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^{1+1/n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n^{1+1/n}} = \lim_{n\to\infty} \frac{n}{n \cdot n^{1/n}} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = \frac{1}{\lim\limits_{n\to\infty} n^{1/n}}.$$

To evaluate $\lim\limits_{n\to\infty} n^{1/n}$, let $y = n^{1/n}$. Then $\ln y = \frac{1}{n} \ln n = \frac{\ln n}{n}$. We have

$$\lim_{n\to\infty} y = \lim_{n\to\infty} e^{\ln y} = \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^{\lim\limits_{n\to\infty} \frac{\ln n}{n}} = e^{\lim\limits_{x\to\infty} \frac{\ln x}{x}} = e^{\lim\limits_{x\to\infty} \frac{1/x}{1}} = e^{\lim\limits_{x\to\infty} \frac{1}{x}} = e^0 = 1,$$

using L'Hôpital's rule on a related function of $\frac{\ln n}{n}$. Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ diverges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^{1+1/k}}$ also $\boxed{\textit{diverges.}}$

80. (a) For large n, the nth term of the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k^2-3k-2}{k^2(k+1)^2}$ behaves like

$$a_n = \frac{n^2 - 3n - 2}{n^2(n+1)^2} = \frac{n^2\left(1 - \frac{3}{n} - \frac{2}{n^2}\right)}{n^2 \cdot n^2\left(1 + \frac{1}{n}\right)^2} \approx \frac{1}{n^2}.$$

So we compare with the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$, which is a convergent p-series (since $p = 2 > 1$). We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\left(1-\frac{3}{n}-\frac{2}{n^2}\right)}{n^2\left(1+\frac{1}{n}\right)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\left(1-\frac{3}{n}-\frac{2}{n^2}\right)}{\left(1+\frac{1}{n}\right)^2} = 1.$$

Since the limit is a positive number and the series $\sum\limits_{k=1}^{\infty} b_k$ converges, by the Limit Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k^2-3k-2}{k^2(k+1)^2}$ also $\boxed{\textit{converges.}}$

(b) We use a partial fraction decomposition on the kth term of the series $\sum\limits_{k=1}^{\infty} a_k$ as follows:

$$\frac{k^2-3k-2}{k^2(k+1)^2} = \frac{A}{k^2} + \frac{B}{(k+1)^2} + \frac{C}{k} + \frac{D}{(k+1)}.$$

Comparing coefficients after taking the common denominator on the right hand side, we obtain $A = -2$; $B = 2$; $C = 1$; $D = -1$. So,

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{k^2-3k-2}{k^2(k+1)^2} &= \sum_{k=1}^{\infty} -\frac{2}{k^2} + \sum_{k=1}^{\infty} \frac{2}{(k+1)^2} + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{(k+1)}\right) \\ &= -2 \cdot \frac{\pi^2}{6} + 2\left(\frac{\pi^2}{6} - 1\right) + \lim_{n\to\infty} \left[\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)\right] \\ &= -2 + \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = -2 + 1 = -1,\end{aligned}$$

where we have used the result $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ (see p.572), and the fact that $\sum\limits_{k=1}^{\infty} \frac{1}{(k+1)^2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum\limits_{k=1}^{\infty} \frac{1}{k^2} - 1$. So the required sum of the series is $\boxed{-1.}$

AP® Practice Problems

1. Analyze each series to determine if it converges:

I. $$\sum_{k=1}^{\infty} \frac{2\pi^k}{3^k \pi} = \frac{2}{\pi} \sum_{k=1}^{\infty} \left(\frac{\pi}{3}\right)^k$$

This is a geometric series that $\boxed{\text{diverges}}$, since $r = \frac{\pi}{3} > 1$.

II. The convergence of the series $\sum\limits_{k=1}^{\infty} \frac{k^2}{2k^3+1}$ can be evaluated using the Integral Test as follows:

A related function to $\frac{k^2}{2k^3+1}$ is $f(x) = \frac{x^2}{2x^3+1}$.

$$f'(x) = \frac{2x\left(2x^3+1\right) - 6x^2 \cdot x^2}{\left(2x^3+1\right)^2} = \frac{4x^4 + 2x - 6x^4}{(2x^3+1)^2} = \frac{-2x^4 + 2x}{\left(2x^3+1\right)^2}$$

$f(x)$ is positive for $x \geq 1$.

$f(x)$ is differentiable for $x \geq 1$ since $f'(x)$, as determined above, is defined for $x \geq 1$.
$f(x)$ is continuous for $x \geq 1$ since a function is continuous where the function is differentiable.
$f(x)$ is decreasing for $x \geq 1$ since $f'(x) < 0$ for $x \geq 1$.
Consider the Improper Integral

$$\int_1^\infty \frac{x^2}{2x^3+1}\,dx = \lim_{b\to\infty}\int_1^b \left(\frac{x^2}{2x^3+1}\,dx\right) = \lim_{b\to\infty}\left[\frac{1}{6}\ln\left|2x^3+1\right|\right]_1^b$$
$$= \frac{1}{6}\lim_{b\to\infty}\left[\ln\left|2b^3+1\right| - \ln 3\right] = \infty$$

Since the improper integral $\int_1^\infty \frac{x^2}{2x^3+1}dx$ diverges, by the Integral Test the series $\sum\limits_{k=1}^{\infty} \frac{k^2}{2k^3+1}$ also $\boxed{\text{diverges}}$.

III. $$\sum_{k=1}^{\infty}\frac{k^2+3\sqrt[3]{k}}{2k^5} = \sum_{k=1}^{\infty}\left(\frac{k^2}{2k^5} + \frac{3k^{\frac{1}{3}}}{2k^5}\right) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^3} + \frac{3}{2}\sum_{k=1}^{\infty}\frac{1}{k^{\frac{14}{3}}}$$

Each of the individual series is a $\boxed{\text{convergent}}$ p-series since, for each, $p > 0$.

CHOICE B

2. Analyze each series to determine if it converges:

I. $$\sum_{k=1}^{\infty}\frac{7k-5}{k^3} = \sum_{k=1}^{\infty}\left(\frac{7k}{k^3} - \frac{5}{k^3}\right) = 7\sum_{k=1}^{\infty}\frac{1}{k^2} - 5\sum_{k=1}^{\infty}\frac{1}{k^3}$$

Each of the series is a $\boxed{\text{convergent}}$ p-series.

II. The convergence of the series $\sum\limits_{k=1}^{\infty} \frac{k^2}{2k^3+1}$ can be evaluated using the Integral Test as follows:
A related function to $\frac{k^2}{2k^3+1}$ is $f(x) = \frac{x^2}{2x^3+1}$.

$$f'(x) = \frac{2x\left(2x^3+1\right) - 6x^2\cdot x^2}{\left(2x^3+1\right)^2} = \frac{4x^4+2x-6x^4}{(2x^3+1)^2} = \frac{-2x^4+2x}{\left(2x^3+1\right)^2}$$

$f(x)$ is positive for $x \geq 1$.
$f(x)$ is differentiable for $x \geq 1$ since $f'(x)$, as determined above, is defined for $x \geq 1$.
$f(x)$ is continuous for $x \geq 1$ since a function is continuous where the function is differentiable.
$f(x)$ is decreasing for $x \geq 1$ since $f'(x) < 0$ for $x \geq 1$.
Consider the Improper Integral

$$\int_1^\infty \frac{x^2}{2x^3+1}\,dx = \lim_{b\to\infty}\int_1^b \left(\frac{x^2}{2x^3+1}\,dx\right) = \lim_{b\to\infty}\left[\frac{1}{6}\ln\left|2x^3+1\right|\right]_1^b$$
$$= \frac{1}{6}\lim_{b\to\infty}\left[\ln\left|2b^3+1\right| - \ln 3\right] = \infty$$

Since the improper integral $\int_1^\infty \frac{x^2}{2x^3+1}dx$ diverges, by the Integral Test the series $\sum\limits_{k=1}^{\infty} \frac{k^2}{2k^3+1}$ also $\boxed{\text{diverges}}$.

III. $\displaystyle\sum_{k=1}^{\infty} \frac{k+3}{(k-3)^2+1} = \sum_{k=1}^{\infty} \frac{k+3}{k^2-6k+10}$

We choose an appropriate p-series to use for comparison by examining the behavior of the series for large values of n.

$$\frac{n+3}{n^2-6n+10} \cdot \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \frac{1+\frac{3}{n}}{n-6+\frac{10}{n}} \approx \frac{1}{n} \text{ for large values of } n.$$

This leads us to choose the p-series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$, which diverges, and use the Limit Comparison Test with $a_n = \frac{n+3}{n^2-6n+10}$ and $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+3}{n^2-6n+10}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+3n}{n^2-6n+10} = \lim_{n\to\infty} \frac{n^2+3n}{n^2-6n+10} \cdot \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)}$$
$$= \lim_{n\to\infty} \frac{1+\frac{3}{n}}{1-\frac{6}{n}+\frac{10}{n^2}} = 1$$

Since the limit is a positive number and the p-series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges, then by the Limit Comparison Test $\sum\limits_{k=1}^{\infty} \frac{k+3}{k^2-6k+10} = \sum\limits_{k=1}^{\infty} \frac{k+3}{(k-3)^2+1}$ also $\boxed{\text{diverges}}$.

CHOICE D

3. $\displaystyle\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2+2k+1} = \sum_{k=1}^{\infty} \frac{\sin^2 k}{(k+1)^2}$

The direct comparison test provides that for $0 < a_n < b_n$

If $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

If $\sum\limits_{n=1}^{\infty} a_n$ diverges, then $\sum\limits_{n=1}^{\infty} b_n$ diverges.

Here, since $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series and $0 < \frac{1}{(k+1)^2} \le \frac{1}{k^2}$, $\sum\limits_{k=1}^{\infty} \frac{1}{(k+1)^2}$ converges by the Direct Comparison Test.

Then, since $\sum\limits_{k=1}^{\infty} \frac{1}{(k+1)^2}$ converges and $0 < \frac{\sin^2 k}{(k+1)^2} \le \frac{1}{(k+1)^2}$, $\sum\limits_{k=1}^{\infty} \frac{\sin^2 k}{(k+1)^2}$ $\boxed{\text{converges}}$ by the Direct Comparison Test.

4. Use the Limit Comparison Test and compare the series $\sum\limits_{k=1}^{\infty} \frac{2k^2-1}{k(k^2+3)}$ to the harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$, which diverges.

Then, with $a_n = \frac{2n^2-1}{n(n^2+3)}$ and $b_n = \frac{1}{n}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n^2-1}{n(n^2+3)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^2-1}{n^2+3} = \lim_{n\to\infty} \frac{2n^2-1}{n^2+3} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2-\frac{1}{n^2}}{1+\frac{3}{n^2}} = 2$$

Since the limit is a positive number, and $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges, by the Limit Comparison Test $\sum\limits_{k=1}^{\infty} \frac{2k^2-1}{k(k^2+3)}$ also $\boxed{\text{diverges}}$.

8.5 Alternating Series; Absolute Convergence

Concepts and Vocabulary

1. False: For odd k, $(-1)^k = -1$ and $\cos k\pi = -1$. So $(-1)^k \cos(k\pi) = (-1)(-1) = +1$. For even k, $(-1)^k = +1$ and $\cos k\pi = +1$. So $(-1)^k \cos(k\pi) = (+1)(+1) = +1$. This means that the terms of the series are all positive for every k and so the series is not an alternating series.

2. False: For odd k, $(-1)^k = -1$, so $\left[1 + (-1)^k\right] = 1 - 1 = 0$. For even k, $(-1)^k = +1$, so $\left[1 + (-1)^k\right] = 1 + 1 = 2$. So the terms of the series are alternately either 0 or positive, but that does not make it an alternating series, which by definition alternates *positive and negative* terms.

3. False: It is not sufficient that $\lim\limits_{n\to\infty} a_n = 0$ for the alternating series $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k$ to be convergent. We also need to require that the a_k are nonincreasing, that is $a_k \geq a_{k+1}$ for all k.

4. False: The correct conclusion under the stated conditions is that the error in using the sum of n terms S_n in order to approximate the sum S of the series is given by $|E_n| \leq a_{n+1}$ and not $|E_n| \leq a_n$.

5. False: It is possible for an alternating series to be convergent even if it is not absolutely convergent. For example, the alternating harmonic series is convergent but not absolutely convergent (as it would then become a divergent harmonic series).

6. True: This follows from the statement of the theorem on p. 684.

Skill Building

7. The series is $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k = \sum\limits_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2}$. We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n^2} = 0$. By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} < 1, \text{ for } n \geq 1.$$

So the a_n are nonincreasing. By the Alternating Series Test, the series $\boxed{\textit{converges.}}$

8. The series is $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k = \sum\limits_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2\sqrt{k}}$. We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{2\sqrt{n}} = 0$. By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2\sqrt{n+1}}}{\frac{1}{2\sqrt{n}}} = \frac{\sqrt{n}}{\sqrt{n+1}} < 1, \text{ for } n \geq 1.$$

So the a_n are nonincreasing. By the Alternating Series Test, the series $\boxed{\textit{converges.}}$

9. The series is $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k = \sum\limits_{k=1}^{\infty} (-1)^{k+1} \frac{k}{2k+1}$. We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}} = \frac{1}{2} \neq 0$. Since the limit is nonzero, by the Divergence Test, the series will $\boxed{\textit{diverge.}}$

10. The series is $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+1}{k}$. We have $\lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \neq 0$. Since the limit is nonzero, by the Divergence Test, the series $\boxed{\textit{diverges.}}$

11. The series is $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{k^2}{5k^2+2}$. We have $\lim_{n\to\infty}\frac{n^2}{5n^2+2} = \lim_{n\to\infty}\frac{1}{5+\frac{2}{n^2}} = \frac{1}{5} \neq 0$. Since the limit is nonzero, by the Divergence Test, the series will $\boxed{\textit{diverge.}}$

12. The series is $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+1}{k^2}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n+1}{n^2} = \lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n^2}\right) = 0+0 = 0.$$

By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+2}{(n+1)^2}}{\frac{n+1}{n^2}} = \lim_{n\to\infty}\frac{(n+2)n^2}{(n+1)^3} = \frac{n^3+2n^2}{n^3+3n^2+3n+1} < 1 \text{ for } n \geq 1.$$

So the a_n are nonincreasing. By the Alternating Series Test, the series $\boxed{\textit{converges.}}$

13. The series is $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k+1)2^k}$. We have $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{(n+1)2^n} = 0$. By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+2)2^{n+1}}}{\frac{1}{(n+1)2^n}} = \frac{1}{2}\frac{n+1}{n+2} < 1, \text{ for } n \geq 1.$$

So the a_n are nonincreasing. By the Alternating Series Test, the series $\boxed{\textit{converges.}}$

14. The series is $\sum_{k=2}^{\infty}(-1)^{k+1}a_k = \sum_{k=2}^{\infty}(-1)^{k+1}\frac{1}{k\ln k}$. We have $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n\ln n} = 0$. By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)\ln(n+1)}}{\frac{1}{n\ln n}} = \frac{n}{n+1}\cdot\frac{\ln n}{\ln(n+1)} < 1, \text{ for } n \geq 2,$$

since both n and $\ln n$ are monotonically increasing functions of n for $n \geq 2$. So the terms of the series are nonincreasing. By the Alternating Series Test, the series $\boxed{\textit{converges.}}$

15. The series is $\sum_{k=2}^{\infty}(-1)^{k+1}a_k = \sum_{k=2}^{\infty}(-1)^{k+1}\frac{1}{1+2^{-k}}$. We have $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{1+2^{-n}} = \frac{1}{1+0} = 1 \neq 0$. Since the limit is nonzero, by the Divergence Test, the series $\boxed{\textit{diverges.}}$

16. The series is $\sum_{k=0}^{\infty}(-1)^{k}a_k = \sum_{k=0}^{\infty}(-1)^{k}\frac{1}{k!}$. We have $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n!} = 0$. By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1} \leq 1, \text{ for } n \geq 0.$$

So the a_n are nonincreasing. By the Alternating Series Test, the series $\boxed{converges.}$

17. The series is $\sum\limits_{k=1}^{\infty}(-1)^{k+1}a_k = \sum\limits_{k=1}^{\infty}(-1)^{k+1}\left(\frac{k}{k+1}\right)^k$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \neq 0,$$

using the standard limit definition of e. Since the limit is nonzero, by the Divergence Test, the series $\boxed{diverges.}$

18. The series is $\sum\limits_{k=1}^{\infty}(-1)^{k+1}a_k = \sum\limits_{k=1}^{\infty}(-1)^{k+1}\frac{k^2}{(k+1)^3}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^2}{(n+1)^3} = \lim_{n\to\infty}\frac{n^2}{n^3\left(1+\frac{1}{n}\right)^3} = \lim_{n\to\infty}\frac{1}{n\left(1+\frac{1}{n}\right)^3} = 0.$$

Let $f(x) = \frac{x^2}{(x+1)^3}$ be a related function of the nth term of the series. Examine the derivative:

$$f'(x) = \frac{(x+1)^3\cdot 2x - x^2\cdot 3(x+1)^2}{(x+1)^6} = \frac{(x+1)^2 x}{(x+1)^6}[2(x+1)-3x] = \frac{(x+1)^2 x(2-x)}{(x+1)^6}.$$

So for $x > 2$, $f'(x) < 0$, which means that for $n \geq 2$, the a_n are nonincreasing. By the Alternating Series Test, the series $\boxed{converges.}$

19. To use the Alternating Series test, note that $a_k = e^{-k} = \frac{1}{e^k} > 0$ for all k.
We find $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} e^{-n} = \lim\limits_{n\to\infty}\frac{1}{e^n} = 0$.
On the interval $[1,\infty)$, the function $f(x) = e^{-x}$ is continuous, positive, and decreasing (f is decreasing because $f'(x) = -e^{-x} < 0$ for $x > 1$). Thus the a_k are nonincreasing, and the alternating series $\sum\limits_{k=1}^{\infty}(-1)^k e^{-k}$ $\boxed{\text{converges}}$.

20. To use the Alternating Series test, note that $a_k = ke^{-k} = \frac{k}{e^k} > 0$ for all k.
We find $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} ne^{-n} = \lim\limits_{n\to\infty}\frac{n}{e^n}$, which is in indeterminate form $\frac{\infty}{\infty}$.
We use L'Hôpital's Rule: $\lim\limits_{n\to\infty}\frac{n}{e^n} = \lim\limits_{n\to\infty}\frac{1}{e^n} = 0$.
On the interval $[1,\infty)$, the function $f(x) = xe^{-x}$ is continuous, positive, and decreasing (f is decreasing because $f'(x) = \frac{1-x}{e^x} < 0$ for $x > 1$). Thus the a_k are nonincreasing, and the alternating series $\sum\limits_{k=1}^{\infty}(-1)^k ke^{-k}$ $\boxed{\text{converges}}$.

21. To use the Alternating Series test, note that $a_k = \tan^{-1}k > 0$ for all k. Next we have $\lim\limits_{n\to\infty}\tan^{-1}n = \frac{\pi}{2}$. By the Test for Divergence, the alternating series $\sum\limits_{k=1}^{\infty}(-1)^k\tan^{-1}k$ $\boxed{\text{diverges}}$.

22. To use the Alternating Series test, note that $a_k = e^{2/k} > 0$ for all k. Next we have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} e^{2/n} = 1$. By the Test for Divergence, the alternating series $\sum\limits_{k=1}^{\infty} (-1)^k e^{2/k}$ $\boxed{\text{diverges}}$.

23. **(a)** $a_k = \frac{1}{k^5} > 0$ for all $k \geq 1$, and $\lim\limits_{n\to\infty} \frac{1}{n^5} = 0$.
$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^5}}{\frac{1}{n^5}} = \frac{n^5}{(n+1)^5} = \left(\frac{n}{n+1}\right)^5 < 1$ for all $n > 1$, so the terms are nonincreasing. (In fact, they are decreasing.)

(b) For a convergent alternating series meeting the conditions above in Part (a), the error E_n in using the sum S_n of the first n terms as an approximation to the sum S is numerically less than or equal to the $(n+1)$st term of the series.
For $a_n = \frac{1}{n^5}$, we must determine n for which

$$\begin{aligned}\frac{1}{(n+1)^5} &\leq 0.001\\ (n+1)^5 &\geq 1000\\ n &\geq -1 + \sqrt[5]{1000} \approx 2.98\end{aligned}$$

So $\boxed{n = 3}$.

(c) As determined above, we will evaluate

$$\begin{aligned}\sum_{k=1}^{3} \frac{(-1)^{k+1}}{k^5} &= \frac{1}{1} - \frac{1}{2^5} + \frac{1}{3^5}\\ &= 1 - \frac{1}{32} + \frac{1}{243} = \boxed{\frac{7565}{7776} \approx 0.973}\end{aligned}$$

24. **(a)** $\lim\limits_{n\to\infty} \frac{1}{n^4} = 0$.
$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^4}}{\frac{1}{n^4}} = \frac{n^4}{(n+1)^4} = \left(\frac{n}{n+1}\right)^4 < 1$ for all $n > 1$, so the terms are nonincreasing.

(In fact, they are decreasing.)

(b) For a convergent alternating series meeting the conditions above in Part (a), the error E_n in using the sum S_n of the first n terms as an approximation to the sum S is numerically less than or equal to the $(n+1)$st term of the series.
For $a_n = \frac{1}{n^4}$, we must determine n for which

$$\begin{aligned}\frac{1}{(n+1)^4} &\leq 0.001\\ (n+1)^4 &\geq 1000\\ n &\geq -1 + \sqrt[4]{1000} \approx 4.62\end{aligned}$$

So $\boxed{n = 5}$.

(c) As determined above, we will evaluate

$$\begin{aligned}\sum_{k=1}^{5} \frac{(-1)^{k+1}}{k^5} &= \frac{1}{1} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4}\\ &= 1 - \frac{1}{64} + \frac{1}{81} - \frac{1}{256} + \frac{1}{625} = \boxed{\frac{12,280,111}{12,960,000} \approx 0.948}\end{aligned}$$

25. **(a)** $a_k = \frac{1}{k!} > 0$ for all $k \geq 1$, and $\lim\limits_{n\to\infty} \frac{1}{n!} = 0$.

$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1} < 1$ for all $n \geq 1$, so the terms are nonincreasing.

(In fact, they are decreasing.)

(b) For a convergent alternating series meeting the conditions above in Part (a), the error E_n in using the sum S_n of the first n terms as an approximation to the sum S is numerically less than or equal to the $(n+1)$st term of the series.

For $a_n = \frac{1}{n^4}$, we must determine n for which

$$\frac{1}{(n+1)!} \leq 0.001$$
$$(n+1)! \geq 1000$$

Now $6! = 720$ and $7! = 5040$. Since $7!$ is the smallest value great than 1000, therefore $n+1 = 7$, or $\boxed{n = 6}$.

(c) As determined above, we will evaluate

$$\sum_{k=1}^{6} \frac{(-1)^{k+1}}{k!} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!}$$
$$= 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \boxed{\frac{91}{144} \approx 0.632}$$

26. **(a)** $a_k = \frac{1}{k^k} > 0$ for all $k \geq 1$, and $\lim\limits_{n\to\infty} \frac{1}{n^n} = 0$.

$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^{n+1}}}{\frac{1}{n^n}} = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)(n+1)^n} = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{1}{n+1}\left(\frac{1}{1+\frac{1}{n}}\right)^n < 1$ for all $n \geq 1$, so the terms are nonincreasing. (In fact, they are decreasing.)

(b) For a convergent alternating series meeting the conditions above in Part (a), the error E_n in using the sum S_n of the first n terms as an approximation to the sum S is numerically less than or equal to the $(n+1)$st term of the series.

For $a_n = \frac{1}{n^n}$, we must determine n for which

$$\frac{1}{(n+1)^{n+1}} \leq 0.001$$
$$(n+1)^{n+1} \geq 1000$$
$$(n+1)\ln(n+1) \geq \ln 1000$$
$$n \geq 3.56, \text{ using technology}$$

So $\boxed{n = 4}$.

(c) As determined above, we will evaluate

$$\sum_{k=1}^{4} \frac{(-1)^{k+1}}{k^k} = \frac{1}{1^1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4}$$
$$= 1 - \frac{1}{4} + \frac{1}{27} - \frac{1}{256} = \boxed{\frac{5413}{6912} \approx 0.783}$$

27. **(a)** $a_k = \frac{1}{2^k} > 0$ for all $k \geq 1$, and $\lim\limits_{n\to\infty} \frac{1}{2^n} = 0$.

$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \frac{2^n}{2^{n+1}} = \frac{1}{2} \cdot \frac{2^n}{2^n} = \frac{1}{2} < 1$ for all $n \geq 1$, so the terms are nonincreasing.

(In fact, they are decreasing.)

(b) For a convergent alternating series meeting the conditions above in Part (a), the error E_n in using the sum S_n of the first n terms as an approximation to the sum S is numerically less than or equal to the $(n+1)$st term of the series.

For $a_n = \frac{1}{2^n}$, we must determine n for which

$$\begin{aligned} \frac{1}{2^{n+1}} &\leq 0.001 \\ 2^{n+1} &\geq 1000 \\ (n+1)\ln 2 &\geq \ln 1000 \\ n\ln 2 + \ln 2 &\geq \ln 1000 \\ n\ln 2 &\geq \ln 1000 - \ln 2 = \ln\frac{1000}{2} = \ln 500 \\ n &\geq \frac{\ln 500}{\ln 2} \approx 8.97 \end{aligned}$$

So $\boxed{n=9}$.

(c) As determined above, we will evaluate

$$\begin{aligned} \sum_{k=1}^{6} \frac{(-1)^{k+1}}{2^k} &= \frac{1}{2^1} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} - \frac{1}{2^6} + \frac{1}{2^7} - \frac{1}{2^8} + \frac{1}{2^9} \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \frac{1}{128} - \frac{1}{256} + \frac{1}{512} = \boxed{\frac{171}{512} \approx 0.334} \end{aligned}$$

Another method: Note that the sum is

$$\sum_{k=1}^{9} \frac{(-1)^{k+1}}{2^k} = \sum_{k=1}^{9} \frac{(-1)(-1)^k}{2^k} = \sum_{k=1}^{9} (-1)\left(-\frac{1}{2}\right)^k.$$

This is the geometric series $\sum\limits_{k=1}^{n} ar^k$, with $a=-1$, $r=-\frac{1}{2}$, $n=9$, and $|r| = \frac{1}{2} < 1$.

So the sum is $\frac{ar(1-r^n)}{1-r} = \frac{(-1)\left(-\frac{1}{2}\right)\left[1-\left(-\frac{1}{2}\right)^9\right]}{1-\left(-\frac{1}{2}\right)}$, which also evaluates to the same result.

28. **(a)** $a_k = \frac{1}{3^k} > 0$ for all $k \geq 1$, and $\lim\limits_{n\to\infty} \frac{1}{3^n} = 0$.

$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{3^{n+1}}}{\frac{1}{3^n}} = \frac{3^n}{3^{n+1}} = \frac{3^n}{3\cdot 3^n} = \frac{1}{3} < 1$ for all $n \geq 1$, so the terms are nonincreasing.

(In fact, they are decreasing.)

(b) For a convergent alternating series meeting the conditions above in Part (a), the error E_n in using the sum S_n of the first n terms as an approximation to the sum S is numerically less than or equal to the $(n+1)$st term of the series.

For $a_n = \frac{1}{3^n}$, we must determine n for which

$$\begin{aligned} \frac{1}{3^{n+1}} &\leq 0.001 \\ 3^{n+1} &\geq 1000 \\ (n+1)\ln 3 &\geq \ln 1000 \\ n\ln 3 + \ln 3 &\geq \ln 1000 \\ n\ln 3 &\geq \ln 1000 - \ln 3 = \ln\left(\frac{1000}{3}\right) \\ n &\geq \frac{\ln\left(\frac{1000}{3}\right)}{\ln 3} \approx 5.72 \end{aligned}$$

So $\boxed{n=6}$.

(c) As determined above, we will evaluate

$$\begin{aligned}\sum_{k=1}^{6}\frac{(-1)^{k+1}}{3^k} &= \frac{1}{3^1}-\frac{1}{3^2}+\frac{1}{3^3}-\frac{1}{3^4}+\frac{1}{3^5}-\frac{1}{3^6}\\ &= \frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\frac{1}{81}+\frac{1}{243}-\frac{1}{729}=\boxed{\frac{182}{729}\approx 0.250}\end{aligned}$$

Another method: Note that the sum is

$$\sum_{k=1}^{6}\frac{(-1)^{k+1}}{3^k}=\sum_{k=1}^{6}\frac{(-1)(-1)^k}{3^k}=\sum_{k=1}^{6}(-1)\left(-\frac{1}{3}\right)^k.$$

This is the geometric series $\sum\limits_{k=1}^{n} ar^k$, with $a=-1$, $r=-\frac{1}{3}$, $n=6$, and $|r|=\frac{1}{3}<1$.

So the sum is $\frac{ar(1-r^n)}{1-r}=\frac{(-1)\left(-\frac{1}{3}\right)\left[1-\left(-\frac{1}{3}\right)^6\right]}{1-\left(-\frac{1}{3}\right)}$, which also evaluates to the same result.

29. We have $S_3=\sum\limits_{k=1}^{3}(-1)^{k+1}\frac{1}{k^2}=(-1)^2\frac{1}{1^2}+(-1)^3\frac{1}{2^2}+(-1)^4\frac{1}{3^2}=1-\frac{1}{4}+\frac{1}{9}\approx\boxed{0.8611.}$

The upper estimate to the error is $|E_3|\le a_4=\frac{1}{4^2}=\frac{1}{16}=\boxed{0.0625.}$

30. We have $S_2=\sum\limits_{k=0}^{2}(-1)^k\frac{1}{k!}=(-1)^0\frac{1}{0!}+(-1)^1\frac{1}{1!}+(-1)^2\frac{1}{2!}=1-1+\frac{1}{2}=\boxed{0.5000.}$

The upper estimate to the error is $|E_2|\le a_3=\frac{1}{3!}=\frac{1}{6}\approx\boxed{0.1667.}$

31. We have $S_3=\sum\limits_{k=1}^{3}(-1)^{k+1}\frac{1}{k^4}=(-1)^2\frac{1}{1^4}+(-1)^3\frac{1}{2^4}+(-1)^4\frac{1}{3^4}=1-\frac{1}{16}+\frac{1}{81}\approx\boxed{0.9498.}$

The upper estimate to the error is $|E_3|\le a_4=\frac{1}{4^4}=\frac{1}{256}\approx\boxed{0.0039.}$.

32. We have

$S_3=\sum\limits_{k=1}^{3}(-1)^{k+1}\left(\frac{1}{\sqrt{k}}\right)^k=(-1)^2\left(\frac{1}{\sqrt{1}}\right)^1+(-1)^3\left(\frac{1}{\sqrt{2}}\right)^2+(-1)^4\left(\frac{1}{\sqrt{3}}\right)^3=1-\frac{1}{2}+\frac{1}{3\sqrt{3}}\approx\boxed{0.6925.}$

The upper estimate to the error is $|E_3|\le a_4=\left(\frac{1}{\sqrt{4}}\right)^4=\boxed{0.0625.}$

33. We have

$S_2=\sum\limits_{k=0}^{2}(-1)^k\frac{1}{k!}\left(\frac{1}{3}\right)^k=(-1)^0\frac{1}{0!}\left(\frac{1}{3}\right)^0+(-1)^1\frac{1}{1!}\left(\frac{1}{3}\right)^1+(-1)^2\frac{1}{2!}\left(\frac{1}{3}\right)^2=1-\frac{1}{3}+\frac{1}{18}\approx\boxed{0.7222.}$

The upper estimate to the error is $|E_2|\le a_3=\frac{1}{3!}\left(\frac{1}{3}\right)^3=\frac{1}{6}\cdot\frac{1}{27}\approx\boxed{0.00617.}$

34. We have

$S_2=\sum\limits_{k=0}^{2}(-1)^k\frac{1}{k!}\left(\frac{1}{2}\right)^k=(-1)^0\frac{1}{0!}\left(\frac{1}{2}\right)^0+(-1)^1\frac{1}{1!}\left(\frac{1}{2}\right)^1+(-1)^2\frac{1}{2!}\left(\frac{1}{2}\right)^2=1-\frac{1}{2}+\frac{1}{8}=\boxed{0.6250.}$

The upper estimate to the error is $|E_2|\le a_3=\frac{1}{3!}\left(\frac{1}{2}\right)^3=\frac{1}{6}\cdot\frac{1}{8}\approx\boxed{0.0208.}$

35. We have

$$\begin{aligned}S_2&=\sum_{k=0}^{2}(-1)^k\frac{1}{2k+1}\left(\frac{1}{3}\right)^{2k+1}\\ &=(-1)^0\frac{1}{2\cdot 0+1}\left(\frac{1}{3}\right)^{2\cdot 0+1}+(-1)^1\frac{1}{2\cdot 1+1}\left(\frac{1}{3}\right)^{2\cdot 1+1}+(-1)^2\frac{1}{2\cdot 2+1}\left(\frac{1}{3}\right)^{2\cdot 2+1}\\ &=\frac{1}{3}-\frac{1}{3}\left(\frac{1}{3}\right)^3+\frac{1}{5}\left(\frac{1}{3}\right)^5=\frac{1}{3}-\frac{1}{81}+\frac{1}{1215}\approx\boxed{0.3218.}\end{aligned}$$

The upper estimate to the error is $|E_2|\le a_3=\frac{1}{2\cdot 3+1}\left(\frac{1}{3}\right)^{2\cdot 3+1}=\frac{1}{7}\left(\frac{1}{3}\right)^7=\frac{1}{15309}\approx\boxed{6.53\times 10^{-5}.}$

36. We have $S_3 = \sum_{k=1}^{3} (-1)^{k+1} \frac{1}{k^k} = (-1)^2 \frac{1}{1^1} + (-1)^3 \frac{1}{2^2} + (-1)^4 \frac{1}{3^3} = 1 - \frac{1}{4} + \frac{1}{27} \approx \boxed{0.7870.}$ The upper estimate to the error is $|E_3| \leq a_4 = \frac{1}{4^4} \approx \boxed{0.0039.}$

37. Since $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k}$ is a constant multiple of an alternating harmonic series which converges, the given series also converges. Further $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{2k}$, the series of absolute values, is a constant multiple of the harmonic series which diverges, so it diverges too. Since the original series converges, but does not converge absolutely, it is $\boxed{\textit{conditionally convergent.}}$

38. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{3k-4}$. We have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3n-4} = 0$. Further, by the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{3(n+1)-4}}{\frac{1}{3n-4}} = \frac{3n-4}{3n-1} < 1 \text{ for } n \geq 1.$$

So the a_n are nonincreasing. By the Alternating Series Test, the original series converges. Consider now the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{3k-4}$. We compare the nth term of this series with that of the harmonic series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{3n-4}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3n-4} = \lim_{n\to\infty} \frac{1}{3-\frac{4}{n}} = \frac{1}{3}.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ diverges, the series $\sum_{k=1}^{\infty} a_k$ also diverges by the Limit Comparison Test. This means that the original series is convergent but not absolutely convergent. So in other words it is $\boxed{\textit{conditionally convergent.}}$

39. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin k}{k^2+1}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sin k}{k^2+1}$. We have $\left|\frac{\sin n}{n^2+1}\right| \leq \frac{1}{n^2+1} < \frac{1}{n^2}$. So, by the Comparison Test, comparing with the p-series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ which (since $p = 2 > 1$) is convergent, we see that the series of absolute values $\sum_{k=1}^{\infty} a_k$ is convergent. This means that the original series is $\boxed{\textit{absolutely convergent.}}$

40. The series is $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\cos k}{k^2}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\cos k}{k^2}$. We have $\left|\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2}$. So, by the Comparison Test, comparing with the p-series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ which (since $p = 2 > 1$) is convergent, we see that the series of absolute values $\sum_{k=1}^{\infty} a_k$ is convergent. This means that the original series is $\boxed{\textit{absolutely convergent.}}$

41. The series is $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = \sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{1}{5}\right)^k$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\left(\frac{1}{5}\right)^k$. It converges since it is a geometric series with $|r| = \frac{1}{5} < 1$. This means that the original series is $\boxed{\textit{absolutely convergent.}}$

42. The series is $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{5^k}{6^{k+1}}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\frac{5^k}{6^{k+1}}$. Rewriting this as $\sum_{k=1}^{\infty}\frac{1}{6}\cdot\frac{5^k}{6^k} = \sum_{k=1}^{\infty}\frac{1}{6}\left(\frac{5}{6}\right)^k$ we see that converges because it is the geometric series $\sum_{k=1}^{\infty} ar^k$ with $a = \frac{1}{6}$ and $|r| = \left|\frac{5}{6}\right| = \frac{5}{6} < 1$. This means that the original series is $\boxed{\text{absolutely convergent}}$.

43. The series is $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{e^k}{k}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{e^n}{n} = \lim_{x\to\infty}\frac{e^x}{x} = \lim_{x\to\infty}\frac{e^x}{1} = \infty \neq 0,$$

using L'Hôpital's rule on a related function of a_n. Since the limit is nonzero, by the Divergence Test, the series $\boxed{\textit{diverges.}}$

44. The series is $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}2^k}{k^2}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2^n}{n^2} = \lim_{x\to\infty}\frac{2^x}{x^2} = \lim_{x\to\infty}\frac{2^x\ln 2}{2x} = \lim_{x\to\infty}\frac{2^x(\ln 2)^2}{2} = \infty \neq 0,$$

using L'Hôpital's rule on a related function of a_n. Since the limit is nonzero, by the Divergence Test, the series $\boxed{\textit{diverges.}}$

45. The series is $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\frac{1}{k(k+1)}$. The nth term of this series satisfies $a_n = \frac{1}{n(n+1)} < \frac{1}{n^2}$ for all $n \geq 1$. Now the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}\frac{1}{k^2}$ converges being a convergent p-series (since $p = 2 > 1$), so by the Comparison Test, the series of absolute values converges. This means that the original series is $\boxed{\textit{absolutely convergent.}}$

46. The series is $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k\sqrt{k+3}}$. Consider the series of absolute values $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\frac{1}{k\sqrt{k+3}}$. The nth term of this series satisfies $a_n = \frac{1}{n\sqrt{n+3}} < \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$ for all $n \geq 1$. Now the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}\frac{1}{k^{3/2}}$ converges being a convergent p-series (since $p = \frac{3}{2} > 1$), so by the Comparison Test, the series of absolute values converges. This means that the original series is $\boxed{\textit{absolutely convergent.}}$

47. The series is $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sqrt{k}}{k^2+1}$. Consider the series of absolute values $\sum_{k=1}^{\infty}a_k = \sum_{k=1}^{\infty}\frac{\sqrt{k}}{k^2+1}$. The nth term of this series behaves like $a_n = \frac{\sqrt{n}}{n^2+1} = \frac{n^{1/2}}{n^2\left(1+\frac{1}{n^2}\right)} \approx \frac{1}{n^{3/2}}$ for large n. We compare with the series $\sum_{k=1}^{\infty}b_k = \sum_{k=1}^{\infty}\frac{1}{k^{3/2}}$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}} = 1.$$

Since the limit is a positive number, and the series $\sum_{k=1}^{\infty}b_k$ is convergent (it is a convergent p-series with $p=\frac{3}{2}>1$), by the Limit Comparison Test, the series of absolute values $\sum_{k=1}^{\infty}a_k$ is also convergent. This means that the original series is $\boxed{\textit{absolutely convergent.}}$

48. The series is $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sqrt{k}}{k+1}$. Consider the series of absolute values $\sum_{k=1}^{\infty}a_k = \sum_{k=1}^{\infty}\frac{\sqrt{k}}{k+1}$. The nth term of this series behaves like $a_n = \frac{\sqrt{n}}{n+1} = \frac{n^{1/2}}{n\left(1+\frac{1}{n}\right)} \approx \frac{1}{n^{1/2}}$ for large n. We compare with the series $\sum_{k=1}^{\infty}b_k = \sum_{k=1}^{\infty}\frac{1}{k^{1/2}}$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1.$$

Since the limit is a positive number and the series $\sum_{k=1}^{\infty}b_k$ is a divergent p-series (since $0<p=\frac{1}{2}<1$), by the Limit Comparison Test, the series of absolute values is divergent as well. Next we need to examine the original series for convergence. We have $\lim_{n\to\infty}a_n = \lim_{n\to\infty}\frac{\sqrt{n}}{n+1} = \lim_{n\to\infty}\frac{1}{\sqrt{n}+\frac{1}{\sqrt{n}}} = 0$. Consider a related function of a_n, $f(x)=\frac{\sqrt{x}}{x+1}$. We examine the derivative:

$$f'(x) = \frac{(x+1)\cdot\frac{1}{2\sqrt{x}} - \sqrt{x}\cdot 1}{(x+1)^2} = \frac{x+1-2x}{2\sqrt{x}(x+1)^2} = \frac{1-x}{2\sqrt{x}(x+1)^2} \le 0, \text{ for } x \ge 1.$$

The function is a nonincreasing one for $x\ge 1$, which means that the terms of the original series are nonincreasing for $n\ge 1$. By the Alternating Series Test, we see that the original series must be convergent. However, since the series of absolute values is divergent, this means that the original series must be $\boxed{\textit{conditionally convergent.}}$

49. The series is $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{\ln k}{k}$. Consider the series of absolute values $\sum_{k=1}^{\infty}a_k = \sum_{k=1}^{\infty}\frac{\ln k}{k}$. This series diverges by the Integral Test. Using $f(x)=\frac{\ln x}{x}$ which is a continuous, positive and decreasing function (since $f'(x)=\frac{x\cdot\frac{1}{x}-\ln x\cdot 1}{x^2}=\frac{1-\ln x}{x^2}\le 0$) on $[e,\infty)$, and for which $a_k=f(k)$ for all values of $k\ge 3$, we have

$$I=\int_3^{\infty}f(x)\,dx = \lim_{b\to\infty}\int_3^b\frac{\ln x\,dx}{x} = \lim_{b\to\infty}\frac{1}{2}\left[\ln x^2\right]\Big|_3^b = \lim_{b\to\infty}\frac{1}{2}\left(\ln b^2-\ln 3^2\right)=\infty.$$

So the improper integral I diverges, showing that the series of absolute values also diverges. Since the series of absolute values diverges, we must continue to see if the original series is convergent. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0,$$

using L'Hôpital's rule on a related function of a_n. Next we examine the derivative of the related function:

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2} < 0, \text{ for } x \geq 3,$$

recalling that $\ln e = 1$. This means that for $n \geq 3$, the a_n are nonincreasing. So by the Alternating Series Test, the original series converges. However, since the series of absolute values is divergent, this means that the original series is $\boxed{\textit{conditionally convergent.}}$

50. The series is $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1} \ln k}{k^3}$. Consider the series of absolute values $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\ln k}{k^3}$. The series $\sum\limits_{k=1}^{\infty} d_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is a convergent p-series since $p = \frac{3}{2} > 1$. We have

$$\lim_{n\to\infty} \frac{a_n}{d_n} = \lim_{n\to\infty} \frac{\frac{\ln n}{n^3}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\ln n}{n^{3/2}} = \lim_{x\to\infty} \frac{\ln x}{x^{3/2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{3}{2}x^{1/2}} = \lim_{x\to\infty} \frac{2}{3x^{3/2}} = 0,$$

using L'Hôpital's rule on a related function of the ratio of nth terms of the two series. By the result of Problem 60, in end of chapter exercises of Section 4, we conclude that the series of absolute values is convergent. But this means that the original series is
$\boxed{\textit{absolutely convergent.}}$

51. The series is $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k = \sum\limits_{k=1}^{\infty} (-1)^{k+1} \frac{1}{ke^k}$. Consider the series of absolute values $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{ke^k}$. We compare with the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ which is a convergent p-series (since $p = 2 > 1$). We have $a_n = \frac{1}{ne^n} < \frac{1}{n^2}$ if $e^n > n$. (To see that this is true for all n, let $f(x) = e^x - x$. Then $f'(x) = e^x - 1 > 0$ for $x > 0$. This means the function $f(x)$ is an increasing one for $x > 0$, and in turn this shows $e^x > x$ for $x > 0$.) We see that the series of absolute values is convergent by the Comparison Test, so the original series must be
$\boxed{\textit{absolutely convergent.}}$

52. The series is $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{e^k}$. Consider the series of absolute values $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{e^k}$. This is a geometric series that converges because $|r| = \frac{1}{e} < 1$, since $e > 1$. Since the series of absolute values converges, the original series must be
$\boxed{\textit{absolutely convergent.}}$

Applications and Extensions

53. The series is $\sum\limits_{k=2}^{\infty} (-1)^k a_k = \sum\limits_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$. The series of absolute values is $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{k \ln k}$. Consider the function $f(x) = \frac{1}{x \ln x}$, which is defined, positive, and decreasing on $[2, \infty)$, and such that $a_k = f(k)$ for all $k \geq 2$. We have $I = \int_2^{\infty} f(x)\,dx = \lim\limits_{b\to\infty} \int_2^b \frac{dx}{x \ln x}$. Let $u = \ln x$, then $du = \frac{dx}{x}$. Continuing,

$$I = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u} = \lim_{b\to\infty} [\ln u]\Big|_{\ln 2}^{\ln b} = \lim_{b\to\infty} [\ln(\ln b) - \ln(\ln 2)] = \infty.$$

Since the improper integral I diverges, by the Integral Test, the series of absolute values also diverges. Next, we examine the original series for convergence. We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n\ln n} = 0$. Also, consider the derivative of the related function of a_n:

$$f'(x) = -\frac{1}{x^2\ln x} - \frac{1}{x^2(\ln x)^2} < 0 \text{ for } x \geq 2.$$

So the a_n are nonincreasing. By the Alternating Series Test, the original series converges. But since it is not absolutely convergent, it is $\boxed{\textit{conditionally convergent.}}$

54. The series is $\sum\limits_{k=2}^{\infty}(-1)^k a_k = \sum\limits_{k=2}^{\infty}\frac{(-1)^k}{k(\ln k)^2}$. The series of absolute values is $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty}\frac{1}{k(\ln k)^2}$. Consider the function $f(x) = \frac{1}{x(\ln x)^2}$, which is defined, positive, and decreasing on $[2,\infty)$, and such that $a_k = f(k)$ for all $k \geq 2$. We have $I = \int_2^{\infty} f(x)\,dx = \lim\limits_{b\to\infty}\int_2^b \frac{dx}{x(\ln x)^2}$. Let $u = \ln x$, then $du = \frac{dx}{x}$. Continuing,

$$I = \lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{du}{u^2} = \lim_{b\to\infty}\int_{\ln 2}^{\ln b} u^{-2}\,du = \lim_{b\to\infty}\left[\frac{u^{-2+1}}{-2+1}\right]_{\ln 2}^{\ln b} = -\lim_{b\to\infty}\left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right) = 0.$$

Since the improper integral I converges, by the Integral Test, the series of absolute values also converges. But this means that the original series is $\boxed{\textit{absolutely convergent.}}$

55. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty}\frac{\cos k}{3^{k-1}}$. The series of absolute values is $\sum\limits_{k=1}^{\infty}|a_k| = \sum\limits_{k=1}^{\infty}\left|\frac{\cos k}{3^{k-1}}\right| = \sum\limits_{k=1}^{\infty} 3\cdot\frac{|\cos k|}{3^k}$. We have $3\cdot\frac{|\cos n|}{n^k} \leq 3\cdot\frac{1}{3^k} = 3\left(\frac{1}{3}\right)^k$ for all $n \geq 1$, since $|\cos n| \leq 1$ for all $n \geq 1$. Since $\left|\frac{1}{3}\right| = \frac{1}{3} < 1$, the geometric series $\sum\limits_{k=1}^{\infty} 3\cdot\left(\frac{1}{3}\right)^k$ converges. So, by the Comparison Test, we see that the series of absolute values is convergent. This means that the original series $\sum\limits_{k=1}^{\infty} a_k$ is $\boxed{\text{absolutely convergent}}$.

56. The series is $\sum\limits_{k=1}^{\infty}\frac{(-1)^{k+1}\tan^{-1}k}{k}$. The series of absolute values is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty}\frac{\tan^{-1}k}{k}$. Its terms are all nonzero since $\tan^{-1}n > 0$ for $n \geq 1$. Comparing this series with $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty}\frac{1}{k}$ we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{\tan^{-1}n}{n}}{\frac{1}{n}} = \lim_{n\to\infty}\tan^{-1}n = \frac{\pi}{2}.$$

Since the limit is a positive real number and the series $\sum\limits_{k=1}^{\infty} b_k$ diverges because it is a harmonic series, we conclude that the series of absolute values $\sum\limits_{k=1}^{\infty} a_k$ also diverges by the Limit Comparison Test. Next, we check to see if the original series converges or not. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\tan^{-1}n}{n} = \lim_{x\to\infty}\frac{\tan^{-1}x}{x} = \lim_{x\to\infty}\frac{\frac{1}{1+x^2}}{1} = 0,$$

by using L'Hôpital's rule on a related function of a_n, $f(x) = \frac{\tan^{-1}x}{x}$. We have, examining its derivative:

$$f'(x) = \frac{x\cdot\frac{1}{1+x^2} - \tan^{-1}x\cdot 1}{x^2} = \frac{x - (1+x^2)\tan^{-1}x}{x^2(1+x^2)} \leq 0$$

whenever $x \leq (1+x^2)\tan^{-1} x$, or equivalently, whenever $\tan^{-1} x \geq \frac{x}{1+x^2}$. We prove that this is true for all $x \geq 0$. Let $x_0 \geq 0$. The value of the function $\tan^{-1} x$ at $x = x_0$ can be represented as a definite integral:

$$\tan^{-1} x_0 = \int_0^{x_0} \frac{dx}{1+x^2}$$

since $\tan^{-1} 0 = 0$. The function $g(x) = \frac{1}{1+x^2}$ is a nonincreasing function of x for $x \geq 0$ since

$$g'(x) = -\frac{2x}{(1+x^2)^2} \leq 0$$

for $x \geq 0$. So the smallest value of $g(x) = \frac{1}{1+x^2}$ on the interval $[0, x_0]$ is $g(x_0) = \frac{1}{1+x_0^2}$. So we have

$$\tan^{-1} x_0 = \int_0^{x_0} \frac{dx}{1+x^2} \geq \int_0^{x_0} \frac{dx}{1+x_0^2} = \frac{1}{1+x_0^2}\int_0^{x_0} dx = \frac{x_0}{1+x_0^2}.$$

That is, $\frac{x_0}{1+x_0^2} \leq \tan^{-1} x_0$, or $x_0 \leq (1+x_0^2)\tan^{-1} x_0$. Since $x_0 \geq 0$ was an arbitrary point, it follows that $x \leq (1+x^2)\tan^{-1} x$ for all $x \geq 0$, i.e., $f'(x) \leq 0$ for all $x \geq 0$. This means in the original series the a_n are nonincreasing for $n \geq 1$ and so the original series is convergent by the Alternating Series Test. However, the series of absolute values has been shown above to be divergent, so the original series is $\boxed{\textit{conditionally convergent.}}$

57. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{1/k}}$. We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n^{1/n}} = \frac{1}{\lim\limits_{n\to\infty} n^{1/n}}$. To evaluate $\lim\limits_{n\to\infty} n^{1/n}$, set $y = n^{1/n}$. Then $\ln y = \frac{1}{n}\ln n$. We have

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0,$$

using L'Hôpital's rule on a related function of the ratio. So $\lim\limits_{n\to\infty} \ln y = \ln \lim\limits_{n\to\infty} y = 0$ gives $\lim\limits_{n\to\infty} y = \lim\limits_{n\to\infty} n^{1/n} = 1$.
Since $\lim\limits_{n\to\infty} a_n = 1 \neq 0$, by the Divergence Test, the series will be $\boxed{\textit{divergent}}$.

58. The series is $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k = \sum\limits_{k=1}^{\infty} (-1)^{k+1}\left(\frac{k}{k+1}\right)^k$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \neq 0,$$

using the limit definition of e. Since the limit is nonzero, by the Divergence Test, the series will be $\boxed{\textit{divergent.}}$

59. The series is $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \cdots = \sum\limits_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k!}$. The series of absolute values is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k!}$. the nth term of this series satisfies $a_n = \frac{1}{n!} \leq \frac{1}{n^2}$ when $n! \geq n^2$. Let $n \geq 4$. We estimate as follows:

$$\begin{aligned} n! &= n\cdot(n-1)\cdot(n-2)(n-3)\cdot\cdots\cdot 3\cdot 2\cdot 1 \\ &\geq n\cdot(n-1)\cdot(n-2)\cdot 1\cdot 1\cdot 1\cdot\cdots\cdot 1\cdot 1\cdot 1 \\ &= n(n-1)(n-2). \end{aligned}$$

Next, we demonstrate (for $n \geq 4$) that $n(n-1)(n-2) \geq n^2$:

$$\begin{aligned} n(n-1)(n-2) &\geq n^2 \\ n(n^2-3n+2) &\geq n^2 \\ n^2-3n+2 &\geq n \\ n^2-4n+2 &\geq 0 \\ n^2-4n+4-2 &\geq 0 \\ (n-2)^2-2 &\geq 0 \\ (n-2)^2 &\geq 2. \end{aligned}$$

That is, $n-2 \leq -\sqrt{2}$ or $n-2 \geq \sqrt{2}$. Keeping the positive root solution, we have $n \geq 2+\sqrt{2} \approx 3.414$ which concurs with our initial assumption that $n \geq 4$. So we have shown the desired inequality, that is $n! > n^2$ for $n \geq 4$. Since the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series (since $p = 2 > 1$), by the Comparison Test, the series of absolute values converges, which means the original series is $\boxed{\textit{absolutely convergent.}}$

60. The series is $1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots = \sum\limits_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(2k-1)^2}$. The series of absolute values is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{(2k-1)^2}$. The nth term of this series satisfies $a_n = \frac{1}{(2n-1)^2} < \frac{1}{n^2}$, since $2n-1 > n$ for $n > 1$. Since the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series (since $p = 2 > 1$), by the Comparison Test, the series of absolute values converges, which means the original series is $\boxed{\textit{absolutely convergent.}}$

61. (a) Since $\frac{1}{2} - \frac{1}{3} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{2^3} - \frac{1}{3^3} + \cdots = \sum\limits_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{3^k}\right)$, the series cannot be written in the form $\sum\limits_{k=1}^{\infty} (-1)^k a_k$. Thus the Alternating Series Test is not applicable.

(b) $\sum\limits_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{3^k}\right) = \sum\limits_{k=1}^{\infty} \frac{1}{2^k} - \sum\limits_{k=1}^{\infty} \frac{1}{3^k}$, which is the difference of two convergent series and thus converges.

(c) $\sum\limits_{k=1}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ and $\sum\limits_{k=1}^{\infty} \frac{1}{3^k} = \sum\limits_{k=1}^{\infty} \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$, so $\sum\limits_{k=1}^{\infty} \left(\frac{1}{2^k} - \frac{1}{3^k}\right) = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$.

62. (a) To apply the Alternating Series Test, a_k must be nonincreasing. But for

$$\sum_{k=1}^{\infty} (-1)^k a_k \qquad \text{where} \quad \begin{cases} a_k = \dfrac{1}{k}, & \text{if } k \text{ is odd} \\ a_k = \dfrac{1}{k^2}, & \text{if } k \text{ is even} \end{cases}$$

$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ is not true, since

$$a_1 = 1, \quad a_2 = \frac{1}{4}, \quad a_3 = \frac{1}{3}, \quad a_4 = \frac{1}{16}, \quad a_5 = \frac{1}{5}, \ldots$$

(b)

$$\begin{aligned}
S &= \sum_{k=1}^{\infty} (-1)^k a_k \qquad \text{where} \begin{cases} a_k = \dfrac{1}{k}, & \text{if } k \text{ is odd} \\ a_k = \dfrac{1}{k^2}, & \text{if } k \text{ is even} \end{cases} \\
&= \sum_{k=1}^{\infty} \left[-\frac{1}{2k-1} + \frac{1}{(2k)^2} \right] \\
&= \sum_{k=1}^{\infty} \left(-\frac{1}{2k-1} \right) + \sum_{k=1}^{\infty} \frac{1}{(2k)^2} \\
&= \sum_{k=1}^{\infty} \frac{1}{(2k)^2} - \sum_{k=1}^{\infty} \frac{1}{2k-1}
\end{aligned}$$

But

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{2k-1} &= \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots \\
&= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{1} + \frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{5} + \cdots \right) \\
&> \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \right)
\end{aligned}$$

So

$$\sum_{k=1}^{\infty} \frac{1}{2k-1} > \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$$

Now $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which diverges.

So $\sum_{k=1}^{\infty} \frac{1}{2k-1}$ diverges, since its terms are greater than those of a divergent series.
Therefore the original series diverges, since it is a convergent series minus a divergent series.

63. The series expands as $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. So positive terms are of the form $\sum_{k=1}^{\infty} \frac{1}{2k-1}$. Since the nth term of this series satisfies $\frac{1}{2n-1} > \frac{1}{2n}$ for $n \geq 1$, and the series $\sum_{k=1}^{\infty} \frac{1}{2k}$, being a constant multiple of the divergent harmonic series is divergent, the series of positive terms also diverges by the Comparison Test, as was needed to be proved.

64. The series expands as $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. So negative terms are of the form $-\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k}$, which, being a constant multiple of the divergent harmonic series is divergent. So the series of negative terms diverges as was needed to be proved.

65. We group terms of the alternating harmonic series as follows: $S_1 = 1$;
$S_2 = 1 + \left(-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8}\right) \approx -0.0416667$; $S_3 = S_1 + S_2 + \left(\frac{1}{3}\right) \approx +0.2916663$;
$S_4 = S_1 + S_2 + S_3 + \left(-\frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16}\right) \approx 0.02559557$, and so on. That is, for odd and even indices,

$$S_{2n-1} = S_1 + S_2 + \cdots + S_{2n-2} + \left(\frac{1}{2n-1}\right)$$

and

$$S_{2n} = S_1 + S_2 + \cdots + S_{2n-1} + \left(-\frac{1}{8n-6} - \frac{1}{8n-4} - \frac{1}{8n-2} - \frac{1}{8n} \right).$$

We show that the terms of the sequence $\{S_n\}$ are decreasing and bounded from below by 0. Using the Algebraic Difference Test,

$$S_{2n} - S_{2n-1} = S_{2n-1} - \frac{1}{8n-6} - \frac{1}{8n-4} - \frac{1}{8n-2} - \frac{1}{8n} - \frac{1}{2n-1}. \tag{1}$$

Examining equation (1), that the quantity on the right side will be negative for $n \geq 1$, proving that the sequence is a decreasing one. By its construction, the sequence is bounded from below by 0, so it converges. The sum to which the series $\sum\limits_{k=1}^{\infty} S_k$ it converges is 0 since the limit of the sequence $\{S_n\}$ of partial sums is 0. So we have managed to find a way to group the terms of the alternating harmonic series to produce a series whose sum is 0.

66. By trial and error, we find that $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 2 < 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{15}$. This means we can come close to 2 using the first 7 terms of the positive odd denominator numbers. Subsequently, we form sum groupings as in Problem 65, starting from $n = 15$. This way, we will be able to form groupings whose eventual sum will be 2.

67. We create blocks of numbers that sum to a little above 1, and then blocks that sum to a little under 1, and so on; adding these blocks up, one gets the sum to be infinite. This can be done because the sum of the positive terms of the conditionally convergent alternating harmonic series diverges, and the sum of the negative terms of the conditionally convergent alternating harmonic series diverges as well.

68. We have the series of absolute values of $e^{-x}\cos x + e^{-2x}\cos 2x + e^{-3x}\cos 3x + \cdots$ satisfy (since $|cos\theta| \leq 1$),

$$|e^{-x}\cos x| + |e^{-2x}\cos 2x| + |e^{-3x}\cos 3x| + \cdots \leq e^{-x} + e^{-2x} + e^{-3x} + \cdots = \sum_{k=1}^{\infty} \left(\tfrac{1}{e}\right)^{kx} = \sum_{k=1}^{\infty} \left[\left(\tfrac{1}{e}\right)^{x}\right]^{k}.$$

This last is a geometric series that converges since $0 < |r| = \left(\frac{1}{e}\right)^x < 1$ for all $x > 0$ (since $e^x > 1$ for $x > 0$). By the Comparison Test, the series of absolute values converges, so the original series converges absolutely, as was needed to be shown.

69. The nth term of the series of absolute values of the series $1 + r\cos\theta + r^2\cos(2\theta) + r^3\cos(3\theta) + \cdots$ satisfies $|r^n \cos n\theta| \leq |r|^n$. Since $\sum\limits_{k=1}^{\infty} |r|^k$ is a geometric series that is convergent when $0 < |r| < 1$, or $-1 < r < 1$, we find that the series of absolute values converges, and the original series $\boxed{\textit{absolutely converges} \text{ for } -1 < r < 1.}$ Also, since the geometric series diverges when $|r| \geq 1$, we conclude that the given series $\boxed{\textit{diverges} \text{ for } |r| \geq 1.}$

70. The alternating harmonic series is conditionally convergent, so its terms can be rearranged to obtain different kinds of sums, as illustrated in the statement of the problem.

71. $N = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, and $N - 1$ are both divergent harmonic series. They can be added and the sums rearranged to produce any number one wishes. (See for example Problems 65-67 where we rearranged the alternating harmonic series, whose series of absolute values is the harmonic series. We were able to rearrange terms to obtain any sum we wanted.) Only series which are absolutely convergent can be added together and rearranged at will to produce a convergent series with a unique sum. (See p.686 for the list of properties of absolutely convergent series.)

72. The series will either converge or diverge. For instance consider the divergent harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ and the divergent (constant term) series $\sum\limits_{k=1}^{\infty} 1$. The sum will diverge, since the partial sum is $S_n = n + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, and $\lim\limits_{n\to\infty} S_n = \infty$. Their difference will diverge as well, since the partial sum will be

$S_n = (1-1) + \left(1-\frac{1}{2}\right) + \left(1-\frac{1}{3}\right) + \left(1-\frac{1}{4}\right) + \cdots + \left(1-\frac{1}{n}\right) = 0 + \frac{1}{2} + \frac{3}{4} + \cdots + \frac{n-1}{n} = \sum_{k=1}^{n} \frac{k}{k+1}$.
Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, by the Divergence Test, the difference series also diverges.

Now consider the divergent harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ and the series $\sum_{k=1}^{\infty} \frac{1}{k+1}$ which, being the harmonic series from the second term forward, also diverges. Then their difference will converge, and this is seen as follows. The nth partial sum of the difference of these series will be

$$\begin{aligned} S_n &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

which converges to 1, since

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1.$$

So the series which is the term by term difference of the two series also converges.
These examples demonstrate that nothing conclusive can be said about the convergence or divergence when two divergent series are added or subtracted term by term from each other.

73. The summand in the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ can be split up into partial fractions as $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$. The nth partial sum has the form
$S_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - \frac{1}{n}$, due to the telescoping.
We have $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1$. Since the limit of the sequence $\{S_n\}$ of partial sums is 1, it means the series sums to 1.

74. To show the series $\sum_{k=1}^{\infty} (a_k + b_k)$ converges absolutely, consider the absolute value of the nth term. We have $|a_n + b_n| \leq |a_n| + |b_n|$, by the Triangle Inequality. Let the sequence of partial sums of $\sum_{k=1}^{\infty} |a_k|$ be $\{S_n\}$ and the sequence of partial sums of $\sum_{k=1}^{\infty} |b_k|$ be $\{S'_n\}$, and the sequence of partial sums of $\sum_{k=1}^{\infty} |a_k + b_k|$ be $\{S''_n\}$. Then, since we have

$$0 \leq S''_n = \sum_{k=1}^{n} |a_k + b_k| \leq \sum_{k=1}^{n} |a_k| + |b_k| = \sum_{k=1}^{n} |a_k| + \sum_{k=1}^{n} |b_k| = S_n + S'_n$$

and (by assumption) the sequences $\{S_n\}$ and $\{S'_n\}$ converge, it follows that the sequence $\{S''_n\}$ is also convergent. Since absolutely convergent series can be rearranged and regrouped at will without changing the sum, it follows that $\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$.

75. An absolutely convergent series can be rearranged at will (see Problem 78 below), so one may group all positive terms together, and all the negative terms together, and the sum of the series will not be affected. Since the entire series has a finite sum, it means that each of the component series will also have a finite sum, and so will converge separately.

76. Let the conditionally convergent series be $\sum_{k=1}^{\infty} a_k$. We assume the terms are nonzero; none of the arguments below will be affected by a few zero terms. Note that $b_n = \frac{a_n + |a_n|}{2}$ are all the positive nth terms, and $c_n = \frac{a_n - |a_n|}{2}$ are all the negative nth terms. Also note that $a_n = b_n + c_n$. Suppose that the series of positive terms is convergent. Then, because its terms are all positive, it is also absolutely convergent. Since $\sum\limits_{k=1}^{\infty} a_k$ converges conditionally and $\sum\limits_{k=1}^{\infty} b_k$ converges absolutely, then $\sum\limits_{k=1}^{\infty} c_k$ converges absolutely, because $|c_n| = \frac{1}{2}(|a_n - |a_n||) \leq \frac{1}{2}(a_n + |a_n|) = b_n$ so by the Comparison Test, if $\sum\limits_{k=1}^{\infty} b_k$ converges absolutely, then so must $\sum\limits_{k=1}^{\infty} c_k$. But since $a_n = b_n + c_n$ this must mean that $\sum\limits_{k=1}^{\infty} a_k$ converges absolutely (by the result of Problem 74) which contradicts its conditional convergence. This contradiction was a result of assuming that $\sum\limits_{k=1}^{\infty} b_k$ was convergent, so that must be false. This means that the series of positive terms diverges. But since $c_n = a_n - b_n$ it means that $\sum\limits_{k=1}^{\infty} c_k$ must also be divergent. So, finally, we have shown that the series of positive terms and the series of negative terms both diverge separately.

77. The nth term of the series $\sum\limits_{k=1}^{\infty} c_k$ is

$$|c_n| = \begin{cases} \frac{1}{a^n}, & n \text{ even} \\ \frac{1}{b^n}, & n \text{ odd} \end{cases}$$

Let $d = \min\{a, b\}$. Since $a > 1$ and $b > 1$, we have $\frac{1}{d} < 1$ and thus $|c_n| \leq \frac{1}{d^n}$. Since $\frac{1}{d} < 1$, the series $\sum\limits_{k=1}^{\infty} \frac{1}{d^k}$ is a convergent geometric series, so $\sum\limits_{k=1}^{\infty} |c_k|$ converges. This means that $\sum\limits_{k=1}^{\infty} c_k$ $\boxed{\textit{converges absolutely.}}$

78. If the series $\sum\limits_{k=1}^{\infty} a_k$ is absolutely convergent, then the series of absolute values $\sum\limits_{k=1}^{\infty} |a_k| = L$ must be convergent to a finite number (say) L. Since the sequence of partial sums $\{S_n\}$ is also convergent to the limit L, this means for every $\epsilon > 0$, there must be an N_1 such that for $n > N_1$ we have

$$|S_n - L| = \left| S_n - \sum_{k=1}^{\infty} |a_k| \right| = \left| \sum_{k \geq N_1 + 1}^{\infty} |a_k| \right| < \frac{\epsilon}{2}.$$

That is, for any $\epsilon > 0$, there exists an N_1 such that $\sum\limits_{k=N_1+1}^{\infty} |a_k| < \frac{\epsilon}{2}$. Let $f : \mathbb{N} \to \mathbb{N}$ be a permutation on the indices of the series, that is a one to one and onto map defined on the positive integers. This achieves the rearrangement of the series from $\sum\limits_{k=1}^{\infty} a_k$ to $\sum\limits_{k=1}^{\infty} a_{f(k)}$. Since the series of absolute values is convergent, it means that the sequence of partial sums $S_n = |a_1| + |a_2| + \cdots + |a_n|$ is bounded from above by some real number, say K. Then the sequence of partial sums of the rearrangements $S'_n = |a_{f(1)}| + |a_{f(2)}| + \cdots + |a_{f(n)}|$ is also bounded from above by the some number K'. Since $\{S'_n\}$ is a bounded from above and monotonically increasing sequence (since all its terms are positive), it will converge.
To show that the limits are the same, we use the fact that there exists some $N_2 \geq N_1$ such that among the terms $a_{f(1)}, a_{f(2)}, \cdots, a_{f(N_2)}$ we shall find all of the terms $a_1, a_2, \cdots, a_{N_1}$. For $n > N_2$, let $\mathcal{D}$ be the (finite) set of subscripts $k \in \{f(1), f(2), f(3), \cdots f(n)\}$ that do not match up with any of the $\{1, 2, \cdots N_1\}$. Then for $n > N_2 \geq N_1$, we have

$$\left| \sum_{k=1}^{n} |a_{f(k)}| - \sum_{k=1}^{N_1} |a_k| \right| = \left| \sum_{\mathcal{D}} |a_{f(k)}| \right| \leq \sum_{\mathcal{D}} |a_{f(k)}| \leq \sum_{k=N_1+1}^{\infty} |a_k| < \frac{\epsilon}{2},$$

where the final inequality follows from the fact that all the terms $|a_{f(k)}|$ with $k \in \mathcal{D}$ are to be found among the set $\{|a_{N_1+1}|, |a_{N_1+2}|, \cdots\}$, since $N_1 \leq N_2 < n$ and $\mathcal{D}$, which is a finite set, only contains elements not in the set $\{1, 2, 3, \cdots, N_1\}$. Now, we compute:

$$\left|\sum_{k=1}^{n} |a_{f(k)}| - L\right| = \left|\sum_{k=1}^{n} |a_{f(k)}| - \sum_{k=1}^{N_1} |a_k| + \sum_{k=1}^{N_1} |a_k| - L\right|$$

$$\leq \left|\sum_{k=1}^{n} |a_{f(k)}| - \sum_{k=1}^{N_1} |a_k|\right| + \left|\sum_{k=1}^{N_1} |a_k| - L\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that the series of absolute values obtained by a rearrangement of the original series converges to L. So we have shown that a rearrangement of the original absolutely convergent series will result in a series which itself is absolutely convergent to the same limit as the original series.

79. The proof of this result is identical in every respect to the proof carried out in the textbook, pages 680-681 (Alternating Series Test); the only change is that instead of writing the alternating series as $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ with $a_k > 0$ in the textbook, it is written as $\sum_{k=1}^{\infty} a_k$ in this problem. (That is, the minus signs are now included in the terms a_k themselves.) So, we have to be careful about putting absolute value signs around the terms while importing the proof from the book to this problem. For instance, in the text, the hypothesis for nondecreasing terms is written as $a_{k+1} \leq a_k$ while in the problem the hypothesis for nondecreasing terms applies to the absolute values: $|a_{k+1}| \leq |a_k|$ for all $k \geq 1$. As in the book, form the series $\{S_{2n}\}$ of partial sums and show that it is nondecreasing; form the series $\{S_{2n+1}\}$ of partial sums and show that it is nonincreasing; finally show that both these series tend to the same limit as $n \to \infty$.

Challenge Problems

80. The series can be written as $\sum_{k=1}^{\infty} (-1)^k a_k = \sum_{k=1}^{\infty} \sin \frac{(-1)^k}{k} = \sum_{k=1}^{\infty} (-1)^k \sin\left(\frac{1}{k}\right)$, since $\sin(-\theta) = -\sin\theta$. The series of absolute values is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$. Note that the terms of the series are nonzero because $\sin\left(\frac{1}{n}\right) > 0$ for $n \geq 1$. Comparing this series to the divergent harmonic series, $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1,$$

employing the substitution $x = \frac{1}{n}$, and using the standard limit result at the end. Since the limit is a positive number and the series $\sum_{k=1}^{\infty} b_k$ is divergent, by the Limit Comparison Test, we see that the series $\sum_{k=1}^{\infty} a_k$ must be divergent. So the series of absolute values diverges. Next, we need to see if the original series itself converges. The nth term satisfies $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\frac{1}{n} = 0$. Also, if $f(x) = \sin\frac{1}{x}$ is a related function of the nth term of the series, examining its derivative yields $f'(x) = -\frac{1}{x^2}\cos\frac{1}{x} < 0$ for $x > 1$ (recall that for θ between 0 and 1 radians, $\cos\theta$ is a positive function). This shows that the reated function, and the sequence it is related to, are nonincreasing. So by the Alternating Series Test, the original series converges. But since the series of absolute values has been shown above to be divergent, this means that the original series is $\boxed{\textit{conditionally convergent.}}$

81. The series is $\sum\limits_{k=2}^{\infty}(-1)^k a_k = \sum\limits_{k=2}^{\infty}\frac{(-1)^k}{\sqrt[p]{k^3+1}}$. The absolute value series is $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty}\frac{1}{(n^3+1)^{1/p}}$. The nth term of the series behaves like

$$a_n = \frac{1}{(n^3+1)^{1/p}} = \frac{1}{n^{3/p}\left(1+\frac{1}{n^3}\right)^{1/p}} \approx \frac{1}{n^{3/p}}$$

for large n. So we compare with the series $\sum\limits_{k=2}^{\infty} b_k = \sum\limits_{k=2}^{\infty}\frac{1}{k^{3/p}}$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{(n^3+1)^{3/p}}}{\frac{1}{n^{3/p}}} = \lim_{n\to\infty}\left(\frac{n^3}{n^3+1}\right)^{1/p} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n^3}\right)^{1/p}} = 1.$$

Since the limit is a positive number and the series $\sum\limits_{k=2}^{\infty} b_k = \sum\limits_{k=2}^{\infty}\frac{1}{k^q}$ is a convergent "q-series" since for $2 < p < 3$, we have $q = \frac{3}{p} > 1$, it means that the absolute value series is also convergent by the Limit Comparison Test. This in turn implies that the original series is $\boxed{\textit{absolutely convergent} \text{ if } 2 < p < 3.}$

If $p \geq 3$, then the large n behavior of the nth term is $\frac{1}{n^{3/p}} = \frac{1}{n^q}$ where $q \leq 1$. We compare with the divergent series $\sum\limits_{k=2}^{\infty} b_k = \frac{1}{k^q}$ for $q \leq 1$; the computation is identical to the one shown above, except the conclusion is that the original series $\boxed{\textit{diverges} \text{ if } p \geq 3.}$

82. The series is $\sum\limits_{k=2}^{\infty}(-1)^k a_k = \sum\limits_{k=2}^{\infty}(-1)^k k^{(1-k)/k}$. The series of absolute values is $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} k^{(1-k)/k} = \sum\limits_{k=2}^{\infty}\frac{k^{1/k}}{k}$. Comparing with the divergent harmonic series $\sum\limits_{k=2}^{\infty} b_k = \sum\limits_{k=2}^{\infty}\frac{1}{k}$ we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{n^{1/n}}{n}}{\frac{1}{n}} = \lim_{n\to\infty} n^{1/n} = 1,$$

where the reader is referred to Problem 57 for the evaluation of the final limit. Since the limit is a positive number, and the series $\sum\limits_{k=2}^{\infty} b_k$ diverges, by the comparison test, the series of absolute values diverges as well. Next, we check to see if the original series is convergent. We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{n^{1/n}}{n} = 0$. Let $f(x) = \frac{x^{1/x}}{x}$ be a related function of the a_n. Let $y = x^{1/x}$. Then $\ln y = \frac{1}{x}\ln x$. Differentiating, we have

$$\frac{y'}{y} = \frac{x\cdot\frac{1}{x} - \ln x\cdot 1}{x^2} = \frac{1-\ln x}{x^2}.$$

Next, examine the derivative of $f(x) = \frac{x^{1/x}}{x} = \frac{y}{x}$:

$$f'(x) = \frac{y'}{x} - \frac{y}{x^2} = \frac{x^{1/x}}{x^2}(1-\ln x) - \frac{x^{1/x}}{x^2} = \frac{x^{1/x}}{x^2}(-\ln x) < 0,$$

whenever $x > 1$. Since the related function is decreasing the series is nonincreasing for $n \geq 1$. By the Alternating Series Test, the original series converges. However, since the series of absolute values diverges as seen above, the original series is $\boxed{\textit{conditionally convergent.}}$

83. The series is $\sum\limits_{k=2}^{\infty} \frac{(-1)^k}{(\ln k)^{\ln k}}$. The series of absolute values is $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{(\ln k)^{\ln k}}$. We have

$$(\ln n)^{\ln n} = \left(e^{\ln(\ln n)}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln(\ln n)} = n^{\ln(\ln n)},$$

since $(e^a)^b = \left(e^b\right)^a$. Since $\ln n$ is an increasing function, so is $\ln(\ln n)$. So there will exist some $n > 2$ for which $\ln(\ln n) \geq 2$. Solving, we find that $n \geq e^{e^2} \approx 1618.18$. So we see that the nth term of this series satisfies $\frac{1}{(\ln n)^{\ln n}} < \frac{1}{n^2}$ for $n \geq 1619$. Since the p-series $\sum\limits_{k=2}^{\infty} b_k = \sum\limits_{k=2}^{\infty} \frac{1}{k^2}$ converges ($p = 2 > 1$), we see by the Comparison Test that the series of absolute values also converges. (We note that convergence does not depend on the omission of a finite number of terms from the comparison, in this instance the first 1618 terms, recalling the series starts with $n = 2$.) But this means that the original series is $\boxed{\textit{absolutely convergent.}}$

84. The series is $\sum\limits_{k=1}^{\infty} (-1)^{k+1} \frac{\sqrt{k}}{k+1}$. We have $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \frac{\sqrt{n}}{n+1} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}+\frac{1}{\sqrt{n}}} = 0$. Let $f(x) = \frac{\sqrt{x}}{x+1}$ be a related function of the absolute value of the nth term of the series. Then, examining the derivative, we have

$$f'(x) = \frac{(x+1)\cdot\frac{1}{2\sqrt{x}} - \sqrt{x}\cdot 1}{(x+1)^2} = \frac{x+1-2x}{2\sqrt{x}(x+1)^2} = \frac{1-x}{2\sqrt{x}(x+1)^2} \leq 0$$

for $x \geq 1$. Since the function is nonincreasing, this means the sequence $\{|a_k|\}$ is nonincreasing, or $|a_{k+1}| \leq |a_k|$ for $k \geq 1$. By the result of Problem 79, we see that the series $\boxed{\textit{converges.}}$

85. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} (-1)^{k+1} \frac{k}{(k+1)^2}$. We have $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \frac{n}{(n+1)^2} = \lim\limits_{n\to\infty} \frac{1}{\left(\sqrt{n}+\frac{1}{\sqrt{n}}\right)^2} = 0$. Let $f(x) = \frac{x}{(x+1)^2}$ be a related function of the absolute value of the nth term of the series. Then, examining the derivative, we have

$$f'(x) = \frac{(x+1)^2\cdot 1 - x\cdot 2(x+1)\cdot 1}{(x+1)^4} = \frac{(x+1)}{(x+1)^4}[x+1-2x] = \frac{1-x}{(1+x)^3} \leq 0$$

for $x \geq 1$. Since the function is nonincreasing, this means the sequence $\{|a_k|\}$ is nonincreasing or $|a_{k+1}| \leq |a_k|$ for $k \geq 1$. By the result of Problem 79, we see that the series $\boxed{\textit{converges.}}$

86. The series is $\sum\limits_{k=2}^{\infty} (-1)^k a_k = \sum\limits_{k=2}^{\infty} (-1)^k \ln \frac{k+1}{k}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln \frac{n+1}{n} = \ln \lim_{n\to\infty} \frac{n+1}{n} = \ln \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = \ln(1+0) = 0.$$

Let $f(x) = \ln \frac{x+1}{x}$ be a related function of the absolute value of the nth term of the series. Then, examining the derivative, we have

$$f'(x) = \frac{x}{x+1} \cdot \frac{x\cdot 1 - (x+1)\cdot 1}{x^2} = -\frac{1}{x(x+1)} \leq 0$$

for $x \geq 1$. Since the function is nonincreasing, this means the sequence $\{|a_k|\}$ is nonincreasing or $|a_{k+1}| \leq |a_k|$ for $k \geq 1$. By the result of Problem 79, we see that the series $\boxed{\textit{converges.}}$

87. (a) Since $\lim\limits_{n\to\infty} a_n = 0$, we have $\lim\limits_{n\to\infty} \Delta a_n = \lim\limits_{n\to\infty} (a_n - a_{n+1}) = 0$. We are given that the sequence $\{\Delta a_n\}$ decreases. Therefore, by the Alternating Series Test, $\sum\limits_{k=1}^{\infty}(-1)^{k+1}\Delta a_k$ is a convergent alternating series.

(b) Let $R_n = \sum\limits_{k=n+1}^{\infty}(-1)^{k+1}a_k$. R_n is a well defined value since $\sum\limits_{k=1}^{\infty}(-1)^{k+1}a_k$ is a convergent alternating series. Expanding R_n, we have

$$\begin{aligned}R_n &= (-1)^{n+2}a_{n+1} + (-1)^{n+3}a_{n+2} + (-1)^{n+4}a_{n+3} + (-1)^{n+5}a_{n+4} + \cdots\\ &= (-1)^{n+2}[a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots]\\ &= (-1)^{n+2}[(a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \cdots].\end{aligned}$$

Since the sequence $\{a_n\}$ decreases, each of the pairs grouped in this last expression is positive, so the absolute value of R_n is given by

$$|R_n| = a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots .$$

Rewriting this produces

$$\begin{aligned}|R_n| &= \frac{a_n}{2} + \frac{1}{2}[-a_n + 2a_{n+1} - 2a_{n+2} + 2a_{n+3} - 2a_{n+4} + \cdots] && (1)\\ &= \frac{a_n}{2} + \frac{1}{2}[(-a_n + a_{n+1}) + (a_{n+1} - a_{n+2}) + (-a_{n+2} + a_{n+3}) + \cdots]\\ &= \frac{a_n}{2} + \frac{1}{2}[-(a_n - a_{n+1}) + (a_{n+1} - a_{n+2}) - (a_{n+2} - a_{n+3}) + \cdots]\\ &= \frac{a_n}{2} + \frac{1}{2}[-\Delta a_n + \Delta a_{n+1} - \Delta a_{n+2} + \Delta a_{n+3} - \cdots] && (2)\\ &= \frac{a_n}{2} + \frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\Delta a_{k+n-1},\end{aligned}$$

as was required to be shown. Rewriting equation (2) produces

$$|R_n| = \frac{a_n}{2} + \frac{1}{2}[(-\Delta a_n + \Delta a_{n+1}) + (-\Delta a_{n+2} + \Delta a_{n+3}) + \cdots].$$

Since $\{\Delta a_n\}$ is a decreasing sequence, each of the pairs grouped in this last expression is negative, so we have

$$|R_n| < \frac{a_n}{2}.$$

(c) Use equation (1) from part (b) to rewrite the absolute value of R_n as follows:

$$\begin{aligned}|R_n| &= \frac{a_{n+1}}{2} + \frac{1}{2}[a_{n+1} - 2a_{n+2} + 2a_{n+3} - 2a_{n+4} + \cdots]\\ &= \frac{a_{n+1}}{2} + \frac{1}{2}[(a_{n+1} - a_{n+2}) + (-a_{n+2} + a_{n+3}) + (a_{n+3} - a_{n+4}) + \cdots]\\ &= \frac{a_{n+1}}{2} + \frac{1}{2}[\Delta a_{n+1} - \Delta a_{n+2} + \Delta a_{n+3} - \cdots] && (3)\\ &= \frac{a_{n+1}}{2} + \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k+1}\Delta a_{k+n},\end{aligned}$$

as was required to be shown. Rewriting equation (3) produces

$$|R_n| = \frac{a_{n+1}}{2} + \frac{1}{2}[(\Delta a_{n+1} - \Delta a_{n+2}) + (\Delta a_{n+3} - \Delta a_{n+4}) + \cdots].$$

Since $\{\Delta a_n\}$ is a decreasing sequence, each of the pairs grouped in this last expression is positive, so we have

$$|R_n| > \frac{a_{n+1}}{2}.$$

AP® Practice Problems

1. Evaluate each of the series as follow:

   I. The series $\sum_{k=1}^{\infty}(-1)^k\frac{1}{k^2}$ is an alternating series, with $a_n=\frac{1}{n^2}$. We begin by confirming that $\lim_{n\to\infty}(a_n)=\lim_{n\to\infty}\frac{1}{n^2}=0$. Next, using the Algebraic Ratio test, we verify that the terms $a_k=\frac{1}{k^2}$ are nonincreasing.

   Since $\frac{a_{n+1}}{a_n}=\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}=\left(\frac{n}{n+1}\right)^2<1$ for all $n\geq 1$, the terms a_k are nonincreasing.

   By the Alternating Series Test, the series $\boxed{\text{converges}}$.

   II. The series $\sum_{k=1}^{\infty}(-1)^k\left(\frac{5}{3}\right)^k$ is an alternating series, where $a_n=\left(\frac{5}{3}\right)^n$. We check $\lim_{n\to\infty}(a_n)=\lim_{n\to\infty}\left(\frac{5}{3}\right)^n=\infty$.

   Since $\lim_{n\to\infty}(a_n)\neq 0$, by the Test for Divergence, the series $\boxed{\text{diverges}}$.

   III. The series $\sum_{k=1}^{\infty}(-1)^k\frac{1}{\sqrt{k}}$ is an alternating series, where $a_n=\frac{1}{\sqrt{n}}=\frac{1}{n^{1/2}}$. We check $\lim_{n\to\infty}(a_n)=\lim_{n\to\infty}\frac{1}{n^{1/2}}=0$.

   Next, using the Algebraic Ratio test, we verify that the terms $a_k=\frac{1}{k^{1/2}}$ are nonincreasing.

   Since $\frac{a_{n+1}}{a_n}=\frac{\frac{1}{(n+1)^{1/2}}}{\frac{1}{n^{1/2}}}=\frac{n^{1/2}}{(n+1)^{1/2}}=\left(\frac{n}{n+1}\right)^{1/2}=1$ for all $n\geq 1$

   the terms a_k are nonincreasing. By the Alternating Series Test, the series $\boxed{\text{converges}}$.

   CHOICE C

2. The error, E_n, of a convergent Alternating Series using n terms is numerically less than or equal to the $(n+1)$st term of the series (i.e., $|E_n|\leq a_{n+1}$).

   Here the maximum error incurred by using the first three non-zero terms to approximate the sum of the series is less than or equal to the 4th term. $|E_n|\leq a_4=\frac{4}{10^4}=\boxed{0.0004}$.

   CHOICE C

3. The error, E_n, of a convergent Alternating Series using n terms is numerically less than or equal to the $(n+1)$st term of the series (i.e., $|E_n|\leq a_{n+1}$).

   Here, $|E_n|<0.001\leq a_{n+1}\leq\frac{1}{(n+1)^3}$.

$$\begin{aligned}\frac{1}{(n+1)^3}&\leq\frac{1}{1000}\\(n+1)^3&\geq 1000\\n+1&\geq 10\\n&\geq 9\end{aligned}$$

$$\boxed{n=9}$$

   CHOICE B

4. I. We begin by testing the series for absolute convergence.

   The series of absolute values is $\sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}3}{k}\right|=3\sum_{k=1}^{\infty}\frac{1}{k}$.

   This is a p-series which converges if $p>1$ and diverges if $p\leq 1$.

   Here, $\sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}3}{k}\right|$ diverges, i.e., it is $\boxed{\text{not absolutely convergent}}$.

We need to apply the Alternating Series test to determine if the given series is conditionally convergent.

The series $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2}$ is an alternating series, where $a_n = \frac{1}{n^2}$.

We begin by confirming that $\lim_{n\to\infty} (a_n) = \lim_{n\to\infty} \frac{3}{n} = 0$. Next, using the Algebraic Ratio test, we verify that the terms $a_k = \frac{3}{k}$ are nonincreasing.

Since $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)}}{\frac{1}{n}} = \left(\frac{n}{n+1}\right) = \left(\frac{1}{1+\frac{1}{n}}\right) = 1$ for all $n \geq 1$ the terms a_k are nonincreasing. By the Alternating Series Test, the series converges.

Thus it is $\boxed{\text{conditionally convergent}}$.

II. We begin by testing the series for absolute convergence.

The series of absolute values is $\sum_{k=1}^{\infty} \left|(-1)^{k+1} \left(\frac{1}{k}\right)^{4/3}\right| = \sum_{k=1}^{\infty} \frac{1}{k^{4/3}}$.

This is a p-series which converges if $p > 1$ and diverges if $p \leq 1$.

Here, since $p = \frac{4}{3} > 1$, $\sum_{k=1}^{\infty} \left|(-1)^{k+1} \left(\frac{1}{k}\right)^{4/3}\right|$ converges.

The series $\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{k}\right)^{4/3}$ is $\boxed{\text{absolutely convergent}}$.

III. We begin by testing the series for absolute convergence.

The series of absolute values is $\sum_{k=1}^{\infty} \left|(-1)^k \left(\frac{3}{4}\right)^k\right| = \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k$.

This is a convergent Geometric Series.

The series $\sum_{k=1}^{\infty} (-1)^k \left(\frac{3}{4}\right)^k$ is $\boxed{\text{absolutely convergent}}$.

CHOICE A

5. (a) $$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!}$$
$$= \boxed{1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}}$$

(b) The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$ is an alternating series, where $a_n = \frac{1}{n!}$. We check $\lim_{n\to\infty} (a_n) = \lim_{n\to\infty} \frac{1}{n!} = 0$.

Next, using the Algebraic Ratio test, we verify that the terms $a_k = \frac{1}{k!}$ are nonincreasing.

Since $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1} \leq 1$ for all $n \geq 1$, the terms a_k are nonincreasing. By the Alternating Series Test, the series $\boxed{\text{converges}}$.

(c) The error E_n of a convergent Alternating Series using n terms is numerically less than or equal to the $(n+1)$st term of the series (i.e., $|E_n| \leq a_{n+1}$).

Here, $|E_n| < 0.0001 \leq a_{n+1} \leq \frac{1}{(n+1)!}$.

$$\frac{1}{10,000} \leq \frac{1}{(n+1)!}$$
$$(n+1)! \leq 10,000$$
$$7! = 5,040$$
$$8! = 40,320$$

Since 8! is the least factorial to exceed 10,000,

$$\begin{aligned} n+1 &= 8 \\ n &= 7 \end{aligned}$$

Since the series begins with $k = 0$, $n = 7$ would be the 8th term.
$\boxed{\text{8 terms}}$

6. This series has both positive and negative terms, but it is not an alternating series. Use the Absolute Convergence Test to investigate the series $\sum\limits_{k=1}^{\infty} \left|\frac{\cos(2k)}{4^k}\right|$. Since $\left|\frac{\cos(2n)}{4^n}\right| \le \frac{1}{4^n}$ for all n, and since $\sum\limits_{k=1}^{\infty} \frac{1}{4^k}$ is a convergent Geometric Series, then by the Comparison Test for Convergence, the series $\sum\limits_{k=1}^{\infty} \left|\frac{\cos(2k)}{4^k}\right|$ converges.

   That is, $\sum\limits_{k=1}^{\infty} \frac{\cos(2k)}{4^k}$ is $\boxed{\text{absolutely convergent}}$.

8.6 Ratio Test; Root Test

Concepts and Vocabulary

1. <u>False:</u> We have $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cos(n+1)\pi}{\cos n\pi}\right| = 1$ for all n, so $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$. This means the Ratio Test is inconclusive, and you cannot conclude divergence as claimed.

2. <u>False:</u> For a counterexample, consider the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$. We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| = \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\right| = \lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right|^2 = 1,$$

whereas (see p.572) $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. So the value of $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ has no relation to the actual value of the sum $\sum\limits_{k=1}^{\infty} a_k$.

3. <u>False:</u> If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive. For a counterexample, consider the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k}$. We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right| = 1.$$

However, $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k}$ being the harmonic series, diverges.

4. <u>False:</u> Since the Root Test works by extracting the nth root, a series that has an nth *power* in its nth term (as opposed to one that has an nth *root* in its nth term) works well with the Root Test (providing, of course, the Root Test is applicable in the first place).

Skill Building

5. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{4k^2-1}{2^k}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{4(n+1)^2-1}{2^{n+1}}}{\frac{4n^2-1}{2^n}} = \frac{2^n}{2^{n+1}} \cdot \left[\frac{4(n^2+2n+1)-1}{4n^2-1}\right] = \frac{1}{2}\left(\frac{4n^2+8n+3}{4n^2-1}\right).$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{2}\left|\frac{4n^2+8n+3}{4n^2-1}\right| = \frac{1}{2}\lim_{n\to\infty}\left|\frac{4+\frac{8}{n}+\frac{3}{n^2}}{4-\frac{1}{n^2}}\right| = \frac{1}{2}\cdot\left|\frac{4+0+0}{4-0}\right| = \frac{1}{2} < 1.$$

Since the limit is less than 1, the series $\boxed{\textit{converges}}$ by the Ratio Test.

6. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{(2k+1)2^k}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(2(n+1)+1)2^{n+1}}}{\frac{1}{(2n+1)2^n}} = \frac{(2n+1)2^n}{(2n+3)2^{n+1}} = \frac{1}{2}\left(\frac{2n+1}{2n+3}\right).$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{2}\left|\frac{2n+1}{2n+3}\right| = \frac{1}{2}\lim_{n\to\infty}\left|\frac{2+\frac{1}{n}}{2+\frac{3}{n}}\right| = \frac{1}{2}\cdot\left|\frac{2+0}{2+0}\right| = \frac{1}{2} < 1.$$

Since the limit is less than 1, the series $\boxed{\textit{converges}}$ by the Ratio Test.

7. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\left(\frac{2}{3}\right)^{n+1}}{n\left(\frac{2}{3}\right)^n} = \left(\frac{2}{3}\right)\frac{n+1}{n}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{2}{3}\right)\left|\frac{n+1}{n}\right| = \lim_{n\to\infty}\frac{2}{3}\cdot\left|1+\frac{1}{n}\right| = \frac{2}{3}\cdot|1+0| = \frac{2}{3} < 1.$$

Since the limit is less than 1, the series $\boxed{\textit{converges}}$ by the Ratio Test.

8. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{5^k}{k^2}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{5^{n+1}}{(n+1)^2}}{\frac{5^n}{n^2}} = \frac{n^2}{(n+1)^2}\cdot\frac{5^{n+1}}{5^n} = 5\frac{n^2}{(n+1)^2}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 5\left|\frac{n^2}{(n+1)^2}\right| = 5\lim_{n\to\infty}\left|\frac{1}{\left(1+\frac{1}{n}\right)^2}\right| = 5\cdot\left|\frac{1}{(1+0)^2}\right| = 5 > 1.$$

By the Ratio Test, since the limit is greater than 1, the series $\boxed{\textit{diverges.}}$

9. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{10^k}{(2k)!}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{10^{n+1}}{(2(n+1))!}}{\frac{10^n}{(2n)!}} = \frac{10^{n+1}}{10^n} \cdot \frac{(2n)!}{(2n+2)!} = 10 \cdot \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{10}{(2n+2)(2n+1)}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{10}{(2n+2)(2n+1)}\right| = 0.$$

Since the limit is less than 1, the series $\boxed{\textit{converges}}$ by the Ratio Test.

10. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(2k)!}{5^k 3^{k-1}}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2(n+1))!}{5^{n+1}.3^{(n+1)-1}}}{\frac{(2n)!}{5^n.3^{n-1}}} = \frac{5^n}{5^{n+1}} \cdot \frac{3^{n-1}}{3^n} \cdot \frac{(2n+2)!}{(2n)!} = \frac{1}{5} \cdot \frac{1}{3} \cdot \frac{(2n+2)(2n+1)(2n)!}{(2n)!} = \frac{1}{15}(2n+2)(2n+1).$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{15}(2n+2)(2n+1)\right| = \infty.$$

Since the limit is ∞, the series $\boxed{\textit{diverges}}$ by the Ratio Test.

11. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k}{(2k-2)!}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{(2(n+1)-2)!}}{\frac{n}{(2n-2)!}} = \frac{n+1}{n} \cdot \frac{(2n-2)!}{(2n)!} = \frac{n+1}{n} \cdot \frac{(2n-2)!}{2n(2n-1)(2n-2)!} = \frac{n+1}{n} \cdot \frac{1}{2n(2n-1)}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n} \cdot \frac{1}{2n(2n-1)}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n}\right| \cdot \lim_{n\to\infty} \left|\frac{1}{2n(2n-1)}\right|$$
$$= \lim_{n\to\infty} \left|1 + \frac{1}{n}\right| \cdot 0 = |1+0| \cdot 0 = 1 \cdot 0 = 0.$$

Since the limit is less than 1, the series $\boxed{\textit{converges}}$ by the Ratio Test.

12. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(k+1)!}{3^k}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+2)!}{3^{n+1}}}{\frac{(n+1)!}{3^n}} = \frac{3^n}{3^{n+1}} \cdot \frac{(n+2)!}{(n+1)!} = \frac{1}{3} \frac{(n+2)(n+1)!}{(n+1)!} = \frac{1}{3}(n+2).$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{3}(n+2)\right| = \infty.$$

By the Ratio Test, since the limit is ∞, the series $\boxed{\textit{diverges.}}$

13. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{2^k}{k(k+1)}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)(n+2)}}{\frac{2^n}{n(n+1)}} = \frac{2^{n+1}}{2^n} \cdot \frac{n(n+1)}{(n+1)(n+2)} = 2 \cdot \frac{n}{n+2}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2n}{n+2}\right| = \lim_{n\to\infty} \left|\frac{2}{1+\frac{2}{n}}\right| = 2 > 1.$$

Since the limit is greater than 1, the series $\boxed{\textit{diverges}}$ by the Ratio Test.

14. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k!}{k^2(k+1)^2}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^2(n+2)^2}}{\frac{n!}{n^2(n+1)^2}} = \frac{(n+1)!}{n!} \cdot \frac{n^2(n+1)^2}{(n+1)^2(n+2)^2} = \frac{(n+1)n!}{n!} \cdot \frac{n^2}{(n+2)^2} = \frac{(n+1)n^2}{(n+2)^2}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)n^2}{(n+2)^2}\right| = \lim_{n\to\infty} \left|(n+1) \cdot \frac{1}{\left(1+\frac{2}{n}\right)^2}\right|$$
$$= \infty.$$

By the Ratio Test, since the limit is ∞, the series $\boxed{\textit{diverges.}}$

15. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^3}{k!}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^3}{(n+1)!}}{\frac{n^3}{n!}} = \frac{n!}{(n+1)!} \cdot \frac{(n+1)^3}{n^3} = \frac{n!}{(n+1)n!} \cdot \frac{(n+1)^3}{n^3} = \frac{1}{n+1} \cdot \frac{(n+1)^3}{n^3}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{n+1} \cdot \frac{(n+1)^3}{n^3}\right| = \lim_{n\to\infty} \left|\frac{1}{n+1}\right| \cdot \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right)^3\right| = |0| \cdot |1| = 0.$$

By the Ratio Test, since the limit is less than 1, the series $\boxed{\textit{converges.}}$

16. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k!}{k^{k+1}}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+2}}}{\frac{n!}{n^{n+1}}} = \frac{(n+1)!}{n!} \cdot \frac{n^{n+1}}{(n+1)^{n+2}} = \frac{(n+1)n!}{n!} \cdot \frac{n^{n+1}}{(n+1)^{n+2}} = \left(\frac{n}{n+1}\right)^{n+1}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1} \cdot \left(\frac{n}{n+1}\right)^n\right| = \lim_{n\to\infty} \left|\frac{1}{1+\frac{1}{n}}\right| \cdot \lim_{n\to\infty} \left|\frac{1}{\left(1+\frac{1}{n}\right)^n}\right| = |1| \cdot \left|\frac{1}{e}\right| = \frac{1}{e} < 1.$$

By the Ratio Test. since the limit is less than 1, the series $\boxed{\textit{converges.}}$

17. $a_k = (-1)^k \frac{k^2}{2^k} \neq 0$ for all $k \geq 1$.
$a_{n+1} = (-1)^{n+1} \frac{(n+1)^2}{2^{n+1}}$ and $a_n = (-1)^n \frac{n^2}{2^n}$.

So

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{n^2+2n+1}{n^2} \cdot \frac{2^n}{2 \cdot 2^n} = \frac{1}{2}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right).$$

Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{1}{2}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\right] = \frac{1}{2}.$$

Since $\frac{1}{2} < 1$, the series $\sum\limits_{k=1}^{\infty} (-1)^k \frac{k^2}{2^k}$ $\boxed{converges}$.

18. $a_k = (-1)^k \frac{k}{e^k} \neq 0$ for all $k \geq 0$.
$a_{n+1} = (-1)^{n+1} \frac{n+1}{e^{n+1}}$ and $a_n = (-1)^n \frac{n}{e^n}$.

So

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}} = \frac{n+1}{e^{n+1}} \cdot \frac{e^n}{n} = \frac{n+1}{n} \cdot \frac{e^n}{e \cdot e^n} = \frac{1}{e}\left(1 + \frac{1}{n}\right).$$

Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{e}\left(1 + \frac{1}{n}\right) = \frac{1}{e}.$$

Since $\frac{1}{e} < 1$, the series $\sum\limits_{k=1}^{\infty} (-1)^k \frac{k}{e^k}$ $\boxed{converges}$.

19. $a_k = (-1)^k \frac{\ln k}{3^k} \neq 0$ for all $k \geq 2$.
$a_{n+1} = (-1)^{n+1} \frac{\ln(n+1)}{3^n}$ and $a_n = (-1)^n \frac{\ln n}{3^n}$.

So

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{\ln(n+1)}{3^{n+1}}}{\frac{\ln n}{3^n}} = \frac{\ln(n+1)}{3^{n+1}} \cdot \frac{3^n}{\ln n} = \cdot \frac{3^n}{3 \cdot 3^n} = \frac{1}{3} \cdot \frac{\ln(n+1)}{\ln n}$$

Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{1}{3} \cdot \frac{\ln(n+1)}{\ln n}\right].$$

This is an indeterminate limit of the form $\frac{\infty}{\infty}$, so we use L'Hôpital's Rule:

$$\lim_{n\to\infty} \left[\frac{1}{3} \cdot \frac{\ln(n+1)}{\ln n}\right] = \lim_{n\to\infty} \left(\frac{1}{3} \cdot \frac{\frac{1}{n+1}}{\frac{1}{n}}\right) = \lim_{n\to\infty} \left(\frac{1}{3} \cdot \frac{n}{n+1}\right) \lim_{n\to\infty} \left(\frac{1}{3} \cdot \frac{1}{1+\frac{1}{n}}\right) = \frac{1}{3}$$

Since $\frac{1}{3} < 1$, the series $\sum\limits_{k=2}^{\infty} (-1)^k \frac{\ln k}{3^k}$ $\boxed{converges}$.

20. $a_k = (-1)^k \frac{\ln k}{k!} \neq 0$ for all $k \geq 2$.
$a_{n+1} = (-1)^{n+1} \frac{\ln(n+1)}{(n+1)!}$ and $a_n = (-1)^n \frac{\ln n}{n!}$.

So

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{\ln(n+1)}{(n+1)!}}{\frac{\ln n}{n!}} = \frac{\ln(n+1)}{(n+1)!} \cdot \frac{n!}{\ln n} = \frac{\ln(n+1)}{\ln n} \cdot \frac{n!}{(n+1)!} = \frac{\ln(n+1)}{\ln n} \cdot \frac{n!}{(n+1) \cdot n!}$$
$$= \frac{\ln(n+1)}{\ln n} \cdot \frac{1}{n+1}.$$

Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{\ln(n+1)}{\ln n} \cdot \frac{1}{n+1}\right] = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \cdot \lim_{n\to\infty} \frac{1}{n+1}.$$

$\lim\limits_{n\to\infty} \frac{\ln(n+1)}{\ln n}$ is in the indeterminate form $\frac{\infty}{\infty}$, so L'Hôpital's Rule is applicable.

$$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$
$$\lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Therefore

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \cdot \lim_{n\to\infty} \frac{1}{n+1} = 1 \cdot 0 = 0.$$

Since $0 < 1$, the series $\sum\limits_{k=2}^{\infty} (-1)^k \frac{\ln k}{k!}$ $\boxed{\textit{converges}}$.

21. $a_k = (-1)^k \frac{5^k}{3^{k-1}} \neq 0$ for all $k \geq 1$
$a_{n+1} = (-1)^{n+1} \frac{5^{n+1}}{3^n} a_n = (-1)^n \frac{5^n}{3^{n-1}}$ and $a_n = (-1)^n \frac{5^n}{3^{n-1}}$.

So

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{5^{n+1}}{3^n}}{\frac{5^n}{3^{n-1}}} = \frac{5^{n+1}}{3^n} \cdot \frac{3^{n-1}}{5^n} = \frac{5 \cdot 5^n}{5^n} \cdot \frac{3^{n-1}}{3 \cdot 3^{n-1}} = \frac{5}{3}.$$

Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{5}{3} = \frac{5}{3}.$$

Since $\frac{5}{3} > 1$, the series $\sum\limits_{k=1}^{\infty} (-1)^k \frac{5^k}{3^{k-1}}$ $\boxed{\textit{diverges}}$.

22. $a_k = (-1)^k \frac{3^k+4}{2^{k-1}} \neq 0$ for all $k > 0$.
$a_{n+1} = (-1)^{n+1} \frac{3^{n+1}+4}{2^{(n+1)-1}}$ and $a_n = (-1)^n \frac{3^n+4}{2^{n-1}}$.

So

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{3^{n+1}+4}{2^n}}{\frac{3^n+4}{2^{n-1}}} = \frac{3^{n+1}+4}{2^n} \cdot \frac{2^{n-1}}{3^n+4} = \frac{3 \cdot 3^n + 4}{2^n} \cdot \frac{2^n}{2(3^n+4)} = \frac{3 \cdot 3^n + 4}{2 \cdot 3^n + 8}.$$

Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3\cdot 3^n+4}{2\cdot 3^n+8} = \lim_{n\to\infty}\frac{3+\frac{4}{3^n}}{2+\frac{8}{3^n}} = \frac{3}{2} > 1.$$

Since $\frac{3}{2} > 1$, the series $\sum\limits_{k=1}^{\infty}(-1)^k\frac{3^k+4}{2^{k-1}}$ $\boxed{\textit{diverges}}$.

23. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty}\frac{3^{k-1}}{k\cdot 2^k}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{3^n}{(n+1)\cdot 2^{n+1}}}{\frac{3^{n-1}}{n\cdot 2^n}} = \frac{3^n}{3^{n-1}}\cdot\frac{2^n}{2^{n+1}}\cdot\frac{n}{n+1} = 3\cdot\frac{1}{2}\cdot\frac{n}{n+1}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3}{2}\cdot\frac{n}{n+1}\right| = \frac{3}{2}\lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right| = \frac{3}{2}\cdot|1| = \frac{3}{2} > 1.$$

By the Ratio Test, since the limit is greater than 1, the series $\boxed{\textit{diverges.}}$

24. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty}\frac{k(k+2)}{3^k}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)(n+3)}{3^{n+1}}}{\frac{n(n+2)}{3^n}} = \frac{3^n}{3^{n+1}}\cdot\frac{(n+1)(n+3)}{n(n+2)} = \frac{1}{3}\cdot\frac{(n+1)(n+3)}{n(n+2)}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{3}\cdot\frac{(n+1)(n+3)}{n(n+2)}\right| = \frac{1}{3}\lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{n}\right)}{\left(1+\frac{2}{n}\right)}\right| = \frac{1}{3}\cdot|1| = \frac{1}{3} < 1.$$

By the Ratio Test, since the limit is less than 1, the series $\boxed{\textit{converges.}}$

25. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty}\frac{k}{e^k}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}} = \frac{e^n}{e^{n+1}}\cdot\frac{n+1}{n} = \frac{1}{e}\cdot\frac{n+1}{n}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{e}\cdot\frac{n+1}{n}\right| = \frac{1}{e}\lim_{n\to\infty}\left|1+\frac{1}{n}\right| = \frac{1}{e}\cdot|1| = \frac{1}{e} < 1.$$

By the Ratio Test, since the limit is less than 1, the series $\boxed{\textit{converges.}}$

26. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty}\frac{e^k}{k^3}$ is a seris of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{e^{n+1}}{(n+1)^3}}{\frac{e^n}{n^3}} = \frac{e^{n+1}}{e^n}\cdot\frac{n^3}{(n+1)^3} = e\cdot\frac{n^3}{(n+1)^3}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|e\cdot\frac{n^3}{(n+1)^3}\right| = e\lim_{n\to\infty}\left|\frac{1}{\left(1+\frac{1}{n}\right)^3}\right| = e\cdot|1| = e > 1.$$

By the Ratio Test, since the limit is greater than 1, the series $\boxed{\textit{diverges.}}$

27. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} k\cdot 2^k$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\cdot 2^{n+1}}{n\cdot 2^n} = 2\cdot\frac{n+1}{n}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|2\cdot\frac{n+1}{n}\right| = 2\lim_{n\to\infty}\left|1+\frac{1}{n}\right| = 2\cdot|1| = 2 > 1.$$

Since the limit is greater than 1, by the Ratio Test, the series $\boxed{\textit{diverges.}}$

28. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{4^k}{k}$ is a series of nonzero terms. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{4^{n+1}}{n+1}}{\frac{4^n}{n}} = \frac{4^{n+1}}{4^n}\cdot\frac{n}{n+1} = 4\cdot\frac{n}{n+1}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|4\cdot\frac{n}{n+1}\right| = 4\lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right| = 4\cdot|1| = 4 > 1.$$

Since the limit is greater than 1, by the Ratio Test, the series $\boxed{\textit{diverges.}}$

29. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \left(\frac{2k+1}{5k+1}\right)^k$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{2n+1}{5n+1}\right)^n\right|} = \lim_{n\to\infty}\left|\frac{2n+1}{5n+1}\right| = \lim_{n\to\infty}\left|\frac{2+\frac{1}{n}}{5+\frac{1}{n}}\right| = \left|\frac{2}{5}\right| = \frac{2}{5} < 1.$$

Since the limit is less than 1, by the Root Test, the series $\boxed{\textit{converges.}}$

30. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \left(\frac{3k-1}{2k+1}\right)^k$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{3n-1}{2n+1}\right)^n\right|} = \lim_{n\to\infty}\left|\frac{3n-1}{2n+1}\right| = \lim_{n\to\infty}\left|\frac{3-\frac{1}{n}}{2+\frac{1}{n}}\right| = \left|\frac{3}{2}\right| = \frac{3}{2} > 1.$$

Since the limit is greater than 1, by the Root Test, the series $\boxed{\textit{diverges.}}$

31. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \left(\frac{k}{5}\right)^k$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{5}\right)^n\right|} = \lim_{n\to\infty} \left|\frac{n}{5}\right| = \infty.$$

Since the limit is greater than 1, by the Root Test, the series $\boxed{\textit{diverges.}}$

32. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\pi^{2k}}{k^k}$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{\pi^{2n}}{n^n}\right|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{\pi^2}{n}\right|^n} = \lim_{n\to\infty} \left|\frac{\pi^2}{n}\right| = 0.$$

Since the limit is less than 1. by the Root Test, the series $\boxed{\textit{converges.}}$

33. $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \left(\frac{\ln k}{k}\right)^k$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{\ln n}{n}\right)^n\right|} = \lim_{n\to\infty} \left|\frac{\ln n}{n}\right| = \lim_{x\to\infty} \left|\frac{\ln x}{x}\right| = \lim_{x\to\infty} \left|\frac{\frac{1}{x}}{1}\right| = \lim_{x\to\infty} \left|\frac{1}{x}\right| = 0,$$

using L'Hôpital's rule on a related function of the nth root of the nth term of the series. Since the limit is less than 1, by the Root Test, the series $\boxed{\textit{converges.}}$

34. $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \left(\frac{1}{\ln k}\right)^k$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{1}{\ln n}\right)^n\right|} = \lim_{n\to\infty} \left|\frac{1}{\ln n}\right| = 0.$$

Since the limit is less than 1, by the Root Test, the series $\boxed{\textit{converges.}}$

35. The nth term of $\sum\limits_{k=2}^{\infty} (-1)^k \frac{(\ln k)^k}{2^k}$ is $a_n = (-1)^n \frac{(\ln n)^n}{2^n} \neq 0$ for $n \geq 2$.

Since a_n involves an nth power, we use the Root Test.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|(-1)^n \frac{(\ln n)^n}{2^n}\right|} = \lim_{n\to\infty} \frac{\ln n}{2} = \infty > 1.$$

The series $\sum\limits_{k=2}^{\infty} (-1)^k \frac{(\ln k)^k}{2^k}$ $\boxed{\textit{diverges}}$.

36. The nth term of $\sum\limits_{k=2}^{\infty} (-1)^k \frac{2^{k+1}}{e^k}$ is $a_n = (-1)^n \frac{2^{n+1}}{e^n} \neq 0$ for $n \geq 1$.

Since a_n involves an nth power, we use the Root Test.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|(-1)^n \frac{2^{n+1}}{e^n}\right|} = \lim_{n\to\infty} \frac{2^{1+\frac{1}{n}}}{e} = \frac{2}{e} < 1.$$

The series $\sum\limits_{k=1}^{\infty}(-1)^k\frac{2^{k+1}}{e^k}$ $\boxed{\textit{converges}}$.

37. The nth term of $\sum\limits_{c=1}^{\infty}(-1)^k\left(\frac{k+1}{k}\right)^k$ is $a_n=(-1)^n\left(\frac{n+1}{n}\right)^n\neq 0$ for $n\geq 1$.

Since a_n involves an nth power, we use the Root Test.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|(-1)^n\left(\frac{n+1}{n}\right)^n\right|}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)=1.$$

The Root Test is $\boxed{\textit{inconclusive}}$ for the series $\sum\limits_{c=1}^{\infty}(-1)^k\left(\frac{k+1}{k}\right)^k$.

38. The nth term of $\sum\limits_{c=1}^{\infty}(-1)^k\left(\frac{2k+3}{k+1}\right)^k$ is $a_n=(-1)^n\left(\frac{2n+3}{n+1}\right)^n\neq 0$ for $n\geq 1$.

Since a_n involves an nth power, we use the Root Test.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|(-1)^n\left(\frac{2n+3}{n+1}\right)^n\right|}=\lim_{n\to\infty}\frac{2n+3}{n+1}=2>1.$$

The series $\sum\limits_{k=1}^{\infty}(-1)^k\left(\frac{2k+3}{k+1}\right)^k$ $\boxed{\textit{diverges}}$.

39. The nth term of $\sum\limits_{c=1}^{\infty}(-1)^k\frac{k^k}{e^{2k}}$ is $a_n=(-1)^n\frac{n^n}{e^{2n}}\neq 0$ for $n\geq 1$.

Since a_n involves an nth power, we use the Root Test.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|(-1)^n\frac{(n)^n}{e^{2n}}\right|}=\lim_{n\to\infty}\frac{n}{e^2}=\frac{1}{e^2}\cdot\lim_{n\to\infty}n=\infty>1.$$

The series $\sum\limits_{k=1}^{\infty}(-1)^k\frac{k^k}{e^{2k}}$ $\boxed{\textit{diverges}}$.

40. The nth term of $\sum\limits_{k=2}^{\infty}(-1)^k\frac{k\ln k}{e^k}$ is $a_n=(-1)^n\frac{n\ln n}{e^n}\neq 0$ for $n\geq 2$.

Since a_n involves an nth power, we use the Root Test.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|(-1)^n\frac{n\ln n}{e^n}\right|}=\lim_{n\to\infty}\sqrt[n]{\left[\frac{(n\ln n)^{1/n}}{e}\right]^n}=\lim_{n\to\infty}\frac{(n\ln n)^{1/n}}{e}$$
$$=\frac{1}{e}\cdot\lim_{n\to\infty}(n\ln n)^{1/n}=\frac{1}{e}\cdot\lim_{n\to\infty}\left[n^{1/n}\cdot(\ln n)^{1/n}\right]=\frac{1}{e}\cdot\lim_{n\to\infty}n^{1/n}\cdot\lim_{n\to\infty}(\ln n)^{1/n}.$$

We consider the two limits separately:

For $\lim\limits_{n\to\infty}n^{1/n}$ let $y=n^{1/n}$, so $\ln y=\ln\left(n^{1/n}\right)=\frac{\ln n}{n}$.

Now $\lim\limits_{n\to\infty}\ln y=\lim\limits_{n\to\infty}\frac{\ln n}{n}$, which is indeterminate of the form $\frac{\infty}{\infty}$, so we use L'Hôpital's Rule:

$$\lim_{n\to\infty}\frac{\ln n}{n}=\lim_{n\to\infty}\frac{\left(\frac{1}{n}\right)}{1}=\lim_{n\to\infty}\frac{1}{n}=0.$$

Then

$$\lim_{n\to\infty}n^{1/n}=\lim_{n\to\infty}y=\lim_{n\to\infty}e^{\ln y}=e^{\lim\limits_{n\to\infty}\ln y}=e^0=1.$$

For $\lim\limits_{n\to\infty}(\ln n)^{1/n}$, let $z=(\ln n)^{1/n}$, so $\ln z=\ln(\ln n)^{1/n}=\frac{\ln(\ln n)}{n}$.

Now $\lim\limits_{n\to\infty}\ln z=\lim\limits_{n\to\infty}\frac{\ln(\ln n)}{n}$, which is indeterminate of the form $\frac{\infty}{\infty}$, so we use L'Hôpital's Rule:

$$\lim_{n\to\infty}\frac{\ln(\ln n)}{n}=\lim_{n\to\infty}\frac{\frac{1}{\ln n}\cdot\frac{1}{n}}{1}=\lim_{n\to\infty}\frac{1}{n\ln n}=0.$$

Then

$$\lim_{n\to\infty}(\ln n)^{1/n}=\lim_{n\to\infty}z=\lim_{n\to\infty}e^{\ln z}=e^{\lim\limits_{n\to\infty}\ln z}=e^0=1.$$

Therefore

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\frac{1}{e}\cdot\lim_{n\to\infty}n^{1/n}\cdot\lim_{n\to\infty}(\ln n)^{1/n}=\frac{1}{3}\cdot 1\cdot 1=\frac{1}{e}<1.$$

The series $\boxed{\textit{converges}}$.

41. $\sum\limits_{k=1}^{\infty}a_k=\sum\limits_{k=1}^{\infty}\left(\frac{\sqrt{k^2+1}}{3k}\right)^k$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{\sqrt{n^2+1}}{3n}\right)^n\right|}=\lim_{n\to\infty}\left|\frac{\sqrt{n^2+1}}{3n}\right|=\lim_{n\to\infty}\left|\sqrt{\frac{1}{9}+\frac{1}{9n^2}}\right|=\left|\sqrt{\frac{1}{9}}\right|=\frac{1}{\sqrt{9}}=\frac{1}{3}<1.$$

Since the limit is less than 1, by the Root Test, the series $\boxed{\textit{converges.}}$

42. $\sum\limits_{k=1}^{\infty}a_k=\sum\limits_{k=1}^{\infty}\left(\frac{\sqrt{4k^2+1}}{k}\right)^k$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt{\left|\left(\frac{\sqrt{4n^2+1}}{n}\right)^n\right|}=\lim_{n\to\infty}\left|\frac{\sqrt{4n^2+1}}{n}\right|=\lim_{n\to\infty}\left|\sqrt{4+\frac{1}{n^2}}\right|=\left|\sqrt{4}\right|=2>1.$$

Since the limit is greater than 1, by the Root Test, the series $\boxed{\textit{diverges.}}$

43. $\sum\limits_{k=1}^{\infty}a_k=\sum\limits_{k=1}^{\infty}\frac{k^2}{2^k}$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\frac{n^2}{2^n}\right|}=\lim_{n\to\infty}\left|\frac{n^{2/n}}{2}\right|=\frac{1}{2}\lim_{n\to\infty}\left(n^{1/n}\right)^2=\frac{1}{2}\left(\lim_{n\to\infty}n^{1/n}\right)^2=\frac{1}{2}\cdot 1^2=\frac{1}{2}<1.$$

(Here, and in the next three problems numbered 44, 45, and 46, and elsewhere below, we use the following result: $\lim\limits_{n\to\infty}n^{1/n}=1$. We can derive this as follows. Let $y=n^{1/n}$. Then $\ln y=\frac{1}{n}\ln n=\frac{\ln n}{n}$. We have

$$\lim_{n\to\infty}y=\lim_{n\to\infty}e^{\ln y}=\lim_{n\to\infty}e^{\frac{\ln n}{n}}=e^{\lim\limits_{n\to\infty}\frac{\ln n}{n}}=e^{\lim\limits_{x\to\infty}\frac{\ln x}{x}}=e^{\lim\limits_{x\to\infty}\frac{1/x}{1}}=e^{\lim\limits_{x\to\infty}\frac{1}{x}}=e^0=1,$$

using L'Hôpital's rule on a related function of $\frac{\ln n}{n}$. This proves the claimed result.)
Since the limit is less than 1, by the Root Test, the series $\boxed{\textit{converges.}}$

44. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k^3}{3^k}$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^3}{3^n}\right|} = \lim_{n\to\infty} \left|\frac{n^{3/n}}{3}\right| = \frac{1}{3} \lim_{n\to\infty} \left(n^{1/n}\right)^3 = \frac{1}{3}\left(\lim_{n\to\infty} n^{1/n}\right)^3 = \frac{1}{3}\cdot 1^3 = \frac{1}{3} < 1,$$

where we used the result (see Problem 43) $\lim\limits_{n\to\infty} n^{1/n} = 1$. Since the limit is less than 1, by the Root Test, the series $\boxed{\textit{converges.}}$

45. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k^4}{5^k}$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^4}{5^n}\right|} = \lim_{n\to\infty} \left|\frac{n^{4/n}}{5}\right| = \frac{1}{5} \lim_{n\to\infty} \left(n^{1/n}\right)^4 = \frac{1}{5}\left(\lim_{n\to\infty} n^{1/n}\right)^4 = \frac{1}{5}\cdot 1^4 = \frac{1}{5} < 1,$$

where we used the result (see Problem 43) $\lim\limits_{n\to\infty} n^{1/n} = 1$. Since the limit is less than 1, by the Root Test, the series $\boxed{\textit{converges.}}$

46. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k}{3^k}$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{3^n}\right|} = \lim_{n\to\infty} \left|\frac{n^{1/n}}{3}\right| = \frac{1}{3} \lim_{n\to\infty} n^{1/n} = \frac{1}{3}\cdot 1 = \frac{1}{3} < 1,$$

where we used the result (see Problem 43) $\lim\limits_{n\to\infty} n^{1/n} = 1$. Since the limit is less than 1, by the Root Test, the series $\boxed{\textit{converges.}}$

47. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{10}{(3k+1)^k}$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{10}{(3n+1)^n}\right|} = \lim_{n\to\infty} \left|\frac{10^{1/n}}{3n+1}\right| = \lim_{n\to\infty} 10^{1/n} \cdot \lim_{n\to\infty} \frac{1}{3n+1} = 10^0 \cdot 0 = 0.$$

Since the limit is less than 1, by the Root Test, the series $\boxed{\textit{converges.}}$

48. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{k^2}$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(1+\frac{1}{n}\right)^{n^2}\right|} = \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right)^{n}\right| = e > 1.$$

Since the limit is greater than 1, by the Root Test, the series $\boxed{\textit{diverges.}}$

49. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(k+1)(k+2)}{k!}$ is a series of nonzero terms. Since the nth term involves the factorial, we could try the Ratio Test. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+2)(n+3)}{(n+1)!}}{\frac{(n+1)(n+2)}{n!}} = \frac{n!}{(n+1)!} \cdot \frac{(n+3)}{(n+1)} = \frac{n!}{(n+1)n!} \cdot \frac{(n+3)}{(n+1)} = \frac{(n+3)}{(n+1)^2}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+3)}{(n+1)^2}\right| = \lim_{n\to\infty} \left|\frac{n\left(1+\frac{3}{n}\right)}{n^2\left(1+\frac{1}{n}\right)^2}\right| = \lim_{n\to\infty} \left|\frac{1}{n}\right| \cdot \lim_{n\to\infty} \left|\frac{\left(1+\frac{3}{n}\right)}{\left(1+\frac{1}{n}\right)^2}\right| = 0 \cdot 1 = 0.$$

Since the limit is less than 1, by the Ratio Test, the series $\boxed{\textit{converges.}}$

50. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k!}{(3k+1)!}$ is a series of nonzero terms. Since the nth term involves the factorial, we could try the Ratio Test. The ratio of the $n+1$st and the nth term is

$$\begin{aligned} \frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(3(n+1)+1)!}}{\frac{n!}{(3n+1)!}} &= \frac{(3n+1)!}{(3n+4)!} \cdot \frac{(n+1)!}{n!} \\ &= \frac{(3n+1)!}{(3n+4)(3n+3)(3n+2)(3n+1)!} \cdot \frac{(n+1)n!}{n!} \\ &= \frac{(n+1)}{(3n+4)(3n+3)(3n+2)}. \end{aligned}$$

Compute the limit of the absolute value of the ratio found above:

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)}{(3n+4)(3n+3)(3n+2)}\right| = \lim_{n\to\infty} \left|\frac{n\left(1+\frac{1}{n}\right)}{n^3\left(3+\frac{4}{n}\right)\left(3+\frac{3}{n}\right)\left(3+\frac{2}{n}\right)}\right| \\ &= \lim_{n\to\infty} \left|\frac{1}{n^2}\right| \cdot \lim_{n\to\infty} \left|\frac{\left(1+\frac{1}{n}\right)}{\left(3+\frac{4}{n}\right)\left(3+\frac{3}{n}\right)\left(3+\frac{2}{n}\right)}\right| \\ &= 0 \cdot \frac{1}{27} = 0. \end{aligned}$$

Since the limit is less than 1, by the Ratio Test, the series $\boxed{\textit{converges.}}$

51. The given series can be written as $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k\ln k}{2^k} = 0 + \sum\limits_{k=2}^{\infty} \frac{k\ln k}{2^k} = \sum\limits_{k=2}^{\infty} \frac{k\ln k}{2^k}$ which is a series of nonzero terms. Since the nth term involves an nth power, we could apply the Root Test. Let us first establish a limit result: $\lim\limits_{n\to\infty} (\ln n)^{1/n} = 1$. To prove this, set $y = (\ln n)^{1/n}$. Then, $\ln y = \frac{1}{n}\ln(\ln n) = \frac{\ln(\ln n)}{n}$. We have

$$\lim_{n\to\infty} y = \lim_{n\to\infty} e^{\ln y} = \lim_{n\to\infty} e^{\frac{\ln(\ln n)}{n}} = e^{\lim\limits_{n\to\infty} \frac{\ln(\ln n)}{n}} = e^{\lim\limits_{x\to\infty} \frac{\ln(\ln x)}{x}} = e^{\lim\limits_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1}} = e^{\lim\limits_{x\to\infty} \frac{1}{x\ln x}} = e^0 = 1,$$

using L'Hôpital's rule on a related function of $\frac{\ln(\ln n)}{n}$. This proves the required result. Applying the Root Test, we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n\ln n}{2^n}\right|} = \lim_{n\to\infty} \left|\frac{n^{1/n}(\ln n)^{1/n}}{2}\right| = \frac{1}{2} \lim_{n\to\infty} n^{1/n} \cdot \lim_{n\to\infty} (\ln n)^{1/n} = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2} < 1.$$

(Here, we also used the result, proved in Problem 43, that $\lim\limits_{n\to\infty} n^{1/n} = 1$, in addition to the limit result proved above.) Since the limit is less than 1, by the Root Test, the series $\boxed{\textit{converges.}}$

The series $\sum\limits_{k=2}^{\infty} \frac{k \ln k}{2^k}$ which is a series of nonzero terms, can also be seen to converge using the Ratio Test. The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)\ln(n+1)}{2^{n+1}}}{\frac{n \ln n}{2^n}} = \frac{2^n}{2^{n+1}} \cdot \frac{n+1}{n} \cdot \frac{\ln(n+1)}{\ln n} = \frac{1}{2} \cdot \frac{n+1}{n} \cdot \frac{\ln(n+1)}{\ln n}.$$

Compute the limit of the absolute value of the ratio found above:

$$\begin{aligned} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{1}{2} \cdot \frac{n+1}{n} \cdot \frac{\ln(n+1)}{\ln n}\right| = \frac{1}{2} \lim_{n\to\infty} \left|1 + \frac{1}{n}\right| \cdot \lim_{n\to\infty} \left|\frac{\ln(n+1)}{\ln n}\right| \\ &= \frac{1}{2} \cdot 1 \cdot \lim_{x\to\infty} \left|\frac{\ln(x+1)}{\ln x}\right| = \frac{1}{2} \lim_{x\to\infty} \left|\frac{\frac{1}{x+1}}{1}\right| \\ &= \frac{1}{2} \lim_{x\to\infty} \left|\frac{1}{x+1}\right| = \frac{1}{2} \cdot 0 = 0, \end{aligned}$$

using L'Hôpital's rule on a related function of the last ratio. Since the limit is less than 1, by the Ratio Test, the series $\boxed{\textit{converges}}$, in agreement with the Root Test above.

52. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \left[\ln\left(e^3 + \frac{1}{k}\right)\right]^k$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\left|\left[\ln\left(e^3 + \frac{1}{n}\right)\right]^n\right|} = \lim_{n\to\infty} \left|\ln\left(e^3 + \frac{1}{n}\right)\right| \\ &= \ln\left[\lim_{n\to\infty} e^3 + \lim_{n\to\infty} \frac{1}{n}\right] \\ &= \ln\left[e^3 + 0\right] = 3 > 1. \end{aligned}$$

Since the limit is greater than 1, by the Root Test, the series $\boxed{\textit{diverges.}}$

53. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \sin^k\left(\frac{1}{k}\right)$ is a series of nonzero terms (since $\sin\theta$ is nonvanishing on the interval $0 < \theta \le 1$ rad). Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\sin^n\left(\frac{1}{n}\right)\right|} = \lim_{n\to\infty} \left|\sin\left(\frac{1}{n}\right)\right| = \left|\sin\left\{\lim_{n\to\infty}\left(\frac{1}{n}\right)\right\}\right| = |\sin 0| = 0.$$

Since the limit is less than 1, by the Root Test, the series $\boxed{\textit{converges.}}$

54. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k^k}{2^{k^2}}$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^n}{2^{n^2}}\right|} = \lim_{n\to\infty} \left|\frac{n^{n/n}}{2^{n^2/n}}\right| = \lim_{n\to\infty} \left|\frac{n}{2^n}\right| = \lim_{x\to\infty} \left|\frac{x}{2^x}\right| = \lim_{x\to\infty} \left|\frac{1}{2^x(\ln 2)}\right| = 0,$$

applying L'Hôpital's rule to a related function of the nth root of the nth term of the series. Since the limit is less than 1, by the Root Test, the series $\boxed{\textit{converges.}}$

55. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^{2k}}{e^k}$ is a series of nonzero terms. Since the nth term involves an nth power, we apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{\left(1+\frac{1}{n}\right)^{2n}}{e^n}\right|} = \lim_{n\to\infty} \left|\frac{\left(1+\frac{1}{n}\right)^2}{e}\right| = \frac{1}{e}\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right]^2 = \frac{1}{e}\cdot 1^2 = \frac{1}{e} < 1.$$

Since the limit is less than 1, by the Root Test, the series $\boxed{\textit{converges.}}$

56. $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{2^k(k+1)}{k^2(k+2)}$ is a series of nonzero terms. It is not clear which test is the better one. The presence of an nth power in the nth term indicates the Root Test; however the linear factors might be easier to handle with the Ratio Test. Let us try the problem both ways.
Ratio Test: The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}(n+2)}{(n+1)^2(n+3)}}{\frac{2^n(n+1)}{n^2(n+2)}} = \frac{2^{n+1}}{2^n}\cdot\frac{n^2(n+2)^2}{(n+1)^3(n+3)} = 2\cdot\frac{n^2(n+2)^2}{(n+1)^3(n+3)}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|2\cdot\frac{n^2(n+2)^2}{(n+1)^3(n+3)}\right| = 2\lim_{n\to\infty}\left|\frac{\left(1+\frac{2}{n}\right)^2}{\left(1+\frac{1}{n}\right)^3\left(1+\frac{3}{n}\right)}\right| = 2\cdot 1 = 2 > 1.$$

Since the limit is greater than 1, the series $\boxed{\textit{diverges}}$ by the Ratio Test.

Root Test: Examine the limit of the absolute value of the nth root of the nth term:

$$\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{|a_n|} &= \lim_{n\to\infty}\sqrt[n]{\left|\frac{2^n(n+1)}{n^2(n+2)}\right|} = \lim_{n\to\infty}\left|\frac{2}{n^{2/n}}\cdot\frac{(n+1)^{1/n}}{(n+2)^{1/n}}\right| \\
&= 2\lim_{n\to\infty}\left(\frac{1}{n^{1/n}}\right)^2\cdot\lim_{n\to\infty}\left|\frac{n^{1/n}\cdot\left(1+\frac{1}{n}\right)^{1/n}}{n^{1/n}\cdot\left(1+\frac{2}{n}\right)^{1/n}}\right| \qquad (1.1)\\
&= 2\cdot 1^2\cdot\left|\frac{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{1/n}}{\lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right)^{1/n}}\right| \\
&= 2\cdot 1^2\cdot\frac{1}{1} = 2 > 1,
\end{aligned}$$

where the limit result from Problem 43 has been used to address the first factor's limit (in equation 8.1). Since the limit is greater than 1, the series $\boxed{\textit{diverges}}$ by the Root test, in agreement with the analysis by the Ratio Test.

Applications and Extensions

57. For $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k}$, the divergent harmonic series (whose divergence can be established using the Integral Test), we have applying the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \lim_{n\to\infty}\left|\frac{1}{\left(1+\frac{1}{n}\right)}\right| = \left|\frac{1}{1}\right| = 1.$$

So the Ratio Test is inconclusive, even though we know by other means that the series diverges.

58. For $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$, a convergent p-series (since $p = 2 > 1$), we have applying the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| = \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\right| = \lim_{n\to\infty}\left|\frac{1}{\left(1+\frac{1}{n}\right)^2}\right| = \left|\frac{1}{1^2}\right| = 1.$$

So the Ratio Test is inconclusive, even though we know by other means that the series converges.

59. Consider the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{|\sin k|}{2^k}$. This series converges, because the nth term satisfies

$$a_n = \frac{|\sin n|}{2^n} \leq \frac{1}{2^n}.$$

Comparing with the convergent geometric series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{2^k}$ (convergent since $|r| = \frac{1}{2} < 1$), we see by the Comparison Test that the series $\sum\limits_{k=1}^{\infty} a_k$ indeed converges. However, computing the limit of the ratio of the absolute value of the $n+1$st and the nth term gives:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{|\sin(n+1)|}{2^{n+1}}}{\frac{|\sin n|}{2^n}}\right| = \lim_{n\to\infty}\left|\frac{2^n}{2^{n+1}}\cdot\frac{\sin(n+1)}{\sin n}\right| \\ &= \frac{1}{2}\lim_{n\to\infty}\left|\frac{\sin n\cos 1 + \cos n \sin 1}{\sin n}\right| \\ &= \frac{1}{2}\lim_{n\to\infty}|\cos 1 + \cot n \sin 1|,\end{aligned}$$

where the addition formula $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ has been used to simplify the limit. As $n \to \infty$, the limit of $\cot n$ does not exist, as on any interval $[n, n+\pi]$, $\cot n$ takes values ranging from $-\infty$ to $+\infty$. So, we have an example of a series $\sum\limits_{k=1}^{\infty} a_k$ where $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \neq \infty$ does not exist despite its being a convergent series.

60. A slight modification of the convergent series in Problem 59 produces an example of a divergent series without a well-defined limit. Consider the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} |\sin k|$. That this series is divergent is seen examining the limit of the nth term: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} |\sin n|$, which does not exist as on any interval $[n, n+2\pi]$, the function $\sin n$ takes all values from -1 to $+1$. Since $\lim\limits_{n\to\infty} a_n \neq 0$, the series $\sum\limits_{k=1}^{\infty} a_k$ is divergent as claimed, using the Divergence Test. Computing the ratio of the absolute value of the $n+1$st and the nth term gives:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sin(n+1)}{\sin n}\right| = \lim_{n\to\infty}\left|\frac{\sin n\cos 1 + \cos n\sin 1}{\sin n}\right| = \lim_{n\to\infty}|\cos 1 + \cot n\sin 1|,$$

where the addition formula $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ has been used to simplify the limit. As $n \to \infty$, the limit of $\cot n$ does not exist, as on any interval $[n, n+\pi]$, $\cot n$ takes values ranging from $-\infty$ to $+\infty$. So, we have an example of a series $\sum\limits_{k=1}^{\infty} a_k$ where $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \neq \infty$ does not exist despite its being a divergent series.

61. (a) To apply the Ratio Test to the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(-1)^k 3^k}{k!}$ (note that the series has nonzero terms) we find the ratio of the $n+1$st and the nth term to be

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(-1)^{n+1}3^{n+1}}{(n+1)!}}{\frac{(-1)^n 3^n}{n!}} = \frac{(-1)^n(-1)}{(-1)^n} \cdot \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)n!} = (-1)\cdot 3 \cdot \frac{1}{n+1} = -\frac{3}{n+1}.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|-\frac{3}{n+1}\right| = |-3| \cdot \lim_{n\to\infty} \left|\frac{1}{n+1}\right| = 3 \cdot 0 = 0.$$

Since the limit is less than 1, the series $\boxed{\textit{converges}}$ by the Ratio Test.

(b) Using a CAS, we directly evaluate the sum to be $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(-1)^k 3^k}{k!} = \boxed{\frac{1-e^3}{e^3}.}$

62. The nth term of the series $\frac{1}{3} - \frac{2^3}{3^2} + \frac{3^3}{3^3} - \frac{4^3}{3^4} + \cdots + \frac{(-1)^{n-1}n^3}{3^n} + \cdots$ is $a_n = \frac{(-1)^{n-1}n^3}{3^n}$. The ratio of the $n+1$st and the nth term of the series is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(-1)^{n+1-1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^{n-1}n^3}{3^n}} = \frac{(-1)^n}{(-1)^{n-1}} \cdot \frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}} = (-1)\cdot\frac{1}{3}\cdot\left(\frac{n+1}{n}\right)^3.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)}{3}\left(\frac{n+1}{n}\right)^3\right| = \frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3}\cdot 1 = \frac{1}{3} < 1.$$

Since the limit is less than 1, by the Ratio Test, we see that the series $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}k^3}{3^k}$ $\boxed{\textit{converges.}}$

63. Observe that for a fixed positive integer n,

$a_n = \frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdots n}{n\cdot n\cdot n\cdots n} = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)\cdots\left(\frac{n}{n}\right) \le \left(\frac{1}{n}\right)\cdot 1\cdot 1\cdots 1 = \frac{1}{n}$, which is convergent since $\lim\limits_{n\to\infty}\frac{1}{n} = 0$. Also, $0 < \frac{n!}{n^n}$, for all $n \ge 1$. By the Squeeze Theorem, since $\lim\limits_{n\to\infty} 0 \le \lim\limits_{n\to\infty}\frac{n!}{n^n} \le \lim\limits_{n\to\infty}\frac{1}{n}$, we have $\lim\limits_{n\to\infty}\frac{n!}{n^n} = 0$.

64. For $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty}\frac{1}{k}$, since the terms are nonzero, the Root Test gives

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{1}{n}\right|} = \lim_{n\to\infty}\left|\frac{1}{n^{1/n}}\right| = 1,$$

using the limit result from Problem 43. Likewise, for $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty}\frac{1}{k^2}$, since the terms are nonzero, the Root Test gives

$$\lim_{n\to\infty}\sqrt[n]{|b_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{1}{n^2}\right|} = \lim_{n\to\infty}\left|\frac{1}{n^{2/n}}\right| = \lim_{n\to\infty}\left(\frac{1}{n^{1/n}}\right)^2 = \left(\lim_{n\to\infty}\frac{1}{n^{1/n}}\right)^2 = 1^2 = 1,$$

using, again, the limit result from Problem 43. So we find that the Root Test is inconclusive for the limit value of 1, because it cannot distinguish between the divergent harmonic series $\sum\limits_{k=1}^{\infty}\frac{1}{k}$ and the convergent p-series (convergent since $p = 2 > 1$) $\sum\limits_{k=1}^{\infty}\frac{1}{k^2}$.

65. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{x^k}{k^2}$ is a series of nonzero terms if $x \neq 0$ (If $x = 0$, then the series becomes $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} 0 = 0$, so the series converges for $x = 0$.) The ratio of the $n+1$st and the nth term is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{x^{n+1}}{(n+1)^2}}{\frac{x^n}{n^2}} = \frac{n^2}{(n+1)^2} \cdot \frac{x^{n+1}}{x^n} = \frac{n^2}{(n+1)^2} \cdot x.$$

Compute the limit of the absolute value of the ratio found above:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n^2}{(n+1)^2} \cdot x\right| = |x| \cdot \lim_{n\to\infty} \left|\frac{n^2}{(n+1)^2}\right| = |x| \cdot \lim_{n\to\infty} \left|\frac{1}{\left(1+\frac{1}{n}\right)^2}\right| = |x| \cdot |1| = |x|.$$

If $|x| < 1$, then the series converges by the Ratio Test, if $|x| > 1$, the series diverges by the Ratio Test, while when $|x| = 1$, the Ratio Test is inconclusive. However, when $|x| = 1$, that is, $x = \pm 1$, the series becomes either the p-series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$, which is known to converge since $p = 2 > 1$, or the alternating series $\sum\limits_{k=1}^{\infty} (-1)^k a_k = \sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ which converges by the Alternating Series Test (since we have the nth term satisfy $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n^2} = 0$, and also $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} < 1$ for $n \geq 1$). So finally, the range of values of x for which there will be convergence is $\boxed{|x| \leq 1,}$ or, equivalently, $\boxed{-1 \leq x \leq 1.}$

66. For the series $\sum\limits_{k=1}^{\infty} a_k$, with the property that $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, let $\left|\frac{a_{n+1}}{a_n}\right| = f(n)$. Then we have

$$|a_{n+1}| = \left|\frac{a_{n+1}}{a_n} \cdot \frac{a_n}{a_{n-1}} \cdot \cdots \cdot \frac{a_2}{a_1} \cdot a_1\right| = f(n)f(n-1)\cdots f(1)|a_1|.$$

We examine the limit of the sequence of partial sums $\{S_{n+1}\} = \sum\limits_{k=1}^{n+1} a_k$ of the series $\sum\limits_{k=1}^{\infty} a_k$. We have

$$\lim_{n\to\infty} |S_{n+1}| = \lim_{n\to\infty} \left|\sum_{k=1}^{n+1} a_k\right| > \lim_{n\to\infty} |a_{n+1}| = \lim_{n\to\infty} f(n)f(n-1)\cdots f(1)|a_1| = \infty,$$

since by hypothesis $\lim\limits_{n\to\infty} f(n) = \infty$. This implies the sequence of partial sums $\{S_{n+1}\}$ diverges, so the series itself will diverge.

67. To prove the root test, let $\sum\limits_{k=1}^{\infty} a_k$ be a series with nonzero terms. Let $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L \neq \infty$ exist. We consider three cases: $0 \leq L < 1$, $L > 1$ and $L = 1$.

$\underline{0 \leq L < 1}$: Let $0 \leq L < r < 1$. Let $\epsilon = r - L > 0$. Since $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} |a_n|^{1/n} = L < \infty$, there exists some $N > 0$, such that if $n > N$, we have

$$\begin{aligned} \left||a_n|^{1/n} - L\right| &< \epsilon = r - L \\ L - r < |a_n|^{1/n} - L &< r - L \\ 2L - r < |a_n|^{1/n} &< r, \end{aligned}$$

or $|a_n| < r^n$. Now, $\sum\limits_{k=0}^{\infty} r^k$ is a geometric series and since $0 < r < 1$, it is convergent to a sum of $\frac{1}{1-r}$. Also, since $|a_n| < r^n$ for $n > N$, by the Comparison Test, the series $\sum\limits_{k=N}^{\infty} |a_n|$ is convergent as well. The series $\sum\limits_{k=1}^{\infty} |a_n|$ differs from the series $\sum\limits_{k=N}^{\infty} |a_n|$ by a finite number of terms, namely $\sum\limits_{k=1}^{N-1} |a_n|$. Also, $\sum\limits_{k=1}^{N-1} |a_n| < \infty$, since the a_n are finite for $n < N$. So, since the convergence or divergence property of a series is not affected by the omission or addition of a finite number of terms whose sum is finite, we conclude that the series $\sum\limits_{k=1}^{\infty} |a_k|$ converges absolutely, and so converges.

$\underline{L > 1:}$ Since $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, we can find some $n \geq N$ such that $|a_n|^{1/n} > 1$. This means $|a_n| > 1^n = 1$ for $n \geq N$, and $\lim\limits_{n\to\infty} |a_n| \geq 1 \neq 0$, so the series diverges by the Divergence Test. This shows that when $L > 1$, the Root Test indicates that the series for which $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L$ is a divergent series.

$\underline{L = 1:}$ To show that the Root Test for this case is inconclusive, we exhibit a convergent, divergent, and a conditionally convergent series, all of which produce the limit 1. From Problem 64, we saw that the divergent harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ and the convergent p-series (with $p = 2 > 1$) $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ both yielded $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$, so the Root Test cannot distinguish between them as far as convergence or divergence is concerned. Consider the conditionally convergent alternating harmonic series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k}$. The terms are nonzero, so the Root Test can be applied. We have $|a_n| = \left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n}$ for $n \geq 1$, which brings us back to the same calculation as for the harmonic series in Problem 64. So the Root Test gives the limit of 1 for this conditionally convergent series as well.

68. (a) Let $n = 2m$ be even, where $m \geq 1$ is any positive integer. Then $n + 1 = 2m + 1$ is odd. Since $a_{2k} = \frac{1}{2^k}$, we have $a_n = a_{2m} = \frac{1}{2^m}$. Since $a_{2k-1} = \frac{1}{2^{k+1}}$, we have

$$a_{n+1} = a_{2m+1} = a_{2(m+1)-1} = \frac{1}{2^{(m+1)+1}} = \frac{1}{2^{m+2}}.$$

By the Ratio Test, which is applicable since the terms of the series are nonzero, we have for even n,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{2m\to\infty}\left|\frac{a_{2m+1}}{a_{2m}}\right| = \lim_{m\to\infty}\left|\frac{\frac{1}{2^{m+2}}}{\frac{1}{2^m}}\right| = \lim_{m\to\infty}\left|\frac{2^m}{2^{m+2}}\right| = \lim_{m\to\infty}\left|\frac{2^m}{2^m\cdot 2^2}\right| = \frac{1}{4}.$$

Let $n = 2m + 1$ be odd, where $m \geq 0$ is zero or any positive integer. Then $n + 1 = 2m + 2 = 2(m + 1)$ is even. Since $a_{2k-1} = \frac{1}{2^{k+1}}$, we have $a_n = a_{2m+1} = a_{2(m+1)-1} = \frac{1}{2^{(m+1)+1}} = \frac{1}{2^{m+2}}$. Since $a_{2k} = \frac{1}{2^k}$, we have $a_{n+1} = a_{2(m+1)} = \frac{1}{2^{m+1}}$. By the Ratio Test, which is applicable since the terms of the series are nonzero, we have for odd n,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{2m+1\to\infty}\left|\frac{a_{2(m+1)}}{a_{2m+1}}\right| = \lim_{m\to\infty}\left|\frac{\frac{1}{2^{m+1}}}{\frac{1}{2^{m+2}}}\right| = \lim_{m\to\infty}\left|\frac{2^{m+2}}{2^{m+1}}\right| = \lim_{m\to\infty}\left|\frac{2^m\cdot 2^2}{2^m\cdot 2}\right| = 2.$$

We see that the Ratio Test produces two different limits for the case of n even or n odd. In other words, $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \neq \infty$ does not exist, so the Ratio Test is not conclusive as to whether the series converges or diverges.

(b) We consider the cases of n even and n odd with the root test. If n is even, then $n = 2m$ for any positive integer m, and as in part (a), $a_n = a_{2m} = \frac{1}{2^m}$. Applying the Root Test, which is applicable since the terms of the series are nonzero, we have for even n

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{2m\to\infty} \sqrt[2m]{|a_{2m}|} = \lim_{m\to\infty} \left(\frac{1}{2^m}\right)^{\frac{1}{2m}} = \left(\frac{1}{2}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

If n is odd, then $n = 2m + 1$ for or any positive integer (or zero) m, and as in part (a), $a_n = a_{2m+1} = \frac{1}{2^{m+2}}$. Applying the Root Test, which is applicable since the terms of the series are nonzero, we have for odd n,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{2m+1\to\infty} \sqrt[2m+1]{|a_{2m+1}|} = \lim_{m\to\infty} \left(\frac{1}{2^{m+2}}\right)^{\frac{1}{(2m+1)}} = \lim_{m\to\infty} \frac{1}{2^{\frac{m+2}{2m+1}}} = \lim_{m\to\infty} 2^{-\frac{1+\frac{2}{m}}{2+\frac{1}{m}}}$$
$$= 2^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Since the limits are the same for both even and odd n, and since the even and the odd cases together comprise the entire series, we see that the Root Test gives us a limit which exists.

(c) Since the limit found by applying the Root Test in part (b) is less than 1, by the Root Test, the series in question $\boxed{\textit{converges.}}$

Challenge Problems

69. We have

$$\sum_{k=1}^{\infty} a_k = 1 + \frac{2}{2^2} + \frac{3}{3^3} + \frac{1}{4^4} + \frac{2}{5^5} + \frac{3}{6^6} + \cdots < \sum_{k=1}^{\infty} b_k = 1 + \frac{2}{2^2} + \frac{3}{3^3} + \frac{4}{4^4} + \frac{5}{5^5} + \frac{6}{6^6} + \cdots + \frac{n}{n^n} + \cdots .$$

The nth term of the series to the right of the inequality is $b_n = \frac{n}{n^n}$. By the Root Test, which applies since the terms of that series are nonzero, we have

$$\lim_{n\to\infty} \sqrt[n]{|b_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n}{n^n}\right|} = \lim_{n\to\infty} \frac{n^{1/n}}{n} = \lim_{n\to\infty} n^{1/n} \cdot \lim_{n\to\infty} \frac{1}{n} = 1 \cdot 0 = 0,$$

where we have used the limit result established in Problem 43. Since the limit is less than 1, by the Root Test, the series $\sum\limits_{k=1}^{\infty} b_k$ converges, and then by the Comparison Test, we see that the original series $\sum\limits_{k=1}^{\infty} a_k$ converges as well.

70. We have

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(k+1)^2}{(k+2)!} = \sum_{k=1}^{\infty} \frac{(k+1)(k+1)}{(k+2)(k+1)(k!)} = \sum_{k=1}^{\infty} \left(\frac{k+1}{k+2}\right)\left(\frac{1}{k!}\right) < \sum_{k=1}^{\infty} \frac{1}{k!},$$

since $\frac{k+1}{k+2} < 1$ for $k \geq 1$. Now, $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k!}$ is convergent since by the Ratio Test,

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to\infty} \left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since the limit is less than 1, by the Ratio Test, the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k!}$ converges, and by the Comparison Test, so does the series $\sum_{k=1}^{\infty} a_k$.

Alternately, using the suggested hint, the Limit Comparison test may be used. The nth term of the original series behaves like

$$a_n = \frac{(n+1)^2}{(n+2)!} = \frac{(n+1)(n+1)}{(n+2)(n+1)(n!)} = \frac{n+1}{n+2} \cdot \frac{1}{n!} \approx \frac{1}{n!} = b_n$$

for large values of n. Comparing the series of positive terms $\sum_{k=1}^{\infty} a_k$ with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k!}$, which is convergent as noted earlier, and also has positive terms, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+1}{n+2} \cdot \frac{1}{n!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{n+1}{n+2} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} = \frac{1+0}{1+0} = 1.$$

Since the limit is a positive real number, by the Limit Comparison Test, the original series $\sum_{k=1}^{\infty} a_k$ converges.

71. The nth term of the series $\sum_{k=1}^{\infty} c_k = a + b + a^2 + b^2 + a^3 + b^3 + \cdots$ is $c_n = a^n + b^n$. Since $0 < a < b$, the terms of the series are all nonzero, and the Root Test may be applied. We have

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{|c_n|} &= \sqrt[n]{(a^n + b^n)} = b\sqrt[n]{1 + \left(\frac{a^n}{b^n}\right)} = b \lim_{n\to\infty} \left[1 + \left(\frac{a}{b}\right)^n\right]^{1/n} \\ &= b\left[1 + \lim_{n\to\infty} \left(\frac{a}{b}\right)^n\right]^{\lim_{n\to\infty} \frac{1}{n}} \\ &= b(1+0)^0 = b < 1,\end{aligned}$$

where, in order to obtain $\lim_{n\to\infty} \left(\frac{a}{b}\right)^n = 0$, we used the fact that $a < b$, so that $\frac{a}{b} < 1$, and to obtain the final inequality we used $b < 1$. Since the limit is less than 1, by the Root Test, the series converges.

72. Since the Ratio Test indicates the series $\sum_{k=1}^{\infty} a_k$ converges, it means $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$. This means for any $\epsilon > 0$ there exists an N such that if $n \geq N$, then $\left|\frac{a_{n+1}}{a_n} - L\right| < \epsilon$, that is, $L - \epsilon < \frac{|a_{n+1}|}{|a_n|} < L + \epsilon$. We have

$$|a_n| = \frac{|a_n|}{|a_{n-1}|} \cdot \frac{|a_{n-1}|}{|a_{n-2}|} \cdots \frac{|a_{N+1}|}{|a_N|} \cdot |a_N|.$$

Applying the inequality above to each of the fractional terms, we get

$$(L-\epsilon)^{n-N} \cdot |a_N| < |a_n| < (L+\epsilon)^{n-N} \cdot |a_N|$$

or, taking the nth root, $$(L-\epsilon)^{1-\frac{N}{n}} \cdot |a_N|^{\frac{1}{n}} < \sqrt[n]{|a_n|} < (L+\epsilon)^{1-\frac{N}{n}} |a_N|^{\frac{1}{n}}.$$

Taking the limit as $n \to \infty$ on this string of inequalities, we have

$$\begin{aligned}
\lim_{n\to\infty}(L-\epsilon)^{1-\frac{N}{n}} \cdot \lim_{n\to\infty}|a_N|^{\frac{1}{n}} \leq \lim_{n\to\infty}\sqrt[n]{|a_n|} \leq \lim_{n\to\infty}(L+\epsilon)^{1-\frac{N}{n}} \cdot \lim_{n\to\infty}|a_N|^{\frac{1}{n}}\\
(L-\epsilon)^{\lim\limits_{n\to\infty}\left(1-\frac{N}{n}\right)} \cdot |a_N|^{\lim\limits_{n\to\infty}\frac{1}{n}} \leq \lim_{n\to\infty}\sqrt[n]{|a_n|} \leq (L+\epsilon)^{\lim\limits_{n\to\infty}\left(1-\frac{N}{n}\right)} \cdot |a_N|^{\lim\limits_{n\to\infty}\frac{1}{n}}\\
(L-\epsilon)^1|a_N|^0 \leq \lim_{n\to\infty}\sqrt[n]{|a_n|} \leq (L+\epsilon)^1|a_N|^0\\
L-\epsilon \leq \lim_{n\to\infty}\sqrt[n]{|a_n|} \leq L+\epsilon.
\end{aligned}$$

Since this is valid for any $\epsilon > 0$, in the limit, as $\epsilon \to 0^+$, since $\lim\limits_{\epsilon\to 0^+}(L-\epsilon) = L$ and $\lim\limits_{\epsilon\to 0^+}(L+\epsilon) = L$, by the Squeeze Theorem, we have $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$ since $L < 1$ by assumption. So by the Root Test, the series converges.
We have shown that if the Ratio Test indicates a series converges, then so will the Root Test. That the converse is not true can be seen from Problem 68, where a series is seen to converge by the Root Test, but the Ratio Test is inconclusive.

AP® Practice Problems

1. I. $\frac{k^2}{(3k+1)!} > 0$ for $k \geq 1$; $a_{n+1} = \frac{(n+1)^2}{[3(n+1)+1]!}$ and $a_n = \frac{n^2}{(3n+1)!}$.

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{(n+1)^2}{[3(n+1)+1]!}}{\frac{n^2}{(3n+1)!}}\right| = \left|\frac{(n+1)^2(3n+1)!}{(3n+4)!n^2}\right| = \left|\frac{(n+1)^2(3n+1)!}{n^2(3n+4)(3n+3)(3n+2)(3n+1)!}\right|\\
&= \left|\left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{(3n+4)(3n+3)(3n+2)}\right|\\
&= \left|\left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{(3n+4)(3n+3)(3n+2)}\right|.
\end{aligned}$$

So $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{(3n+4)(3n+3)(3n+2)}\right| = 0 < 1$.

Since the limit is less than 1, the series $\sum\limits_{k=1}^{\infty}\frac{k^2}{(3k+1)!}$ $\boxed{\text{converges}}$.

II. $\frac{k^k}{k!} > 0$ for $k \geq 1$; $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$ and $a_n = \frac{n^n}{n!}$.

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}\right| = \left|\frac{(n+1)^{n+1}n!}{n^n(n+1)!}\right| = \left|\frac{(n+1)^n(n+1)n!}{n^n(n+1)!}\right|\\
&= \left|\left(\frac{n+1}{n}\right)^n \cdot \frac{(n+1)!}{(n+1)!}\right| = \left|\left(1+\frac{1}{n}\right)^n\right|, \text{ so } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^n\right| = e > 1.
\end{aligned}$$

Since the limit is greater than 1, the series $\sum\limits_{k=1}^{\infty}\frac{k^k}{k!}$ $\boxed{\text{diverges}}$.

III. $k\left(\frac{2}{3}\right)^k > 0$ for $k \geq 1$; $a_{n+1} = (n+1)\left(\frac{2}{3}\right)^{n+1}$ and $a_n = n\left(\frac{2}{3}\right)^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)\left(\frac{2}{3}\right)^{n+1}}{n\left(\frac{2}{3}\right)^n}\right| = \left|\frac{2}{3}\left(\frac{n+1}{n}\right)\right| = \left|\frac{2}{3}\left(1+\frac{1}{n}\right)\right|.$$

So $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2}{3}\left(1+\frac{1}{n}\right)\right| = \frac{2}{3} < 1$.

Since the limit is less than 1, the series $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$ $\boxed{\text{converges}}$.

CHOICE C: I and III only

2. I. $\frac{(-1)^k\sqrt{k}}{k+1} \neq 0$ for $k \geq 1$; $a_{n+1} = \frac{(-1)^{n+1}\sqrt{n+1}}{n+2}$ and $a_n = \frac{(-1)^n\sqrt{n}}{n+1}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}\sqrt{n+1}}{n+2}}{\frac{(-1)^k\sqrt{n}}{n+1}}\right| = \left|\left(\frac{n+1}{n+2}\right)\sqrt{\frac{n+1}{n}}\right| = \left|\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\cdot\left(1+\frac{1}{n}\right)^{1/2}\right|.$$

So $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\cdot\left(1+\frac{1}{n}\right)^{\frac{1}{2}}\right| = 1$.

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test provides $\boxed{\text{no information}}$ about the series $\sum_{k=1}^{\infty}\frac{(-1)^k\sqrt{k}}{k+1}$.

II. $\frac{1}{k\ln k} > 0$ for $k \geq 2$; $a_{n+1} = \frac{1}{(n+1)\ln(n+1)}$ and $a_n = \frac{1}{n\ln n}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)\ln(n+1)}}{\frac{1}{n\ln n}}\right| = \lim_{n\to\infty}\left|\frac{n\ln n}{(n+1)\ln(n+1)}\right|,$$

which is indeterminate of the form $\frac{\infty}{\infty}$, so we use L'Hôpital's Rule:

$$\lim_{n\to\infty}\left|\frac{n\ln n}{(n+1)\ln(n+1)}\right| = \lim_{n\to\infty}\left|\frac{\ln n + n\cdot\frac{1}{n}}{\ln(n+1)+\frac{n+1}{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1+\ln n}{1+\ln(n+1)}\right|,$$

which is indeterminate of the from $\frac{\infty}{\infty}$, so we use L'Hôpital's Rule again:

$$\lim_{n\to\infty}\left|\frac{1+\ln n}{1+\ln(n+1)}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n}}{\frac{1}{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\right| = \lim_{n\to\infty}\left|1+\frac{1}{n}\right| = 1$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test provides $\boxed{\text{no information}}$ about the series $\sum_{k=2}^{\infty}\frac{1}{k\ln k}$.

III. $\frac{k^2+3k}{k+1} > 0$ for $k \geq 1$; $a_{n+1} = \frac{(n+1)^2+3(n+1)}{n+2}$ and $a_n = \frac{n^2+3n}{n+1}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^2+3(n+1)}{n+2}}{\frac{n^2+3n}{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(n+4)(n+1)}{n(n+3)(n+2)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{4}{n}\right)\left(1+\frac{1}{n}\right)}{1\left(1+\frac{3}{n}\right)\left(1+\frac{2}{n}\right)}\right| = 1$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test provides $\boxed{\text{no information}}$ about the series $\sum_{k=1}^{\infty}\frac{k^2+3k}{k+1}$.

CHOICE D: I, II, and III

3. I. $\frac{k!}{100^k} > 0$ for $k \geq 1$; $a_{n+1} = \frac{(n+1)!}{100^{n+1}}$ and $a_n = \frac{n!}{100^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}}\right| = \left|\frac{(n+1)!100^n}{100^{n+1}n!}\right| = \left|\frac{(n+1)n!100^n}{100^n(100)n!}\right| = \left|\frac{n+1}{100}\right|.$$

So $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\frac{n+1}{100}\right| = \infty > 1$.

Since $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, by the Ratio Test the series $\sum\limits_{k=1}^{\infty}\frac{k!}{100^k}$ $\boxed{\text{diverges}}$.

II. $\frac{20^k}{2^{k^2}} > 0$ for $k \geq 1$; $a_n = \frac{20^n}{2^{n^2}}$.
Since a_n involves an nth power, we use the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{20^n}{2^{n^2}}} = \lim_{n\to\infty}\left|\frac{20^{n/n}}{2^{n^2/n}}\right| = \lim_{n\to\infty}\left|\frac{20}{2^n}\right| = 0 < 1$$

Since $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} < 1$, by the Root Test, the series $\sum\limits_{k=1}^{\infty}\frac{20^k}{2^{k^2}}$ $\boxed{\text{converges}}$.

III. We choose a p-series for comparison by examining how the terms of the series behave for large values of n:

$$\frac{\sqrt{n}}{n^3+1} = \frac{\sqrt{n}}{\sqrt{n}\left(n^{5/2}+\frac{1}{\sqrt{n}}\right)} \approx \frac{1}{n^{5/2}}$$

So we compare the series $\sum\limits_{k=1}^{\infty}\frac{\sqrt{k}}{k^3+1}$ to the convergent p-series $\sum\limits_{k=1}^{\infty}\frac{1}{k^{5/2}}$ and use the Limit Comparison Test, with $a_n = \frac{\sqrt{n}}{n^3+1}$ and $b_n = \frac{1}{n^{5/2}}$.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n^3+1}}{\frac{1}{n^{5/2}}} = \lim_{n\to\infty}\frac{n^3}{n^3+1} = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n^3}}\right) = 1$$

Since $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = 1$ and $0 < 1 < \infty$ and $\sum\limits_{k=1}^{\infty}\frac{1}{k^{5/2}}$ converges, by the Limit Comparison Test $\sum\limits_{k=1}^{\infty}\frac{\sqrt{k}}{k^3+1}$ also $\boxed{\text{converges}}$.

CHOICE A: I only

4. $\frac{e^k}{k^k} > 0$ for $k \geq 1$; $a_n = \frac{e^n}{n^n}$.
Since a_n involves an nth power, we use the Root Test.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1 = \lim_{n\to\infty}\sqrt[n]{\frac{e^n}{n^n}} = \lim_{n\to\infty}\left|\frac{e^{n/n}}{n^{n/n}}\right| = \lim_{n\to\infty}\left|\frac{e}{n}\right| = 0 < 1.$$

Since $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} < 1$, by the Root Test the series $\sum\limits_{k=1}^{\infty}\frac{e^k}{k^k}$ $\boxed{\text{converges}}$.

5. $\frac{1}{k^p} > 0$ for $k \geq 1$; $a_{n+1} = \frac{1}{(n+1)^p}$ and $a_n = \frac{1}{n^p}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}}\right| = \lim_{n\to\infty}\left|\frac{n^p}{(n+1)^p}\right| = \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^p\right| = \lim_{n\to\infty}\left|\left(\frac{1}{1+\frac{1}{n}}\right)^p\right| = 1^p = 1$$

Since $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test $\boxed{\text{provides no information}}$ about the series $\sum\limits_{k=1}^{\infty}\frac{1}{k^p}$.

8.7 Summary of Tests

Concepts and Vocabulary

1. **False:** The series $\sum\limits_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ but diverges if $p = 1$ (it then becomes the divergent harmonic series).

2. **False:** According to the Test for Divergence, $\sum\limits_{k=1}^{\infty} a_k$ diverges if $\lim\limits_{n\to\infty} a_n \neq 0$. The test says nothing if $\lim\limits_{n\to\infty} a_n = 0$. For instance, the convergent p-series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ (convergent because $p = 2 > 1$), and the divergent harmonic series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k}$ both satisfy $\lim\limits_{n\to\infty} a_n = 0$.

3. **True:** This is the statement of the theorem on Absolute Convergence Test, see Chapter 8, Section 8.5, p.684.

4. **False:** For a counterexample, consider the alternating harmonic series, $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k = \sum\limits_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$. Then the absolute value series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k}$ is divergent as it is the harmonic series. However, since $\lim\limits_{n\to\infty} a_n = 0$ and by the Algebraic Ratio test $\frac{a_{n+1}}{a_n} < 1$ for $n \geq 1$ means the terms of the series are nonincreasing, by the Alternating Series Test, we see that the series $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k$ converges. (This is called conditional convergence.)

5. **False:** For deciding convergence via the Ratio Test, it is not important that $\left|\frac{a_{n+1}}{a_n}\right| < 1$; we need to require that $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. For example the divergent harmonic series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k}$ satisfies $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n}{n+1}\right| < 1$ for $n \geq 1$, while $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{n}{n+1}\right| = \lim\limits_{n\to\infty} \left|\frac{1}{1+\frac{1}{n}}\right| = 1$ means that the Ratio Test is in fact inconclusive in regards to the harmonic series' convergence or divergence (rather the Integral Test can be used to show the harmonic series diverges).

6. $\underline{0 < a_k \leq b_k}$: This follows from the statement of the Comparison Test for Convergence, see Section 8.4, p. 672.

Skill Building

7. The series is $\sum\limits_{k=1}^{\infty} \frac{9k^3+5k^2}{k^{5/2}+4}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{9n^3+5n^2}{n^{5/2}+4} = \lim_{n\to\infty} \frac{n^3\left(9+\frac{5}{n}\right)}{n^{5/2}\left(1+\frac{4}{n^{5/2}}\right)} = \lim_{n\to\infty} \left[n^{1/2} \cdot \frac{9+\frac{5}{n}}{1+\frac{4}{n^{5/2}}}\right] = \infty.$$

Since the limit of the nth term is nonzero, the series $\boxed{\textit{diverges}}$ by the Divergence Test.

8. The series is $\sum\limits_{k=1}^{\infty}(-1)^{k+1}a_k = \sum\limits_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{2k+1}}$. The series of absolute values is $\sum\limits_{k=1}^{\infty}a_k = \sum\limits_{k=1}^{\infty}\frac{1}{\sqrt{2k+1}}$. The nth term behaves like

$$a_n = \frac{1}{\sqrt{2n+1}} = \frac{1}{n^{1/2}\sqrt{2+\frac{1}{n^{1/2}}}} \approx \frac{1}{n^{1/2}}$$

for large values of n. So we compare with the divergent p-series $\sum\limits_{k=1}^{\infty}b_k = \sum\limits_{k=1}^{\infty}\frac{1}{k^{1/2}}$, which is divergent since $0 < p = \frac{1}{2} < 1$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{n^{1/2}\sqrt{2+\frac{1}{n^{1/2}}}}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty}\frac{1}{\sqrt{2+\frac{1}{n^{1/2}}}} = \frac{1}{\sqrt{2}}.$$

Since the limit is a positive real number, by the Limit Comparison Test, the series of absolute values $\sum\limits_{k=1}^{\infty}a_k$ diverges.

Let's investigate whether the original series converges. We have $\lim\limits_{n\to\infty}a_n = \lim\limits_{n\to\infty}\frac{1}{\sqrt{2n+1}} = 0$. Also by the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{\sqrt{2(n+1)+1}}}{\frac{1}{\sqrt{2n+1}}} = \frac{\sqrt{2n+1}}{\sqrt{2n+3}} < 1$$

for all $n \geq 1$. So the terms of the series are nonincreasing as well. By the Alternating Series Test, the original series converges. But since the series of absolute values diverges, the original series is $\boxed{\textit{conditionally convergent.}}$

9. The series is $6 + 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots = 6 + 2 + \sum\limits_{k=1}^{\infty}\frac{2}{3^k} = 8 + \sum\limits_{k=1}^{\infty}\frac{2}{3}\cdot\left(\frac{1}{3}\right)^{k-1}$. The latter term is a geometric series of the form $\sum\limits_{k=1}^{\infty}ar^{k-1}$ with $a = \frac{2}{3}$ and $r = \frac{1}{3}$. Since $|r| < 1$, it converges to a sum of $\frac{a}{1-r} = \frac{2/3}{1-1/3} = \frac{2/3}{2/3} = 1$. The original series converges to a sum of $8 + 1 = 9$, which is finite, so the series is convergent. Since all terms of the original series are positive, the series is also $\boxed{\textit{absolutely convergent.}}$

10. The series is $\sum\limits_{k=1}^{\infty}a_k = \sum\limits_{k=1}^{\infty}\frac{1}{k^2}\sin\frac{\pi}{k}$. The series of absolute values is $\sum\limits_{k=1}^{\infty}|a_k| = \sum\limits_{k=1}^{\infty}\frac{1}{k^2}\left|sin\frac{\pi}{k}\right|$. We have

$$|a_n| = \frac{1}{n^2}\left|\sin\frac{\pi}{n}\right| \leq \frac{1}{n^2} = b_n$$

since $|\sin\theta| \leq 1$ for $0 < \theta = \frac{\pi}{n} \leq \pi$. By comparing with the series $\sum\limits_{k=1}^{\infty}b_k = \sum\limits_{k=1}^{\infty}\frac{1}{k^2}$, which is a convergent p-series (since $p = 2 > 1$), the series of absolute values converges by the Comparison Test, which means the original series $\boxed{\textit{converges absolutely.}}$

11. The series is $\sum\limits_{k=1}^{\infty}a_k = \sum\limits_{k=1}^{\infty}\frac{3k+1}{k^3+1}$. The nth term behaves like

$$a_n = \frac{3n+2}{n^3+1} = \frac{n\left(3+\frac{2}{n}\right)}{n^3\left(1+\frac{1}{n^3}\right)} \approx \frac{1}{n^2}$$

for large values of n. We compare with the convergent p-series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$, which is convergent since $p = 2 > 1$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3n+2}{n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^3+2n^2}{n^3+1} = \lim_{n\to\infty} \frac{3+\frac{2}{n}}{1+\frac{1}{n^3}} = 3.$$

Since the limit is a positive real number, the series is convergent by the Limit Comparison Test. And since all the terms of the series are positive, it is also $\boxed{\textit{absolutely convergent.}}$

12. The series is $1 + \frac{2^2+1}{2^3+1} + \frac{3^2+1}{3^3+1} + \frac{4^2+1}{4^3+1} + \cdots = \sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k^2+1}{k^3+1}$. The nth term behaves like

$$a_n = \frac{n^2+1}{n^3+1} = \frac{n^2\left(1+\frac{1}{n^2}\right)}{n^3\left(1+\frac{1}{n^3}\right)} \approx \frac{1}{n}$$

for large values of n. We compare with the divergent harmonic series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k}$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2+1}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3+n}{n^3+1} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{1+\frac{1}{n^3}} = 1.$$

Since the limit is a positive real number, by the Limit Comparison Test, the series $\boxed{\textit{diverges.}}$

13. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k+4}{k\sqrt{3k-2}}$. The nth term behaves like

$$a_n = \frac{n+4}{n\sqrt{3n-2}} = \frac{n\left(1+\frac{4}{n}\right)}{n^{3/2}\sqrt{3-\frac{2}{n}}} \approx \frac{1}{n^{1/2}}$$

for large values of n. We compare with the divergent p-series (divergent since $0 < p = \frac{1}{2} < 1$) $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^{1/2}}$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+4}{n\sqrt{3n-2}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n+4}{\sqrt{n}\sqrt{3n-2}} = \lim_{n\to\infty} \frac{1+\frac{4}{n}}{\sqrt{3-\frac{2}{n}}} = \frac{1}{\sqrt{3}}.$$

Since the limit is a positive real number, by the Limit Comparison Test, the series $\boxed{\textit{diverges.}}$

14. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\sin k}{k^3}$. The series of absolute values is $\sum\limits_{k=1}^{\infty} |a_k| = \sum\limits_{k=1}^{\infty} \frac{|\sin k|}{k^3}$. We have

$$|a_n| = \frac{|\sin n|}{n^3} \leq \frac{1}{n^3} = b_n,$$

since $|\sin n| \leq 1$ for $n \geq 1$. Comparing with the convergent p-series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^3}$, which is convergent since $p = 3 > 1$, we see by the Comparison Test that the series of absolute values converges. This means the original series is $\boxed{\textit{absolutely convergent.}}$

15. The series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{3^{2k-1}}{k^2+2k}$ is a series of nonzero terms. We apply the Ratio Test. The ratio of the $n+1$st and the nth term of the series is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{3^{2(n+1)-1}}{(n+1)^2+2(n+1)}}{\frac{3^{2n-1}}{n^2+2n}} = \frac{3^{2n+1}}{3^{2n-1}} \cdot \frac{n^2+2n}{(n^2+2n+1)+2n+1} = \frac{9n^2+18n}{n^2+4n+2}.$$

The limit of the absolute value of the ratio found above is

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{9n^2+18n}{n^2+4n+2} = \lim_{n\to\infty} \frac{9+\frac{18}{n}}{1+\frac{4}{n}+\frac{2}{n^2}} = 9 > 1.$$

Since the limit is greater than 1, by the Ratio Test, the series $\boxed{\textit{diverges.}}$

16. The series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{5^k}{k!}$ is a series of nonzero terms. By the Ratio Test, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{5^{n+1}}{(n+1)!}}{\frac{5^n}{n!}}\right| = \lim_{n\to\infty} \left|\frac{5^{n+1}}{5^n} \cdot \frac{n!}{(n+1)!}\right| = 5 \lim_{n\to\infty} \left|\frac{n!}{(n+1)n!}\right| = 0.$$

Since the limit is less than 1, by the Ratio Test, the series converges. Since all the terms are positive, the series is also $\boxed{\textit{absolutely convergent.}}$

17. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \left(1+\frac{2}{k}\right)^k$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = \lim_{n'\to\infty} \left(1+\frac{1}{n'}\right)^{2n'} = \lim_{n'\to\infty} \left[\left(1+\frac{1}{n'}\right)^{n'}\right]^2 = \left[\lim_{n'\to\infty} \left(1+\frac{1}{n'}\right)^{n'}\right]^2 = e^2 \neq 0$$

where $n' = \frac{n}{2}$. Since the limit of the nth term is nonzero, the series $\boxed{\textit{diverges}}$ by the Divergence Test.

18. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k^2+4}{e^k}$ is a series of nonzero terms. By the Ratio Test, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^2+4}{e^{n+1}}}{\frac{n^2+4}{e^n}}\right| = \lim_{n\to\infty} \left|\frac{e^n}{e^{n+1}} \cdot \frac{(n+1)^2+4}{n^2+4}\right| = \frac{1}{e} \lim_{n\to\infty} \left|\frac{\left(1+\frac{1}{n}\right)^2+\frac{4}{n^2}}{1+\frac{4}{n^2}}\right| = \frac{1}{e} \cdot |1| < 1.$$

Since the limit is less than 1, by the Ratio Test, the series converges. Also, since the terms of the series are all positive, this means the series is $\boxed{\textit{absolutely convergent.}}$

19. The series is $\frac{2}{3} - \frac{3}{4}\cdot\frac{1}{2} + \frac{4}{5}\cdot\frac{1}{3} - \frac{5}{6}\cdot\frac{1}{4} + \cdots = \sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k = \sum\limits_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{k+2} \cdot \frac{1}{k}$. The series of absolute values is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k+1}{k+2} \cdot \frac{1}{k}$. Comparing with the divergent harmonic series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+1}{n+2}\cdot\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n+1}{n+2} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} = 1.$$

Since the limit is a positive real number, by the Limit Comparison Test, the absolute value series diverges.

Next, we examine the original series $\sum\limits_{k=1}^{\infty}(-1)^{k+1}a_k$ for convergence. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n+1}{n+2}\cdot\frac{1}{n} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\cdot\lim_{n\to\infty}\frac{1}{n} = 1\cdot 0 = 0.$$

By the Algebraic Ratio Test, we have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+2}{n+3}\cdot\frac{1}{n+1}}{\frac{n+1}{n+2}\cdot\frac{1}{n}} = \frac{n(n+2)^2}{(n+1)^2(n+3)} < 1$$

provided the following holds:

$$\begin{aligned} n(n+2)^2 &< (n+1)^2(n+3) \\ n(n^2+4n+4) &< (n^2+2n+1)(n+3) \\ n^3+4n^2+4n &< n^3+2n^2+n+3n^2+6n+3 \\ 4n^2+4n &< 5n^2+7n+3 \\ \text{that is,}\qquad n^2+3n+3 &> 0, \end{aligned}$$

which is the case for $n \geq 1$. So the terms of the series are nonincreasing. By the Alternating series Test, the original series converges. Since the series of absolute values diverges, the original series is $\boxed{\textit{conditionally convergent.}}$

20. The series is $2+\frac{3}{2}\cdot\frac{1}{4}+\frac{4}{3}\cdot\frac{1}{4^2}+\frac{5}{4}\cdot\frac{1}{4^3}+\cdots = \sum\limits_{k=1}^{\infty}\frac{k+1}{k}\cdot\frac{1}{4^{k-1}} = \sum\limits_{k=1}^{\infty}\frac{k+1}{k\,4^{k-1}} = \sum\limits_{k=1}^{\infty}a_k$. Since the nth term behaves like

$$a_n = \frac{n+1}{n\,4^{n-1}} = \frac{1}{4^{n-1}}\left(1+\frac{1}{n}\right) \approx \frac{1}{4^{n-1}}$$

for large n, we compare with the series $\sum\limits_{k=1}^{\infty}b_k = \sum\limits_{k=1}^{\infty}\frac{3}{4^{k-1}} = \sum\limits_{k=1}^{\infty}3\left(\frac{1}{4}\right)^{k-1}$ which is a convergent geometric series since $|r| = \frac{1}{4} < 1$. Now, we have

$$a_n = \frac{n+1}{n}\left(\frac{1}{4^{n-1}}\right) = \frac{n+1}{3n}\left(\frac{3}{4^{n-1}}\right) = \frac{n+1}{3n}\,b_n < \frac{n+n}{3n}\,b_n = \frac{2}{3}\,b_n < b_n$$

for all $n \geq 1$. So by the Comparison Test, the series $\sum\limits_{k=1}^{\infty}a_k$ converges. The terms of the series are positive, so this means that the series also $\boxed{\textit{converges absolutely.}}$

21. The series is $1+\frac{1\cdot3}{2!}+\frac{1\cdot3\cdot5}{3!}+\frac{1\cdot3\cdot5\cdot7}{4!}+\cdots$. Note that

$$\frac{1\cdot3\cdot5\cdot7}{4!} = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}{(2\cdot4\cdot6\cdot8)4!} = \frac{8!}{2^4[1\cdot2\cdot3\cdot4]4!} = \frac{8!}{2^4\cdot(4!)(4!)},$$

so the nth term of the series can be written as $a_n = \frac{(2n)!}{2^n\cdot(n!)(n!)}$. By the Ratio Test, which is indicated since factorials are involved, and the terms of the series are nonzero, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(2(n+1))!}{2^{n+1}\cdot(n+1)!(n+1)!}}{\frac{(2n)!}{2^n\cdot(n!)(n!)}}\right| = \lim_{n\to\infty}\left|\frac{2^n}{2^{n+1}}\cdot\frac{n!}{(n+1)!}\cdot\frac{n!}{(n+1)!}\cdot\frac{(2(n+1))!}{(2n)!}\right|$$

$$= \frac{1}{2}\lim_{n\to\infty}\left|\frac{n!}{(n+1)n!}\cdot\frac{n!}{(n+1)n!}\cdot\frac{(2n+2)(2n+1)(2n)!}{(2n)!}\right|$$

$$= \frac{1}{2}\lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)}{(n+1)(n+1)}\right| = \frac{1}{2}\lim_{n\to\infty}\left|\frac{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}\right|$$

$$= \frac{1}{2}\cdot 4 = 2 > 1.$$

Since the limit is greater than 1, by the Ratio Test, the series $\sum_{k=1}^{\infty} a_k$ $\boxed{\textit{diverges.}}$

22. The series is $\frac{1}{\sqrt{1\cdot2\cdot3}}+\frac{1}{\sqrt{2\cdot3\cdot4}}+\frac{1}{\sqrt{3\cdot4\cdot5}}+\cdots=\sum_{k=1}^{\infty}a_k=\sum_{k=1}^{\infty}\frac{1}{\sqrt{k(k+1)(k+2)}}$. The nth term satisfies

$$a_n = \frac{1}{\sqrt{n(n+1)(n+2)}} = \frac{1}{n^{3/2}\sqrt{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}} < \frac{1}{n^{3/2}} = b_n$$

for $n \geq 1$. By comparing with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}\frac{1}{k^{3/2}}$, which is a convergent p-series (since $p=\frac{3}{2}>1$), by the Comparison Test, we see that the original series $\sum_{k=1}^{\infty} a_k$ converges. Since the terms of the series $\sum_{k=1}^{\infty} a_k$ are all positive, this series is also $\boxed{\textit{absolutely convergent.}}$

23. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\frac{k!}{(2k)!}$ is a series of nonzero terms and has factorials, so the use of the Ratio Test is indicated. The ratio of the $n+1$st and nth term of the series is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(2(n+1))!}}{\frac{n!}{(2n)!}} = \frac{(n+1)!}{n!}\cdot\frac{(2n)!}{(2(n+1))!} = \frac{(n+1)n!}{n!}\cdot\frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{(n+1)}{(2n+2)(2n+1)}.$$

Taking the limit of the absolute value of the ratio found above gives

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{(2n+2)(2n+1)}\right| = \lim_{n\to\infty}\frac{1}{2n+2}\cdot\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2+\frac{1}{n}} = 0\cdot\frac{1}{2} = 0.$$

Since the limit is less than 1, by the Ratio Test, the series is convergent. Additionally, the terms of the series are all positive, so this means that the series is also $\boxed{\textit{absolutely convergent.}}$

24. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} k^3e^{-k^4}$. Now the function $f(x)=x^3e^{-x^4}$ is defined on $[1,\infty)$, is continuous, and decreasing for all $x \geq 1$, since

$$f'(x) = 3x^2e^{-x^4} + x^3(-4x^3)e^{-x^4} = (3x^2-4x^6)e^{-x^4} < 0$$

when $3x^2-4x^6<0$, or $3<4x^4$, or $x>\sqrt[4]{\frac{3}{4}}$, but $\sqrt[4]{\frac{3}{4}}<1$. So, for $x\geq1$, the series decreases, as claimed. Also, $f(k)=a_k$ for all integers $k\geq1$. By the Integral Test, we have

$$I = \int_1^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_1^b x^3e^{-x^4}\,dx.$$

Let $u = e^{-x^4}$. Then $du = -4x^3 e^{-x^4}\,dx$. Continuing, we have

$$I = -\frac{1}{4}\lim_{b\to\infty}\int_{e^{-1}}^{e^{-b^4}} du = -\frac{1}{4}\lim_{b\to\infty}[u]\Big|_{e^{-1}}^{e^{-b^4}} = -\frac{1}{4}\lim_{b\to\infty}\left(e^{-b^4} - e^{-1}\right) = -\frac{1}{4}\left(0 - e^{-1}\right) = \frac{1}{4e}.$$

Since the improper integral I converges, by the Integral Test, the series converges as well. Since the terms of the series are all positive, the series is $\boxed{\textit{absolutely convergent.}}$

25. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k}+100}$. Comparing with the divergent p-series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k}}$, which diverges since $0 < p = \frac{1}{2} < 1$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{\sqrt{n}+100}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}+100} = \lim_{n\to\infty}\frac{1}{1+\frac{100}{\sqrt{n}}} = 1.$$

Since the limit is a positive real number, the series $\boxed{\textit{diverges}}$ by the Limit Comparison Test.

26. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k^2+5k}{3+5k^2}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^2+5n}{3+5n^2} = \lim_{n\to\infty}\frac{1+\frac{5}{n}}{\frac{3}{n^2}+5} = \frac{1}{5} \neq 0.$$

Since the limit of the nth term is nonzero, the series $\boxed{\textit{diverges}}$ by the Divergence Test.

27. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^4+4}}$. We have

$$a_n = \frac{1}{\sqrt[3]{n^4+4}} < \frac{1}{\sqrt[3]{n^4}} = \frac{1}{n^{4/3}} = b_n.$$

Comparing the series with the convergent p-series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^{4/3}}$, which is convergent since $p = \frac{4}{3} > 1$, we see that the original series $\sum\limits_{k=1}^{\infty} a_k$ is convergent by the Comparison Test. Further, since all the terms of $\sum\limits_{k=1}^{\infty} a_k$ are positive, the series is $\boxed{\textit{absolutely convergent.}}$

28. The series can be written as $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{11}\left(-\frac{3}{2}\right)^k = \sum\limits_{k=1}^{\infty} (-1)^k \frac{1}{11}\left(\frac{3}{2}\right)^k$. We have

$$\lim_{n\to\infty}|a_n| = \frac{1}{11}\lim_{n\to\infty}\left(\frac{3}{2}\right)^n = \infty \neq 0,$$

using the result stated in Section 8.1, p. 639, and proved in Section 8.1, Problem 140, that $\lim\limits_{n\to\infty} r^n = \infty$ if $|r| > 1$. Since the limit of the absolute value of the nth term is nonzero, the series $\boxed{\textit{diverges}}$ by the Divergence Test.

29. The series is $\frac{1}{3} - \frac{2}{4} + \frac{3}{5} - \frac{4}{6} + \cdots = \sum\limits_{k=1}^{\infty} (-1)^{k+1}\frac{k}{k+2}$. We have

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{n}{n+2} = \lim_{n\to\infty}\frac{1}{1+\frac{2}{n}} = 1 \neq 0.$$

Since the limit of the absolute value of the nth term is nonzero, the series $\boxed{\textit{diverges}}$ by the Divergence Test.

30. The series can be written

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k(-4)^{3k}}{5^k} = \sum_{k=1}^{\infty} (-1)^{3k} \frac{k(4)^{3k}}{5^k} = \sum_{k=1}^{\infty} [(-1)^3]^k \frac{k(4^3)^k}{5^k} = \sum_{k=1}^{\infty} (-1)^k k \left(\frac{64}{5}\right)^k .$$

We have

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} n \cdot \left(\frac{64}{5}\right)^n = \lim_{n\to\infty} n \cdot \lim_{n\to\infty} \left(\frac{64}{5}\right)^n = \infty,$$

since both limits are separately ∞ (for the second limit, we use the result quoted in Problem 28 above). Since the limit of the absolute value of the nth term is nonzero, the series $\boxed{\textit{diverges}}$ by the Divergence Test.

31. The series is $\sum_{k=1}^{\infty} (-1)^k a_k = \sum_{k=1}^{\infty} \left(-\frac{1}{k}\right)^k = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k^k}$. The series of absolute values is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^k}$, the "k to the k series". You could use the Table 6, p.698 to assert convergence. However, that this is convergent can be directly deduced using the root test (note that the terms of the series of absolute values are nonzero, and the nth term has an nth power, so the Root Test is applicable) :

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^n}\right|} = \lim_{n\to\infty} \left|\frac{1}{n}\right| = 0 < 1.$$

Since the limit is less than 1, by the Root Test, the series of absolute values converges, which means the original series is $\boxed{\textit{absolutely convergent.}}$

32. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{5}{2^k+1}$. We have

$$a_n = \frac{5}{2^n+1} < \frac{5}{2^n} = b_n.$$

Comparing with the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{5}{2^k} = \sum_{k=1}^{\infty} \frac{5}{2}\left(\frac{1}{2}\right)^{k-1}$, which is a convergent geometric series (since $|r| = \frac{1}{2} < 1$), we see that the original series converges by the Comparison Test. Since the terms of the original series are all positive, we conclude that series must be $\boxed{\textit{absolutely convergent.}}$

33. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} e^{-k^2}$ is a series of nonzero terms. Since the nth term has an nth power, the Root Test would be ideal to use:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\left|e^{-n^2}\right|\right)^{1/n} = \lim_{n\to\infty} \left|e^{-n}\right| = 0 < 1.$$

Since the limit is less than 1, by the Root Test, the series converges. Since its terms are all positive, the series also $\boxed{\textit{converges absolutely.}}$

34. The series is $\frac{\sin\sqrt{1}}{1^{3/2}} + \frac{\sin\sqrt{2}}{2^{3/2}} + \frac{\sin\sqrt{3}}{3^{3/2}} + \cdots = \sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\sin\sqrt{k}}{k^{3/2}}$. The series of absolute values is $\sum\limits_{k=1}^{\infty} |a_k| = \sum\limits_{k=1}^{\infty} \frac{|\sin\sqrt{k}|}{k^{3/2}}$. We have

$$|a_n| = \frac{|\sin\sqrt{n}|}{n^{3/2}} \le \frac{1}{n^{3/2}} = b_n,$$

since $|\sin\sqrt{n}| \le 1$ for all $n \ge 1$. Comparing with the convergent (since $p = \frac{3}{2} > 1$) p-series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^{3/2}}$, by the Comparison Test, we see the series of absolute values is convergent, which means that the original series must be $\boxed{\textit{absolutely convergent.}}$

35. The series is $\sum\limits_{k=2}^{\infty} (-1)^{k+1} a_k = \sum\limits_{k=2}^{\infty} \frac{(-1)^{k+1}}{k(\ln k)^3}$. The series of absolute values is $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$. Consider the function $f(x) = \frac{1}{x(\ln x)^3}$ which is defined, decreasing and continuous on $[2, \infty)$, and for which $a_k = f(k)$ for all $k \ge 2$. Applying the Integral Test, we have

$$I = \int_2^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^3}.$$

Let $u = \ln x$. Then $du = \frac{dx}{x}$. Continuing,

$$\begin{aligned} I &= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u^3} = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u^{-3}\,du = \lim_{b\to\infty} \left[\frac{u^{-3+1}}{-3+1}\right]_{\ln 2}^{\ln b} \\ &= -\frac{1}{2} \lim_{b\to\infty} \left[\frac{1}{u^2}\right]_{\ln 2}^{\ln b} = -\frac{1}{2} \lim_{b\to\infty} \left[\frac{1}{(\ln b)^2} - \frac{1}{(\ln 2)^2}\right] \\ &= -\frac{1}{2}\left[0 - \frac{1}{(\ln 2)^2}\right] = \frac{1}{2(\ln 2)^2} \end{aligned}$$

Since the improper integral I converges, by the Integral Test, the series also converges. Moreover, the terms of the series are all positive, so that the original series is $\boxed{\textit{absolutely convergent.}}$

36. The series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{(2k)^k}$ is a series of nonzero terms. Since its nth term involves an nth power, the Root Test is the best choice:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{(2n)^n}\right|} = \lim_{n\to\infty} \left|\frac{1}{2n}\right| = 0.$$

Since the limit is less than 1, by the Root Test, the series converges. The terms of the series are all positive, so this series also is $\boxed{\textit{absolutely convergent.}}$

37. The series $\sum\limits_{k=2}^{\infty} a_k = \sum\limits_{k=2}^{\infty} \left(\frac{\ln k}{1000}\right)^k$ is a series of nonzero terms. Since the nth term involves an nth power, we use the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{\ln n}{1000}\right)^n\right|} = \lim_{n\to\infty} \left|\frac{\ln n}{1000}\right| = \infty.$$

By the Root Test, since the limit is greater than 1, the series $\boxed{\textit{diverges.}}$

38. The series is

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{\cosh^2 k} = \sum_{k=1}^{\infty} \frac{1}{\left(\frac{e^k+e^{-k}}{2}\right)^2} = \sum_{k=1}^{\infty} \frac{4}{e^{2k}+e^{-2k}+2}.$$

Here the standard definition $\cosh k = \frac{e^k+e^{-k}}{2}$ was used. We have

$$a_n = \frac{4}{e^{2n}+e^{-2n}+2} < \frac{4}{e^{2n}} = b_n$$

since $e^{-2n}+2 > 0$ for $n \geq 1$. Now, we can write the series $\sum\limits_{k=1}^{\infty} b_k$ as follows:

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{4}{e^{2k}} = \sum_{k=1}^{\infty} \frac{4}{(e^2)^k} = \sum_{k=1}^{\infty} \frac{4}{e^2} \cdot \left(\frac{1}{e^2}\right)^{k-1}.$$

We see that it is a geometric series of the form $\sum\limits_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{4}{e^2}$ and $|r| = \frac{1}{e^2} < 1$ so it converges. By the Comparison Test, the original series also converges. Since the terms of the original series are positive, it is $\boxed{\textit{absolutely convergent.}}$

39. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\tan^{-1} k}{k^2}$. The nth term satisfies

$$a_n = \frac{\tan^{-1} n}{n^2} < \frac{\pi}{2n^2} = b_n$$

since $\tan^{-1} n < \frac{\pi}{2}$ for any $n \geq 1$. Since the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{\pi}{2k^2}$ is a constant multiple of a convergent p-series (convergent since $p = 2 > 1$), the original series also converges by the Comparison Test. Since $\tan^{-1} n > 0$ for all $n \geq 1$, the terms of the original series are all positive, which in turn means that the original series is also $\boxed{\textit{absolutely convergent.}}$

40. The nth partial sum of the series $\sum\limits_{k=1}^{\infty} (\sqrt{k+1} - \sqrt{k})$ is

$$\begin{aligned} S_n &= \sum_{k=1}^{n} (\sqrt{k+1} - \sqrt{k}) \\ &= (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{n} - \sqrt{n-1}) + (\sqrt{n+1} - \sqrt{n}) \\ &= \sqrt{n+1} - \sqrt{1}. \end{aligned}$$

Since $\lim\limits_{n\to\infty} S_n = \infty$, the sequence $\{S_n\}$ of partial sums diverges, which means that the series $\boxed{\textit{diverges}}$ as well.

41. The nth partial sum of the series $\sum_{k=4}^{\infty}\left(\frac{1}{k-3}-\frac{1}{k}\right)$ is

$$\begin{aligned}
S_n &= \sum_{k=4}^{n}\left(\frac{1}{k-3}-\frac{1}{k}\right)\\
&= \left(\frac{1}{1}-\frac{1}{4}\right)+\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{3}-\frac{1}{6}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\\
&\quad +\cdots+\left(\frac{1}{n-7}-\frac{1}{n-4}\right)+\left(\frac{1}{n-6}-\frac{1}{n-3}\right)+\left(\frac{1}{n-5}-\frac{1}{n-2}\right)+\\
&\quad +\left(\frac{1}{n-4}-\frac{1}{n-1}\right)+\left(\frac{1}{n-3}-\frac{1}{n}\right)\\
&= 1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n-2}-\frac{1}{n-1}-\frac{1}{n}.
\end{aligned}$$

The limit of the sequence $\{S_n\}$ is:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n-2}-\frac{1}{n-1}-\frac{1}{n}\right) = 1+\frac{1}{2}+\frac{1}{3} = \frac{6+3+2}{6} = \frac{11}{6}.$$

Since the sequence of partial sums $\{S_n\}$ converges, the series $\boxed{\textit{converges}}$ as well, and the limit of the sequence of partial sums is also the sum of the series, which is $\boxed{\frac{11}{6}.}$

42. The series is $\sum_{k=2}^{\infty}\ln\left(\frac{k}{k+1}\right) = \sum_{k=2}^{\infty}(\ln k-\ln(k+1))$. The nth partial sum of the series is

$$\begin{aligned}
S_n &= \sum_{k=2}^{n}(\ln k-\ln(k+1))\\
&= (\ln 1-\ln 2)+(\ln 2-\ln 3)+\cdots+(\ln(n-1)-\ln n)+(\ln n-\ln(n+1))\\
&= \ln 1-\ln(n+1) = -\ln(n+1).
\end{aligned}$$

Since $\lim_{n\to\infty} S_n = -\infty$, the sequence of partial sums $\{S_n\}$ diverges, which in turn means that the series $\boxed{\textit{diverges.}}$

43. The series is

$$1+\frac{1\cdot 2}{1\cdot 3}+\frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5}+\frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7}+\cdots = \sum_{k=1}^{\infty}\frac{2^k\cdot(k!)(k!)}{(2k)!},$$

where we use the fact that

$$\frac{1}{1\cdot 3\cdot 5\cdot 7\cdots(2k-1)} = \frac{2\cdot 4\cdot 6\cdot 8\cdots(2k)}{1\cdot 2\cdot 3\cdot 4\cdots(2k)} = \frac{2^k\cdot[1\cdot 2\cdot 3\cdot 4\cdots k]}{(2k)!} = \frac{2^k\cdot(k!)}{(2k)!}.$$

Since the terms of the series are nonzero, and factorials are involved in the nth term, the Ratio Test suggests itself to be used. The ratio of the $n+1$st and the nth term of the series is

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\frac{2^{n+1}\cdot(n+1)!(n+1)!}{(2(n+1))!}}{\frac{2^n\cdot(n!)(n!)}{(2n)!}} = \frac{2^{n+1}}{2^n}\cdot\frac{(n+1)!}{n!}\cdot\frac{(n+1)!}{n!}\cdot\frac{(2n)!}{(2(n+1))!}\\
&= 2\cdot\frac{(n+1)n!}{n!}\cdot\frac{(n+1)n!}{n!}\cdot\frac{(2n)!}{(2n+2)(2n+1)(2n)!}\\
&= 2\cdot\frac{(n+1)(n+1)}{(2n+2)(2n+1)}.
\end{aligned}$$

The limit of the absolute value of the ratio found above is

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2\lim_{n\to\infty}\left|\frac{(n+1)(n+1)}{(2n+2)(2n+1)}\right| = 2\lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}\right| = 2\left|\frac{1\cdot 1}{2\cdot 2}\right| = 2\cdot\frac{1}{4} = \frac{1}{2} < 1.$$

Since the limit is less than 1, by the Ratio Test, the series $\boxed{\textit{converges.}}$

44. (a) The series

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\left[\left(\frac{2}{3}\right)^k - \frac{2}{k^2+2k}\right] = \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k - \sum_{k=1}^{\infty}\frac{2}{k^2+2k}$$

is the difference between two series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k$ and $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty}\frac{2}{k^2+2k}$. We show below that each of these series converges, so the original series must converge, which in turn justifies our splitting the series into two parts (which is allowed only for absolutely convergent series). The series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{k-1}$ is a geometric series of the form $\sum_{k=1}^{\infty} ar^{k-1}$, with $a = \frac{2}{3}$, and $r = \frac{2}{3}$. Since $|r| < 1$, this series converges to a sum of

$$\frac{a}{1-r} = \frac{2/3}{1-2/3} = \frac{2/3}{1/3} = 2.$$

The nth term of the series $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty}\frac{2}{k^2+2k}$ satisfies

$$c_n = \frac{2}{n^2+2n} < \frac{2}{n^2} = d_n$$

for $n \geq 1$. Since the series $\sum_{k=1}^{\infty} d_k = \sum_{k=1}^{\infty}\frac{2}{k^2}$ is a constant multiple of a convergent p-series (since $p = 2 > 1$), by the Comparison Test, we see that the series $\sum_{k=1}^{\infty} c_k$ converges. These calculations show that the original series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k - \sum_{k=1}^{\infty} c_k$ converges.

(b) We have already summed the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = 2$ above. To sum the series $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty}\frac{2}{k^2+2k}$, consider its nth partial sum, which can be simplified using partial fractions and the telescoping nature of the terms as follows:

$$\begin{aligned}
S_n &= \sum_{k=1}^{n}\frac{2}{k^2+2k} = \sum_{k=1}^{n}\frac{2}{k(k+2)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+2}\right)\\
&= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots +\\
&\quad + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right)\\
&= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}.
\end{aligned}$$

We have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2}.$$

Since the sequence $\{S_n\}$ of partial sums converges to a sum, the series converges to the same sum, that is, $\sum\limits_{k=1}^{\infty} c_k = \frac{3}{2}$. Finally, the required sum of the original series is

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k - \sum_{k=1}^{\infty} c_k = 2 - \frac{3}{2} = \boxed{\frac{1}{2}.}$$

45. (a) The series

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left[\left(-\frac{1}{4}\right)^k + \frac{3}{k(k+1)}\right] = \sum_{k=1}^{\infty} \left(-\frac{1}{4}\right)^k + \sum_{k=1}^{\infty} \frac{3}{k(k+1)}$$

is the sum of two series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \left(-\frac{1}{4}\right)^k$ and $\sum\limits_{k=1}^{\infty} c_k = \sum\limits_{k=1}^{\infty} \frac{3}{k(k+1)}$. We show below that each of these series converges, so the original series converges as well, which justifies our splitting the series into two parts (which is allowed only for absolutely convergent series). The series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \left(-\frac{1}{4}\right)^k = \sum\limits_{k=1}^{\infty} \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{k-1}$ is a geometric series of the form $\sum\limits_{k=1}^{\infty} ar^{k-1}$, with $a = -\frac{1}{4}$ and $r = -\frac{1}{4}$. Since $|r| < 1$, the series converges to a sum of

$$\frac{a}{1-r} = \frac{-\frac{1}{4}}{1-\left(-\frac{1}{4}\right)} = \frac{-\frac{1}{4}}{1+\frac{1}{4}} = \frac{-\frac{1}{4}}{\frac{5}{4}} = -\frac{1}{5}.$$

The nth term of the series $\sum\limits_{k=1}^{\infty} c_k = \sum\limits_{k=1}^{\infty} \frac{3}{k(k+1)}$ satisfies

$$c_n = \frac{3}{n(n+1)} < \frac{3}{n^2} = d_n$$

for $n \geq 1$. Since the series $\sum\limits_{k=1}^{\infty} d_k = \sum\limits_{k=1}^{\infty} \frac{3}{k^2}$ is a constant multiple of a convergent p-series (since $p = 2 > 1$), by the Comparison Test, we see that the series $\sum\limits_{k=1}^{\infty} c_k$ converges. These calculations demonstrate that the original series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} b_k + \sum\limits_{k=1}^{\infty} c_k$ converges.

(b) We have already summed the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \left(-\frac{1}{4}\right)^k = -\frac{1}{5}$ above. To sum the series $\sum\limits_{k=1}^{\infty} c_k = \sum\limits_{k=1}^{\infty} \frac{3}{k(k+1)}$ consider its nth partial sum, which can be simplified using partial fractions and the telescoping nature of the terms as follows:

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{3}{k(k+1)} = \sum_{k=1}^{n} 3\left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= 3\left(\frac{1}{1} - \frac{1}{2}\right) + 3\left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + 3\left(\frac{1}{n-1} - \frac{1}{n}\right) + 3\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 3 - \frac{3}{n+1}. \end{aligned}$$

We have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(3 - \frac{3}{n+1}\right) = 3.$$

Since the sequence $\{S_n\}$ of partial sums converges to a sum, the series converges to the same sum, that is $\sum\limits_{k=1}^{\infty} c_k = 3$. Finally, the required sum of the original series is

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k + \sum_{k=1}^{\infty} c_k = -\frac{1}{5} + 3 = \boxed{\frac{14}{5}.}$$

Challenge Problems

46. **(a)** The series is

$$1 - 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{3} - \frac{1}{9} - \frac{1}{4} + \frac{1}{27} + \frac{1}{5} - \frac{1}{81} - \cdots .$$

The nth term of this series is

$$a_n = \left[(-1)^{n+1}\frac{1}{n} + (-1)^n \frac{1}{3^{n-1}}\right] = (-1)^{n-1}\left[\frac{1}{n} - \frac{1}{3^n}\right].$$

To show the series converges, we apply the Alternating Series Test. We have

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{3^n}\right) = 0 - 0 = 0.$$

Also, by the Algebraic Difference Test,

$$|a_{n+1}| - |a_n| = \frac{1}{n+1} - \frac{1}{3^{n+1}} - \frac{1}{n} + \frac{1}{3^n} = \frac{1}{n+1} - \frac{1}{n} - \frac{1}{3}\cdot\frac{1}{3^n} + \frac{1}{3^n} = \frac{n-n-1}{n(n+1)} + \frac{2}{3}\cdot\frac{1}{3^n} = -\frac{1}{n(n+1)} + \frac{2}{3^{n+1}} < 0$$

provided

$$\frac{2}{3^{n+1}} < \frac{1}{n(n+1)}$$

$$\text{or,} \qquad 2n(n+1) < 3^{n+1}.$$

The left side of the inequality is a polynomial whose terms are $\{4, 12, 24, \cdots\}$ for $n = 1, 2, 3, \cdots$, while the right side of the inequality is an exponential series with terms $\{9, 27, 81, \cdots\}$ for $n = 1, 2, 3, \cdots$. So we see that the inequality is satisfied for $n \geq 1$, so by the Alternating Series Test, the series converges.

(b) Since the series converges to a sum, we can split up the nth term as two terms and sum them separately, so that the sum of the series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} b_k + \sum\limits_{k=1}^{\infty} c_k$. Here, the first sum

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

is the alternating harmonic series, which we know to be a convergent series (using, for example, the Alternating Series Test, or consulting the Table 6 on p.698). The alternating harmonic series has a sum $\sum\limits_{k=1}^{\infty} b_k = \ln 2$ (see the note on p.681).

The second sum is

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{3^{k-1}} = \frac{-1}{3^0} + \frac{1}{3^1} + \frac{-1}{3^2} + \frac{1}{3^3} + \frac{-1}{3^4} \cdots = -1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \cdots ,$$

is a geometric series $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{3^{k-1}} = \sum_{k=1}^{\infty} \left(-\frac{(-1)^{k-1}}{3^{k-1}}\right) = \sum_{k=1}^{\infty} -\left(-\frac{1}{3}\right)^{k-1}$ of the form $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = -1$ and $r = -\frac{1}{3}$. Since $|r| < 1$, the geometric series also converges to a sum

$$\frac{a}{1-r} = \frac{-1}{1-\left(-\frac{1}{3}\right)} = \frac{-1}{1+1/3} = -\frac{3}{4}.$$

So the sum of the original series is

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k + \sum_{k=1}^{\infty} c_k = \boxed{\ln 2 - \frac{3}{4}.}$$

47. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\ln k}{2k^3-1}$. The nth term of this series satisfies

$$a_n = \frac{\ln n}{2n^3-1} < \frac{n}{2n^3-1} = b_n$$

since $\ln n < n$ for $n \geq 1$. (To see this, let $f(x) = \ln x - x$ be a related function of $\ln n - n$. Then $f'(x) = \frac{1}{x} - 1 \leq 0$ when $x \geq 1$. So the function is decreasing for $x \geq 1$. Since $f(1) = \ln 1 - 1 = 0 - 1 = -1 < 0$, and the function is decreasing for $x \geq 1$, it means that $f(x) < 0$ for all $x \geq 1$, that is, $\ln x < x$ for all $x \geq 1$. So, we have $\ln n < n$ for $n \geq 1$ as well.) Now the series $\sum_{k=1}^{\infty} b_k$ converges by comparing it with the convergent p-series $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$, which is convergent since $p = 2 > 1$. We have

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{\frac{n}{2n^3-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3}{2n^3-1} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}.$$

Since the limit is a positive real number, by the Limit Comparison Test, the series $\sum_{k=1}^{\infty} b_k$ converges; and this in turn, via the Comparison Test, means that the original series $\sum_{k=1}^{\infty} a_k$ $\boxed{\textit{converges}}$ as well.

48. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \sin^3\left(\frac{1}{k}\right)$. Note that the terms of the series are positive, since $0 < \frac{1}{n} \leq 1$ for $1 \leq n < \infty$, and $\sin\theta$ is positive for $\theta = \frac{1}{n}$ that satisfies $0 < \theta \leq 1$ (radians). We compare the series to $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^3}$, which is a convergent p-series since $p = 3 > 1$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin^3\left(\frac{1}{n}\right)}{\frac{1}{n^3}} = \lim_{n\to\infty} \left[\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\right]^3 = \left[\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\right]^3 = \left[\lim_{x\to 0} \frac{\sin x}{x}\right]^3 = 1^3 = 1,$$

where we made the substitution $x = \frac{1}{n}$ and used the standard limits result $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Since the limit is a positive real number, by the Limit Comparison Test, the original series $\sum_{k=1}^{\infty} a_k$ is seen to $\boxed{\textit{converge.}}$

Note: We could instead have used the Comparison Test for convergence using the series $\sum_{k=1}^{\infty} b_k$, since the nth term of the absolute value series $\sum_{k=1}^{\infty} |a_k|$ satisfies

$$|a_n| = \left|\sin\left(\frac{1}{n}\right)\right|^3 \leq \frac{1}{n^3}$$

for all $n \geq 1$, since $|\sin\theta| < \theta$ for $\theta = \frac{1}{n}$ in the interval $(0, 1]$. Since the absolute value series converges by the Comparison Test, the original series converges absolutely, and therefore converges.

8.8 Power Series

Concepts and Vocabulary

1. True: The theorem regarding a power series in the form $\sum_{k=0}^{\infty} a_k(x-c)^k$ states that the series converges either at $x = c$, or converges absolutely for all x, or converges absolutely on $|x - c| < R$, where R is a positive number. (For a full statement, see p.602.)

2. True: This follows from the Ratio Test for convergence of a series.

3. True: This follows from the theorem concerning a power series in the form $\sum_{k=0}^{\infty} a_k(x-c)^k$ with $c = 0$; see Problem 1 for the reasoning.

4. False: There is no reason that this happen. For example, the power series $\sum_{k=1}^{\infty} \frac{x^k}{k}$ converges absolutely if $|x| < 1$ since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{n}{n+1}\right| = |x|\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |x|\lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right| = |x| < 1$$

for convergence by the Ratio Test, as claimed. This means the interval of convergence is $(-1, 1)$.
At $x = 1$, the series becomes $\sum_{k=1}^{\infty} \frac{1}{k}$ which, being the harmonic series, diverges.
At $x = -1$, the series becomes $-\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ which, being a constant multiple of the convergent alternating harmonic series also converges. So the power series is convergent on the interval $[-1, 1)$: that is, it is convergent at the one endpoint of its interval of convergence but not at the other endpoint.

5. True: This follows from the theorem on the radius of convergence of a power series, which says the power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges absolutely for all $|x - c| < R$. The radius R here is a property of the series and not of the value of c, which means that the power series will have the same radius of convergence for any value of c.

6. False: If R is the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k(x-c)^k$, then $\sum_{k=0}^{\infty} a_k x^k$ converges in the interval $-R < x < R$, while $\sum_{k=0}^{\infty} a_k(x-3)^k$ converges in the interval

$-R < x - 3 < R$, or simplifying, $-R + 3 < x < R + 3$. This shows that the two series have different intervals of convergence.

7. False: This follows from the theorem on convergence and divergence of a power series, which says that if a power series $\sum\limits_{k=0}^{\infty} a_k x_k$ converges for $x_0 \neq 0$, then it converges absolutely for all numbers x for which $|x| < |x_0|$, or $-|x_0| < x < |x_0|$. The endpoints $-|x_0|$ and $|x_0|$ have to be examined separately, and if the series converges at $|x_0| = 8$, then it does not follow that the series also converges at $-|x_0| = -8$.

8. True: From the theorem quoted in Problem 7, if $|x_0|$=3, then the power series converges in the interval $-3 < x < 3$, so it also converges for $x = 1$.

9. True: As in Problems 7 and 8, if the series converges for $x = -4$, this means $|x_0| = 4$, so since the series converges in the interval $-4 < x < 4$, it converges for $x = 3$ as well.

10. False: If the series converges for $x = 3$, then it may converge or diverge at $x = 5$ because the radius of convergence of the series $\sum\limits_{k=0}^{\infty} a_k x^k$ is not given or known. For instance, if the radius of convergence is 4, then the series will diverge at $x = 5$, but if the radius of convergence is 7, then the series will converge at $x = 5$.

11. False: Intervals of convergence for the power series $\sum\limits_{k=0}^{\infty} a_k x^k$ could be of the form $|x| \leq R$, or $[-R, R]$, if convergence at the endpoints can be demonstrated. So we can have intervals of the form $[-2, 2]$ or $[-4, 4]$, but not $[-2, 4]$. This is because if the interval of convergence were $[-2, 4]$ it would mean the power series diverges for $x < -2$, but by the theorem on convergence and divergence of a power series, it would then mean that the series also diverges for $x > 2$, which would contradict the statement that it converges in the interval $[-2, 4]$.

12. False: All that can be said is that if a power series diverges for $x = x_1$, then it diverges for all $x > x_1$. It may or may not converge for $x < x_1$. As an illustration, the power series considered in Problem 4 above, $\sum\limits_{k=1}^{\infty} \frac{x^k}{k}$ diverges for $x_1 = 3$, but it also diverges for $x = 2$, and $2 < 3$. On the other hand, it converges at $x = \frac{1}{2}$, (since the radius of convergence of the power series is 1), and $\frac{1}{2} < 3$ as well. So this shows that a series can either converge or diverge for x values less than a certain value x_1 that leads to divergence of the series.

Skill Building

13. The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} kx^k$ is a series of nonzero terms for $k \geq 1$ if $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{nx^n}\right| = |x| \lim_{n\to\infty} \left(\frac{n+1}{n}\right) = |x| \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = |x| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 1$ or $-1 < x < 1$.
For $x = -1$, the series becomes

$$\sum_{k=0}^{\infty} kx^k = \sum_{k=0}^{\infty} (-1)^k k = 0 + -1 + 2 - 3 + 4 - \cdots$$

which diverges since the sequence of partial sums is $\{S_n\} = \{0, -1, 1, -2, 2, -3, 3, \cdots\}$, and it oscillates between growing positive and negative numbers as n grows large. Since the sequence of partial sums is not convergent, the series is not convergent.
For $x = 1$, the series becomes

$$\sum_{k=0}^{\infty} kx^k = \sum_{k=0}^{\infty} k(1)^k = \sum_{k=0}^{\infty} k = \infty,$$

so it diverges.

Concluding, the power series $\sum\limits_{k=0}^{\infty} kx^k$ converges for $\boxed{-1 < x < 1.}$

14. The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{kx^k}{3^k}$ is a series of nonzero terms for $k \geq 1$ if $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)x^{n+1}}{3^{n+1}}}{\frac{nx^n}{3^n}}\right| = \frac{|x|}{3} \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x|}{3} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \frac{|x|}{3} < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 3$, or $-3 < x < 3$.
For $x = -3$, the series becomes

$$\sum_{k=0}^{\infty} \frac{kx^k}{3^k} = \sum_{k=0}^{\infty} \frac{(-k)(-3)^k}{3^k} = \sum_{k=0}^{\infty} (-1)^{k+1} k = 0 + 1 - 2 + 3 - 4 + \cdots .$$

The sequence of partial sums of the series is $\{S_n\} = \{0, 1, -1, 2, -2, \cdots\}$ which oscillates between growing positive and negative values as n grows large. Since the sequence of partial sums does not converge, neither does the series.
For $x = 3$, the series becomes

$$\sum_{k=0}^{\infty} \frac{kx^k}{3^k} = \sum_{k=0}^{\infty} \frac{k3^k}{3^k} = \sum_{k=0}^{\infty} k = \infty,$$

so this series diverges.

Concluding, the power series $\sum\limits_{k=0}^{\infty} \frac{kx^k}{3^k}$ converges for $\boxed{-3 < x < 3.}$

15. The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{(x+1)^k}{3^k} = 1 + \sum\limits_{k=1}^{\infty} \frac{(x+1)^k}{3^k}$ is a series of nonzero terms if $x \neq -1$. (At $x = -1$, the series converges to 1.) We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(x+1)^{n+1}}{3^{n+1}}}{\frac{(x+1)^n}{3^n}}\right| = \frac{1}{3}|x+1| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x+1| < 3$, or $-3 < x + 1 < 3$, or $-4 < x < 2$.
At $x = -4$, the series becomes

$$\sum_{k=0}^{\infty} \frac{(x+1)^k}{3^k} = \sum_{k=0}^{\infty} \frac{(-4+1)^k}{3^k} = \sum_{k=0}^{\infty} \frac{(-3)^k}{3^k} = \sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \cdots .$$

The sequence of partial sums of the series is $\{S_n\} = \{1, 0, 1, 0, \cdots\}$ which oscillates between 1 and 0 for all n. Since the sequence of partial sums does not converge, neither does the series.

At $x = 2$, the series becomes

$$\sum_{k=0}^{\infty} \frac{(x+1)^k}{3^k} = \sum_{k=0}^{\infty} \frac{(2+1)^k}{3^k} = \sum_{k=0}^{\infty} \frac{3^k}{3^k} = \sum_{k=0}^{\infty} 1 = \infty,$$

so this series diverges.

Concluding, the power series $\sum\limits_{k=0}^{\infty} \frac{(x+1)^k}{3^k}$ converges for $\boxed{-4 < x < 2.}$

16. The power series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(x-2)^k}{k^2}$ is a series of nonzero terms for $x \neq 2$. (At $x = 2$, the series converges to 0.) We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+1)^2}}{\frac{(x-2)^n}{n^2}} \right| = |x-2| \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$

$$= |x-2| \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^2} = |x-2| \frac{1}{(1+0)^2} = |x-2| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $-1 < x-2 < 1$, or $1 < x < 3$.

At $x = 1$, the series becomes

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(1-2)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

which is an alternating series whose nth term goes to zero as $n \to \infty$; also the terms of the series are nonincreasing, because by the Algebraic Ratio Test $\left| \frac{a_{n+1}}{a_n} \right| = \frac{n^2}{(n+1)^2} < 1$ for $n \geq 1$. By the Alternating Series Test, the power series converges for $x = 1$.

At $x = 3$, the series becomes

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(3-2)^k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

which is a convergent p-series (since $p = 2 > 1$), so the power series converges for $x = 3$.

Concluding, the power series $\sum\limits_{k=1}^{\infty} \frac{(x-2)^k}{k^2}$ converges for $\boxed{1 \leq x \leq 3.}$

17. (a) The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{x^k}{2^k(k+1)} = 1 + \sum\limits_{k=1}^{\infty} \frac{x^k}{2^k(k+1)}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{2^{n+1}(n+2)}}{\frac{x^n}{2^n(n+1)}} \right| = \frac{|x|}{2} \lim_{n\to\infty} \frac{n+1}{n+2} = \frac{|x|}{2} \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} = \frac{|x|}{2} < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 2$, which means the radius of convergence $\boxed{R = 2.}$

At $x = -2$, the series becomes

$$\sum_{k=0}^{\infty} \frac{x^k}{2^k(k+1)} = \sum_{k=0}^{\infty} \frac{(-2)^k}{2^k(k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \sum_{k'=1}^{\infty} \frac{(-1)^{k'-1}}{k'} = \sum_{k'=1}^{\infty} \frac{(-1)^{k'+1}}{(-1)^2 k'} = \sum_{k'=1}^{\infty} \frac{(-1)^{k'+1}}{k'}$$

where $k' = k+1$; the series is the alternating harmonic series which converges. So the power series converges for $x = -2$.
At $x = 2$, the series becomes

$$\sum_{k=0}^{\infty} \frac{x^k}{2^k(k+1)} = \sum_{k=0}^{\infty} \frac{2^k}{2^k(k+1)} = \sum_{k=0}^{\infty} \frac{1}{k+1} = \sum_{k'=1}^{\infty} \frac{1}{k'}$$

where $k' = k+1$; the series is the harmonic series which diverges. So the power series diverges for $x = 2$.
Concluding, the interval of convergence of the power series is $\boxed{-2 \le x < 2.}$

(b) Before we do the root test, it would be useful to prove the following result which shall be used in several problems below: $\lim\limits_{n\to\infty}(n+a)^{1/n} = 1$. To prove this, set $y = (n+a)^{1/n}$. Then $\ln y = \frac{1}{n}\ln(n+a) = \frac{\ln(n+a)}{n}$. We have, using L'Hôpital's rule on a related function of the ratio:

$$\begin{aligned}\lim_{n\to\infty} y = \lim_{n\to\infty} e^{\ln y} = \lim_{n\to\infty} e^{\ln(n+a)/n} &= \lim_{x\to\infty} e^{\ln(x+a)/x} \\ &= e^{\lim\limits_{x\to\infty}\ln(x+a)/x} = e^{\lim\limits_{x\to\infty}\frac{1}{x+a}/1} = e^{\lim\limits_{x\to\infty}\frac{1}{x+a}} = e^0 = 1.\end{aligned}$$

This proves the claimed result. (Note that this formula holds for any value of a, as the proof does not restrict the value of a. For $a < 0$, we require $a > -n$, which for any fixed a occurs as $n \to \infty$.)
By the Root Test, using the limit result above for $a = 1$,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{x^n}{2^n(n+1)}\right|} = \lim_{n\to\infty} \frac{|x|}{2(n+1)^{1/n}} = \frac{|x|}{2}\frac{1}{\lim\limits_{n\to\infty}(n+1)^{1/n}} = \frac{|x|}{2} < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also, the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = 2}$ and the interval of convergence is $\boxed{-2 \le x < 2.}$

(c) Answers will vary.

18. (a) The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty}(-1)^k\frac{x^k}{2^k(k+1)} = 1 + \sum\limits_{k=1}^{\infty}(-1)^k\frac{x^k}{2^k(k+1)}$ is a series of nonzero terms for $x \ne 0$. (At $x = 0$, the series converges to 1.) We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{n+1}}{2^{n+1}(n+2)}}{\frac{(-1)^n x^n}{2^n(n+1)}}\right| = \frac{|x|}{2}|(-1)|\lim_{n\to\infty}\frac{n+1}{n+2} = \frac{|x|}{2}\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{2}{n}} = \frac{|x|}{2} < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 2$, which means the radius of convergence is $\boxed{R = 2.}$
At $x = -2$, the series becomes

$$\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{2^k(k+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k(-2)^k}{2^k(k+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k(-1)^k2^k}{2^k(k+1)} = \sum_{k=0}^{\infty}\frac{(-1)^{2k}}{k+1} = \sum_{k=0}^{\infty}\frac{1}{k+1} = \sum_{k'=1}^{\infty}\frac{1}{k'}$$

using $k' = k+1$; the series is the harmonic series which diverges, so the power series diverges for $x = -2$.
At $x = 2$, the series becomes

$$\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{2^k(k+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k2^k}{2^k(k+1)} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1} = \sum_{k'=1}^{\infty}\frac{(-1)^{k'-1}}{k'} = \sum_{k'=1}^{\infty}\frac{(-1)^{k'+1}}{(-1)^2k'} = \sum_{k'=1}^{\infty}\frac{(-1)^{k'+1}}{k'}$$

where $k' = k + 1$; the series is the alternative harmonic series, which converges. So the power series converges for $x = 2$. Concluding, the interval of convergence of the power series is $\boxed{-2 < x \leq 2.}$

(b) By the Root Test, using the limit result proved in Problem 17 (b) for $a = 1$, that is, $\lim\limits_{n\to\infty} (n+1)^{1/n} = 1$, we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|(-1)^n \frac{x^n}{2^n(n+1)}|} = \frac{|x|}{2} \lim_{n\to\infty} \frac{1}{(n+1)^{1/n}} = \frac{|x|}{2} < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also, the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = 2}$ and the interval of convergence is $\boxed{-2 < x \leq 2.}$

(c) Answers will vary.

19. (a) The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{x^k}{k+5} = \frac{1}{5} + \sum\limits_{k=1}^{\infty} \frac{x^k}{k+5}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to $\frac{1}{5}$.) We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+6}}{\frac{x^n}{n+5}} \right| = |x| \lim_{n\to\infty} \frac{n+5}{n+6} = |x| \lim_{n\to\infty} \frac{1+\frac{5}{n}}{1+\frac{6}{n}} = |x| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 1$, which means the radius of convergence is $\boxed{R = 1.}$
At $x = -1$, the power series becomes

$$\sum_{k=0}^{\infty} \frac{x^k}{k+5} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+5} = \sum_{k'=5}^{\infty} \frac{(-1)^{k'-5}}{k'} = \sum_{k'=5}^{\infty} \frac{(-1)^{k'+1}}{(-1)^6 k'} = \sum_{k'=5}^{\infty} \frac{(-1)^{k'+1}}{k'}$$

where $k' = k + 5$; this is an alternating harmonic series from the fifth term onward. Since the alternating harmonic series $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges, and its convergence property is unaffected by the removal of a finite number of terms, the power series converges for $x = -1$.
At $x = 1$, the power series becomes

$$\sum_{k=0}^{\infty} \frac{x^k}{k+5} = \sum_{k=0}^{\infty} \frac{(1)^k}{k+5} = \sum_{k=0}^{\infty} \frac{1}{k+5} = \sum_{k'=5}^{\infty} \frac{1}{k'}$$

using $k' = k + 5$; this is the harmonic series from the fifth term forward. Since the harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges, and the divergence property remains unaffected by the removal of a finite number of terms, the power series diverges for $x = 1$.
Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x < 1.}$

(b) By the Root Test, using the limit result proved in Problem 17 (b) for $a = 5$, that is, $\lim\limits_{n\to\infty} (n+5)^{1/n} = 1$, we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left| \frac{x^n}{n+5} \right|} = |x| \lim_{n\to\infty} \frac{1}{(n+5)^{1/n}} = |x| < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also, the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = 1}$ and the interval of convergence is $\boxed{-1 \leq x < 1.}$

(c) Answers will vary.

20. **(a)** The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{x^k}{1+k^2} = 1 + \sum\limits_{k=1}^{\infty} \frac{x^k}{1+k^2}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{1+(n+1)^2}}{\frac{x^n}{1+n^2}}\right| = |x|\lim_{n\to\infty}\frac{1+n^2}{1+(n+1)^2}\\ &= |x|\lim_{n\to\infty}\frac{\frac{1}{n^2}+1}{\frac{1}{n^2}+\left(1+\frac{1}{n}\right)^2} = |x|\frac{0+1}{0+(1+0)^2}\\ &= |x| < 1\end{aligned}$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 1$, which means the radius of convergence is $\boxed{R = 1.}$
At $x = -1$, the power series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty}\frac{x^k}{1+k^2} = \sum_{k=0}^{\infty}\frac{(-1)^k}{1+k^2}.$$

This is an alternating series. The nth term satisfies

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{1+n^2} = 0.$$

Also, by the Algebraic Ratio Test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{1+(n+1)^2}}{\frac{1}{1+n^2}} = \frac{1+n^2}{1+(n+1)^2} < 1 \text{ for } n \geq 0.$$

So the terms are nonincreasing. By the Alternating Series Test, the power series converges for $x = -1$.
At $x = 1$, the power series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty}\frac{x^k}{1+k^2} = \sum_{k=0}^{\infty}\frac{1}{1+k^2}.$$

The nth term of the series satisfies

$$a_n = \frac{1}{1+n^2} < \frac{1}{n^2} = b_n \text{ for } n \geq 1.$$

Comparing with the convergent p-series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty}\frac{1}{k^2}$, which is convergent since $p = 2 > 1$, the series $\sum\limits_{k=1}^{\infty}\frac{1}{1+k^2}$ converges. This means the series

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty}\frac{1}{1+k^2} = 1 + \sum_{k=1}^{\infty}\frac{1}{1+k^2}$$

also converges, which means the power series converges for $x = 1$.
Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x \leq 1.}$

(b) By the Root Test, using the limit result proved in Problem 17 (b) for $a = 0$, that is, $\lim_{n\to\infty} n^{1/n} = 1$, we have

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\left|\frac{x^n}{1+n^2}\right|} = |x| \lim_{n\to\infty} \frac{1}{(1+n^2)^{1/n}} = |x| \lim_{n\to\infty} \frac{1}{(n^2)^{1/n}\left(\frac{1}{n^2}+1\right)^{1/n}} \\ &= |x| \lim_{n\to\infty} \left(\frac{1}{n^{1/n}}\right)^2 \cdot \frac{1}{\left[\lim_{n\to\infty}\left(\frac{1}{n^2}+1\right)\right]^{\lim_{n\to\infty}\frac{1}{n}}} \\ &= |x| \cdot 1^2 \cdot \frac{1}{[0+1]^0} = |x| < 1\end{aligned}$$

for convergence of the series. So we get the same result as with the Ratio Test. Also, the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = 1}$ and the interval of convergence is $\boxed{-1 \le x \le 1.}$

(c) Answers will vary.

21. (a) The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{k^2x^k}{3^k}$ is a series of nonzero terms for $k \ge 1$ if $x \ne 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^2x^{n+1}}{3^{n+1}}}{\frac{n^2x^n}{3^n}}\right| = \frac{|x|}{3} \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \frac{|x|}{3} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 = \frac{|x|}{3} < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 3$, which means the radius of convergence is $\boxed{R = 3.}$
At $x = -3$, the power series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k^2x^k}{3^k} = \sum_{k=0}^{\infty} \frac{k^2(-3)^k}{3^k} = \sum_{k=0}^{\infty} (-1)^k k^2.$$

We have

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} n^2 = \infty,$$

so the series diverges by the Divergence Test. This means the power series diverges for $x = -3$.
At $x = 3$, the power series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k^2 3^k}{3^k} = \sum_{k=0}^{\infty} k^2.$$

Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty,$$

the series diverges by the Divergence Test. So the power series diverges at $x = 3$.
Concluding, the interval of convergence of the power series is $\boxed{-3 < x < 3.}$

(b) By the Root Test, using the limit result proved in Problem 17 (b) for $a = 0$, that is, $\lim_{n\to\infty} n^{1/n} = 1$, we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^2x^n}{3^n}\right|} = \frac{|x|}{3} \lim_{n\to\infty} \left(n^{1/n}\right)^2 = \frac{|x|}{3} \left(\lim_{n\to\infty} n^{1/n}\right)^2 = \frac{|x|}{3} \cdot 1^2 = \frac{|x|}{3} < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = 3}$ and the interval of convergence is $\boxed{-3 < x < 3.}$

(c) Answers will vary.

22. **(a)** The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{2^k x^k}{3^k} = 1 + \sum\limits_{k=1}^{\infty} \frac{2^k x^k}{3^k}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}x^{n+1}}{3^{n+1}}}{\frac{2^n x^n}{3^n}} \right| = \frac{2}{3}|x| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < \frac{3}{2}$, which means the radius of convergence is $\boxed{R = \frac{3}{2}.}$

At $x = -\frac{3}{2}$, the power series becomes

$$\sum_{k=0}^{\infty} \frac{2^k x^k}{3^k} = \sum_{k=0}^{\infty} \frac{2^k \left(-\frac{3}{2}\right)^k}{3^k} = \sum_{k=0}^{\infty} (-1)^k$$

which diverges since the sequence of partial sums $\{S_n\} = \{1, 0, 1, 0, \cdots\}$ has no limit, because its terms oscillate between 1 and 0 for all n. So the power series diverges for $x = -\frac{3}{2}$.

At $x = \frac{3}{2}$, the series becomes

$$\sum_{k=0}^{\infty} \frac{2^k x^k}{3^k} = \sum_{k=0}^{\infty} \frac{2^k \left(\frac{3}{2}\right)}{3^k} = \sum_{k=0}^{\infty} 1 = \infty.$$

So the power series diverges at $x = \frac{3}{2}$.

Concluding, the interval of convergence of the power series is $\boxed{-\frac{3}{2} < x < \frac{3}{2}.}$

(b) By the Root Test, we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left| \frac{2^n x^n}{3^n} \right|} = \frac{2}{3}|x| < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = \frac{3}{2}}$ and the interval of convergence is $\boxed{-\frac{3}{2} < x < \frac{3}{2}.}$

(c) Answers will vary.

23. **(a)** The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{kx^k}{2k+1}$ is a series of nonzero terms for $k \geq 1$ if $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)x^{n+1}}{2(n+1)+1}}{\frac{nx^n}{2n+1}} \right| &= |x| \cdot \lim_{n\to\infty} \frac{n+1}{n} \cdot \lim_{n\to\infty} \frac{2n+1}{2n+3} \\
&= |x| \cdot \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \cdot \lim_{n\to\infty} \left(\frac{2 + \frac{1}{n}}{2 + \frac{3}{n}} \right) \\
&= |x| \cdot 1 \cdot 1 < 1
\end{aligned}$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 1$, which means the radius of convergence is $\boxed{R = 1.}$
At $x = -1$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{kx^k}{2k+1} = \sum_{k=0}^{\infty} \frac{k(-1)^k}{2k+1}.$$

We have

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2} \neq 0.$$

By the Divergence Test, the series diverges. So the power series diverges for $x = -1$.
At $x = 1$, the power series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{kx^k}{2k+1} = \sum_{k=0}^{\infty} \frac{k(1)^k}{2k+1} = \sum_{k=0}^{\infty} \frac{k}{2k+1}.$$

We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$$

as above. By the Divergence Test, the series diverges. So the power series diverges for $x = 1$.
Concluding, the interval of convergence of the power series is $\boxed{-1 < x < 1.}$

(b) By the Root Test, we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{nx^n}{2n+1}\right|} = |x| \cdot \lim_{n\to\infty} \frac{1}{\left(2+\frac{1}{n}\right)^{1/n}} = |x| \cdot \frac{1}{\lim\limits_{n\to\infty} \left(2+\frac{1}{n}\right)^{\lim_{n\to\infty} \frac{1}{n}}} = |x| \cdot \frac{1}{2^0} = |x| < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = 1}$ and the interval of convergence is $\boxed{-1 < x < 1.}$

(c) Answers will vary.

24. **(a)** The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} (6x)^k = 1 + \sum\limits_{k=1}^{\infty} (6x)^k$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(6x)^{n+1}}{(6x)^n}\right| = |6x| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < \frac{1}{6}$, which means the radius of convergence is $\boxed{R = \frac{1}{6}.}$

At $x = -\frac{1}{6}$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (6x)^k = \sum_{k=0}^{\infty} \left(6 \cdot \left[-\frac{1}{6}\right]\right)^k = \sum_{k=0}^{\infty} (-1)^k$$

which diverges since the sequence of partial sums of the series, $\{S_n\} = \{1, 0, 1, 0, \cdots\}$ diverges because the terms oscillate between 1 and 0 for all n. So the power series diverges when $x = -\frac{1}{6}$.

At $x = \frac{1}{6}$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (6k)^k = \sum_{k=0}^{\infty} \left(6\left[\frac{1}{6}\right]\right)^k = \sum_{k=0}^{\infty} (1)^k = \infty.$$

So the series diverges when $x = \frac{1}{6}$.

Concluding, the interval of convergence of the power series is $\boxed{-\frac{1}{6} < x < \frac{1}{6}.}$

(b) By the Root Test, we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|(6x)^n|} = |6x| < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = \frac{1}{6}}$ and the interval of convergence is $\boxed{-\frac{1}{6} < x < \frac{1}{6}.}$

(c) Answers will vary.

25. (a) The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} (x-3)^k = 1 + \sum\limits_{k=1}^{\infty} (x-3)^k$ is a series of nonzero terms for $x \neq 3$. (At $x = 3$, the series converges to 1.) We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{(x-3)^n}\right| = |x-3| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $-1 < x-3 < 1$, or $2 < x < 4$. So the radius of convergence of the series is $\boxed{R = 1.}$
At $x = 2$, the power series becomes

$$\sum_{k=0}^{\infty} (x-3)^k = \sum_{k=0}^{\infty} (2-3)^k = \sum_{k=0}^{\infty} (-1)^k$$

which diverges since the sequence of partial sums of the series, $\{S_n\} = \{1, 0, 1, 0, \cdots\}$ diverges because the terms oscillate between 1 and 0 for all n. So the power series diverges when $x = 2$.
At $x = 4$, the power series becomes

$$\sum_{k=0}^{\infty} (x-3)^k = \sum_{k=0}^{\infty} (4-3)^k = \sum_{k=0}^{\infty} (1)^k = \sum_{k=0}^{\infty} 1 = \infty.$$

So the series diverges when $x = 4$.
Concluding, the interval of convergence of the power series is $\boxed{2 < x < 4.}$.

(b) By the Root Test, we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|(x-3)^n|} = |x-3| < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = 1}$ and the interval of convergence is $\boxed{2 < x < 4.}$

(c) Answers will vary.

26. (a) The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{k(2x)^k}{3^k}$ is a series of nonzero terms for $k \geq 1$ if $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)(2x)^{n+1}}{3^{n+1}}}{\frac{n(2x)^n}{3^n}} \right| = \frac{|2x|}{3} \lim_{n\to\infty} \frac{n+1}{n} = \frac{|2x|}{3} \cdot \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \frac{|2x|}{3} \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < \frac{3}{2}$ which means that the radius of convergence of the series is $\boxed{R = \frac{3}{2}.}$

At $x = -\frac{3}{2}$, the series reduces to

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k(2x)^k}{3^k} = \sum_{k=0}^{\infty} \frac{k\left(2\left[-\frac{3}{2}\right]\right)^k}{3^k} = \sum_{k=0}^{\infty} (-1)^k k.$$

We have

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} n = \infty.$$

By the Divergence Test, the series diverges. So the power series diverges at $x = -\frac{3}{2}$.

At $x = \frac{3}{2}$, the series reduces to

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k(2x)^k}{3^k} = \sum_{k=0}^{\infty} \frac{k\left(2 \cdot \frac{3}{2}\right)^k}{3^k} = \sum_{k=0}^{\infty} k = \infty.$$

This means the power series diverges at $x = \frac{3}{2}$

Concluding, the interval of convergence of the power series is $\boxed{-\frac{3}{2} < x < \frac{3}{2}.}$

(b) By the Root Test, using the limit result proved in Problem 17 (b) for $a = 0$, that is, $\lim\limits_{n\to\infty} n^{1/n} = 1$, we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left| \frac{n(2x)^n}{3^n} \right|} = \left| \frac{2x}{3} \right| \cdot \lim_{n\to\infty} n^{1/n} = \left| \frac{2x}{3} \right| \cdot 1 < 1$$

for convergence of the series. So we get the same result as with the Ratio Test. Also the endpoint computations remain the same, so the radius of convergence of the power series is $\boxed{R = \frac{3}{2}}$ and the interval of convergence is $\boxed{-\frac{3}{2} < x < \frac{3}{2}.}$

(c) Answers will vary.

27. The power series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{x^k}{k^3}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)^3}}{\frac{x^n}{n^3}} \right| = |x| \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = |x| \lim_{n\to\infty} \frac{1}{\left(1 + \frac{1}{n}\right)} = |x| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x| < 1$, and the radius of convergence is $\boxed{R = 1.}$.

At $x = -1$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$. The nth term of this series satisfies

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{|(-1)^n|}{n^3} = \lim_{n\to\infty} \frac{1}{n^3} = 0.$$

By the Algebraic Ratio Test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)^3}}{\frac{1}{n^3}} = \frac{n^3}{(n+1)^3} < 1$$

for $n \geq 1$. Since the terms of the series are nonincreasing, and the nth term has 0 limit, the series converges by the Alternating Series Test. So the power series converges for $x = -1$.
At $x = 1$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^3}$. This is a convergent p-series which converges since $p = 3 > 1$, so the power series converges for $x = 1$.
Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x \leq 1.}$

28. The power series $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{x^k}{\ln k}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{\ln(n+1)}}{\frac{x^n}{\ln n}}\right| = |x| \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \\ &= |x| \lim_{y\to\infty} \frac{\ln y}{\ln(y+1)} = |x| \lim_{y\to\infty} \frac{\frac{1}{y}}{\frac{1}{y+1}} = |x| \lim_{y\to\infty} \frac{y+1}{y} \\ &= |x| \lim_{y\to\infty} \left(1 + \frac{1}{y}\right) = |x| < 1,\end{aligned}$$

for convergence of the series by the Ratio Test, where we used a related function of the ratio $\frac{\ln n}{\ln(n+1)}$ in order to apply L'Hôpital's rule. So the series converges for $|x| < 1$, and the radius of convergence is $\boxed{R = 1.}$
At $x = -1$, the series becomes $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$. The nth term of this series satisfies

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{|(-1)^n|}{\ln n} = \lim_{n\to\infty} \frac{1}{\ln n} = 0.$$

By the Algebraic Ratio Test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{\ln(n+1)}}{\frac{1}{\ln n}} = \frac{\ln n}{\ln(n+1)} < 1$$

if $n \geq 2$, since the natural logarithm function is an increasing function. Since the terms of the series are nonincreasing, and the nth term has 0 limit, the series converges by the Alternating Series Test. So the power series converges for $x = -1$.
At $x = 1$, the series becomes $\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{1}{\ln k}$. The nth term of this series satisfies

$$a_n = \frac{1}{\ln n} > \frac{1}{n} = b_n$$

for $n \geq 2$ since $\ln n < n$ for $n \geq 2$. (This is proved as follows: $f(x) = \ln x - x$ is a decreasing function of x for $x \geq 1$, since $f'(x) = \frac{1}{x} - 1 \leq 0$ for $x \geq 1$, and since

$f(1) = \ln 1 - 1 = 0 - 1 = -1 < 0$, we have, because $f(x)$ is decreasing, $f(x) < 0$ for all $x \geq 1$, that is, $\ln x < x$ for all $x \geq 1$. This in turn means $\ln n < n$ for $n \geq 1$. We restrict to $n \geq 2$ to ensure the denominator of the nth term stays nonzero.) So, comparing with the series $\sum\limits_{k=2}^{\infty} b_k = \sum\limits_{k=2}^{\infty} \frac{1}{k}$, which diverges being the harmonic series from the second term onward, we see by the Comparison Test that the original series diverges. So the power series is divergent for $x = 1$.
Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x < 1.}$

29. The power series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(x-2)^k}{k^3}$ is a series of nonzero terms for $x \neq 2$. (At $x = 2$, the series converges to 0.) We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{(n+1)^3}}{\frac{(x-2)^n}{n^3}}\right| = |x-2|\lim_{n\to\infty}\frac{n^3}{(n+1)^3} = |x-2|\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^3}$$
$$= |x-2|\frac{1}{(1+0)^3} = |x-2| < 1$$

for convergence of the series by the Ratio Test. So the series converges for $-1 < x - 2 < 1$, or $1 < x < 3$. The radius of convergence of the series is $\boxed{R = 1.}$
At $x = 1$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\frac{(x-2)^k}{k^3} = \sum_{k=1}^{\infty}\frac{(1-2)^k}{k^3} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}.$$

This series is convergent because it is absolutely convergent: the series of absolute values is $\sum\limits_{k=1}^{\infty} |a_k| = \sum\limits_{k=1}^{\infty} \frac{1}{k^3}$, which converges as it is a convergent p-series with $p = 3 > 1$. So the power series converges for $x = 1$.
At $x = 3$, the power series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\frac{(x-2)^k}{k^3} = \sum_{k=1}^{\infty}\frac{(3-2)^k}{k^3} = \sum_{k=1}^{\infty}\frac{(1)^k}{k^3} = \sum_{k=1}^{\infty}\frac{1}{k^3}$$

which converges as it is a convergent p-series with $p = 3 > 1$. So the power series converges for $x = 3$.
Concluding, the interval of convergence of the power series is $\boxed{1 \leq x \leq 3.}$

30. The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{k(x-2)^k}{3^k}$ is a series of nonzero terms for $k \geq 1$ if $x \neq 2$. (At $x = 2$, the series converges to 0.) We have

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{n(x-2)^n}{3^n}\right|} = \left|\frac{x-2}{3}\right|\lim_{n\to\infty} n^{1/n} = \frac{|x-2|}{3}\cdot 1 < 1$$

for convergence of the series by the Root Test (where we have used the result proved in problem 17(b), that $\lim\limits_{n\to\infty} n^{1/n} = 1$). So the series converges for $|x-2| < 3$, or $-3 < x - 2 < 3$, or $-1 < x < 5$. The radius of convergence of the series is $\boxed{R = 3.}$
At $x = -1$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty}\frac{k(x-2)^k}{3^k} = \sum_{k=0}^{\infty}\frac{k(-1-2)^k}{3^k} = \sum_{k=0}^{\infty}\frac{k(-3)^k}{3^k} = \sum_{k=0}^{\infty}(-1)^k k$$

which diverges by the Divergence Test since $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} n = \infty$. So the power series diverges for $x = -1$.
At $x = 5$, the power series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k(x-2)^k}{3^k} = \sum_{k=0}^{\infty} \frac{k(5-2)^k}{3^k} = \sum_{k=0}^{\infty} \frac{k(3^k)}{3^k} = \sum_{k=0}^{\infty} k = \infty$$

since the sum of all the nonnegative integers is infinite. This means the power series diverges for $x = 5$.
Concluding, the interval of convergence of the power series is $\boxed{-1 < x < 5.}$

31. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1}x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}}\right| = \lim_{n\to\infty} \left|(-1)\cdot \frac{x^{2n+3}}{x^{2n+1}} \cdot \frac{(2n+1)!}{(2n+3)!}\right| \\ &= |x|^2 \lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} = |x|^2 \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} \\ &= |x|^2 \cdot 0 = 0\end{aligned}$$

which means the series converges by the Ratio Test regardless of the value of x, that is, for all values of x. So the radius of convergence $\boxed{R = \infty}$, and the interval of convergence is $\boxed{-\infty < x < \infty.}$

32. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (kx)^k$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|(nx)^n|} = |x| \lim_{n\to\infty} n = \infty$$

if $x \neq 0$, so the series diverges by the Root Test if $x \neq 0$. When $x = 0$, the series becomes $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} 0 = 0$, so the series converges for $x = 0$. Concluding, the radius of convergence of the series $\boxed{R = 0}$ and the series converges for $\boxed{x = 0.}$

33. The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{kx^k}{\ln(k+1)}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{(n+1)x^{n+1}}{\ln(n+2)}}{\frac{nx^n}{\ln(n+1)}}\right| = |x| \lim_{n\to\infty} \frac{n+1}{n} \cdot \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n+2)} \\ &= |x| \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \cdot \lim_{y\to\infty} \frac{\ln(y+1)}{\ln(y+2)} = |x| \cdot 1 \cdot \lim_{y\to\infty} \frac{\frac{1}{y+1}}{\frac{1}{y+2}} \\ &= |x| \lim_{y\to\infty} \frac{y+2}{y+1} = |x| \lim_{y\to\infty} \frac{1+\frac{2}{y}}{1+\frac{1}{y}} = |x| \cdot \frac{1+0}{1+0} = |x| \cdot 1 \\ &= |x| < 1\end{aligned}$$

for convergence of the series by the Ratio Test, where we have used a related function of the ratio in the last factor of the limit, in order to apply L'Hôpital's rule to it. So the series converges for $-1 < x < 1$, or the radius of convergence $\boxed{R = 1.}$

At $x = -1$, the series becomes $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k(-1)^k}{\ln(k+1)}$. The absolute value of the nth term satisfies

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{\ln(n+1)} = \lim_{z\to\infty} \frac{z}{\ln(z+1)} = \lim_{z\to\infty} \frac{1}{\frac{1}{(z+1)}} = \lim_{z\to\infty} (z+1) = \infty$$

where we apply L'Hôpital's rule to a related function of the absolute value of the nth term of the series. By the Divergence Test, the power series diverges for $x = -1$.

At $x = 1$, the power series becomes $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k}{\ln(k+1)}$. Since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} |a_n| = \infty$, we see that it diverges too.

Concluding, the interval of convergence of the power series is $\boxed{-1 < x < 1.}$

34. The power series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{x^k}{\ln(k+1)}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{\ln(n+2)}}{\frac{x^n}{\ln(n+1)}} \right| = |x| \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n+2)} = |x| \lim_{y\to\infty} \frac{\ln(y+1)}{\ln(y+2)} \\
&= |x| \lim_{y\to\infty} \frac{\frac{1}{y+1}}{\frac{1}{y+2}} = |x| \lim_{y\to\infty} \frac{y+2}{y+1} \\
&= |x| \lim_{y\to\infty} \frac{1+\frac{2}{y}}{1+\frac{1}{y}} = |x| \cdot \frac{1+0}{1+0} = |x| \cdot 1 < 1
\end{aligned}$$

for convergence of the series by the Ratio Test, where we have used a related function of the ratio in the limit in order to apply L'Hôpital's rule to it. So the series converges for $-1 < x < 1$, or the radius of convergence $\boxed{R = 1.}$

At $x = -1$, the series becomes $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(-1)^k}{\ln(k+1)}$. The nth term of this series satisfies $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \frac{1}{\ln(n+1)} = 0$. Also by the Algebraic Ratio Test, we have

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1}{\ln(n+1)}}{\frac{1}{\ln n}} \right| = \left| \frac{\ln n}{\ln(n+1)} \right| < 1$$

for $n \geq 1$, since the natural logarithm function is an increasing function. So the series is nonincreasing. By the Alternating Series Test, the series converges. So the power series converges for $x = -1$.

At $x = 1$, the series reduces to $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{\ln(n+1)}$. The nth term satisfies

$$a_n = \frac{1}{\ln(n+1)} > \frac{1}{n+1} = b_n$$

for $n \geq 1$, where we have invoked the fact that $\ln(n+1) < n+1$ for $n \geq 1$ (see Problem 28 for a proof of a similar fact, $\ln n < n$ for $n \geq 2$; the proof here is analogous). Comparing with the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which is the divergent harmonic series from the

second term onward, by the Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k$ diverges. So the power series diverges for $x = 1$.
Concluding, the interval of convergence of the power series is $\boxed{-1 \le x < 1.}$

35. The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{k(k+1)x^k}{4^k}$ is a series of nonzero terms for $k \ge 1$ if $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)(n+2)x^{n+1}}{4^{n+1}}}{\frac{n(n+1)x^n}{4^n}}\right| = \frac{|x|}{4} \lim_{n\to\infty} \left(\frac{n+2}{n}\right) = \frac{|x|}{4} \lim_{n\to\infty} \left(1 + \frac{2}{n}\right) = \frac{|x|}{4} \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the series converges for $-4 < x < 4$, or the radius of convergence is $\boxed{R = 4.}$.
At $x = -4$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k(k+1)(-4)^k}{4^k} = \sum_{k=0}^{\infty} (-1)^k k(k+1)$$

which diverges by the Divergence Test since $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} |n(n+1)| = \infty$, that is, a nonzero limit. So the power series diverges for $x = -4$.
At $x = 4$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{k(k+1)4^k}{4^k} = \sum_{k=0}^{\infty} k(k+1) = \infty$$

because the sum of product of consecutive nonnegative integers is infinite. (Alternately, we could use $\lim\limits_{n\to\infty} a_n \neq 0$ and conclude divergence based on the Divergence Test.) So the power series diverges for $x = 4$.
Concluding, the interval of convergence of the power series is $\boxed{-4 < x < 4.}$

36. The power series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(-1)^k (x-5)^k}{k(k+1)}$ is a series of nonzero terms for $x \neq 5$. (At $x = 5$, the series converges to 0.) We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1}(x-5)^{n+1}}{(n+1)(n+2)}}{\frac{(-1)^n(x-5)^n}{n(n+1)}}\right| = |x-5| \lim_{n\to\infty} \frac{n}{n+2} = |x-5| \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} = |x-5| \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the series converges if $-1 < x - 5 < 1$, or $4 < x < 6$. The radius of convergence is $\boxed{R = 1.}$
At $x = 4$, the series reduces to

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k (4-5)^k}{k(k+1)} = \sum_{k=1}^{\infty} \frac{(-1)^k (-1)^k}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

The nth term of this series satisfies

$$a_n = \frac{1}{n(n+1)} < \frac{1}{n^2} = b_n$$

for $n \ge 1$. By the Comparison Test, since $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ converges, being a p-series for $p = 2 > 1$, the series $\sum\limits_{k=1}^{\infty} a_k$ converges as well. So the power series converges for $x = 4$.

At $x = 6$, the series reduces to

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k(6-5)^k}{k(k+1)} = \sum_{k=1}^{\infty} \frac{(-1)^k(1)^k}{k(k+1)} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)}.$$

We could deduce convergence by noting that the series of absolute values $\sum\limits_{k=1}^{\infty} |a_k|$ is just the series at $x = 4$, which converges from the calculation above; since the series absolutely converges, it converges. Alternately, the Alternating Series Test may be invoked: the nth term of the series satisfies $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \frac{1}{n(n+1)} = 0$; by the Algebraic Ratio Test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}}\right| = \left|\frac{n}{n+2}\right| < 1$$

for $n \geq 1$, which means the absolute values of the terms of the series are nonincreasing. So it converges by the Alternating Series Test, and the power series converges for $x = 6$.
Concluding, the interval of convergence of the power series is $\boxed{4 \leq x \leq 6.}$

37. The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} (-1)^k \frac{(x-3)^{2k}}{9^k} = 1 + \sum\limits_{k=1}^{\infty} (-1)^k \frac{(x-3)^{2k}}{9^k}$ is a series of nonzero terms if $x \neq 3$. (At $x = 3$, the series converges to 1.) We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1}(x-3)^{2(n+1)}}{9^{n+1}}}{\frac{(-1)^n(x-3)^{2n}}{9^n}}\right| = \frac{|x-3|^2}{9} < 1$$

for convergence of the series by the Ratio Test. So the series converges for $|x-3| < 3$, or $-3 < x-3 < 3$, or $0 < x < 6$. The radius of convergence is $\boxed{R = 3.}$
At $x = 0$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-1)^k \frac{(0-3)^{2k}}{9^k} = \sum_{k=0}^{\infty} (-1)^k \frac{(-1)^{2k}(3^2)^k}{9^k} = \sum_{k=0}^{\infty} [(-1)^3]^k = \sum_{k=0}^{\infty} (-1)^k.$$

The sequence of partial sums is $\{S_n\} = \{1, 0, 1, 0, \cdots\}$. Since the sequence of partial sums oscillates between 0 and 1 for large n, it does not attain a limit, that is, it diverges, which in turn means the series diverges. So the power series diverges for $x = 0$.
At $x = 6$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-1)^k \frac{(6-3)^{2k}}{9^k} = \sum_{k=0}^{\infty} (-1)^k \frac{(3^2)^k}{9^k} = \sum_{k=0}^{\infty} (-1)^k.$$

This is identical to the series at $x = 0$, and since we showed that series diverges, this one does too. So the power series diverges for $x = 6$.
Concluding, the interval of convergence of the power series is $\boxed{0 < x < 6.}$

38. The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{x^k}{e^k} = 1 + \sum\limits_{k=1}^{\infty} \frac{x^k}{e^k}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{x^n}{e^n}\right|} = \frac{|x|}{e} < 1$$

for convergence of the series by the Root Test. So the series converges for $|x| < e$, or $-e < x < e$, and the radius of convergence is $\boxed{R = e.}$.

At $x = -e$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(-e)^k}{e^k} = \sum_{k=0}^{\infty} (-1)^k.$$

The sequence of partial sums of this series has the form $\{S_n\} = \{1, 0, 1, 0, \cdots\}$, which is nonconvergent since the terms of the series oscillate between 0 and 1 for large n. Since the sequence of partial sums diverges, the series diverges. So the power series diverges for $x = -e$. At $x = e$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{e^k}{e^k} = \sum_{k=0}^{\infty} 1 = \infty$$

since the sum of all the nonnegative integers is infinite. So the power series diverges for $x = e$. Concluding, the interval of convergence of the power series is $\boxed{-e < x < e.}$

39. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-1)^k \frac{(2x)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{(2x)^k}{k!}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{(2x)^{n+1}}{(n+1)!}}{(-1)^n \frac{(2x)^n}{n!}} \right| \\ &= \lim_{n\to\infty} \left| (-1) \cdot 2x \cdot \frac{n!}{(n+1)!} \right| = |2x| \lim_{n\to\infty} \frac{n!}{(n+1)n!} \\ &= |2x| \lim_{n\to\infty} \frac{1}{n+1} = 0\end{aligned}$$

for any value of x. Since the limit is less than 1, by the Ratio Test, the series converges for any value of x. This means the radius of convergence $\boxed{R = \infty}$ and the interval of convergence of the power series is $\boxed{-\infty < x < \infty}$.

40. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(x+1)^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{(x+1)^k}{k!}$ is a series of nonzero terms for $x \neq -1$. (At $x = -1$, the series converges to 1.) We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x+1)^{n+1}}{(n+1)!}}{\frac{(x+1)^n}{n!}} \right| = |x+1| \lim_{n\to\infty} \frac{n!}{(n+1)!} = |x+1| \lim_{n\to\infty} \frac{n!}{(n+1)n!} = |x+1| \lim_{n\to\infty} \frac{1}{n+1} = 0$$

for any value of x. Since the limit is less than 1, by the Ratio Test, the series converges for any value of x. This means the radius of convergence $\boxed{R = \infty}$ and the interval of convergence of the power series is $\boxed{-\infty < x < \infty}$.

41. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-1)^k \frac{(x-1)^{4k}}{k!} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(x-1)^{4k}}{k!}$ is a series of nonzero terms for $x \neq 1$. (At $x = 1$, the series converges to 1.) We have

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{(x-1)^{4(n+1)}}{(n+1)!}}{\frac{(x-1)^{4n}}{n!}} \right| = |(x-1)^4| \lim_{n\to\infty} \frac{n!}{(n+1)!} \\ &= |x-1|^4 \lim_{n\to\infty} \frac{n!}{(n+1)n!} = |x-1|^4 \lim_{n\to\infty} \frac{1}{n+1} = 0\end{aligned}$$

for any value of x. Since the limit is less than 1, by the Ratio Test, the series converges for any value of x. This means the radius of convergence $\boxed{R = \infty}$ and the interval of convergence of the power series is $\boxed{-\infty < x < \infty}$.

42. The power series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(x+1)^k}{k(k+1)(k+2)}$ is a series of nonzero terms for $x \neq -1$ (At $x = -1$, the series converges to 0.) We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x+1)^{n+1}}{(n+1)(n+2)(n+3)}}{\frac{(x+1)^n}{n(n+1)(n+2)}}\right| = |x+1| \lim_{n\to\infty}\frac{n}{n+3} = |x+1| \lim_{n\to\infty}\frac{1}{1+\frac{3}{n}} = |x+1|\cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the series converges for $-1 < x+1 < 1$ or $-2 < x < 0$, and the radius of convergence is $\boxed{R = 1.}$

At $x = -2$, the series becomes $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k(k+1)(k+2)}$. The nth term of this series satisfies $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \frac{1}{n(n+1)(n+2)} = 0$. By the Algebraic Ratio Test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(n+1)(n+2)(n+3)}}{\frac{1}{n(n+1)(n+2)}}\right| = \left|\frac{n}{n+3}\right| < 1$$

for $n \geq 1$. This means the absolute value of the terms of the series are nonincreasing. By the Alternating Series Test, the series converges. So the power series converges for $x = -2$.

At $x = 0$, the power series reduces to $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)}$. The nth term of this series satisfies

$$a_n = \frac{1}{n(n+1)(n+2)} < \frac{1}{n^3} = b_n$$

for all $n \geq 1$. Since the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series because $p = 3 > 1$, by the Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k$ converges. So the power series converges for $x = 0$.

Concluding, the interval of convergence of the power series is $\boxed{-2 \leq x \leq 0.}$

43. The power series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k^k x^k}{k!}$ is a series of nonzero terms for $x \neq 0$ (At $x = 0$, the series converges to 0.) We have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}}{\frac{n^n x^n}{n!}}\right| = |x| \lim_{n\to\infty}\left|\frac{(n+1)(n+1)^n}{n^n}\cdot\frac{n!}{(n+1)!}\right| \\ &= |x| \lim_{n\to\infty}\left|(n+1)\cdot\left(1+\frac{1}{n}\right)^n\cdot\frac{n!}{(n+1)n!}\right| = |x| \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^n\right| \\ &= |x|e < 1\end{aligned}$$

for convergence of the series by the Ratio Test. So the series converges if $|x| < \frac{1}{e}$, or $-\frac{1}{e} < x < \frac{1}{e}$, and the radius of convergence is $\boxed{R = \frac{1}{e}.}$ In what follows below, we shall use a useful approximation called Stirling's approximation to help us analyze the various limits, and to put bounds on the nth terms. This sates that for large n, the value of $n!$ is given by

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n}\right) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

A proof of this approximation is found in most books on analysis, and shall not be given here, as it is not within the scope of the textbook. Without this result, one cannot perform the analyses to follow.

At $x = -\frac{1}{e}$, the series reduces to

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^k \left(-\frac{1}{e}\right)^k}{k!} = \sum_{k=1}^{\infty} (-1)^k \frac{k^k}{k!\, e^k}.$$

The nth term of this series satisfies

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^n}{n!\, e^n} = \lim_{n\to\infty} \frac{\left(\frac{n}{e}\right)^n}{n!} = \lim_{n\to\infty} \frac{\left(\frac{n}{e}\right)^n}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi n}} = 0$$

using Stirling's approximation quoted above. By the Algebraic Ratio Test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{n^{n+1}}{(n+1)!\, e^{n+1}}}{\frac{n^n}{n! e^n}}\right| = \frac{n^n \cdot n}{n^n} \cdot \frac{n!}{(n+1)!} \cdot \frac{1}{e} = \frac{n}{n+1} \cdot \frac{1}{e} < 1$$

if $(n+1)e > n$, which is satisfied for all $n \geq 1$, since $e > 1$. This means the terms of the series are nonincreasing. By the Alternating Series Test, the series converges. So the power series converges for $x = -\frac{1}{e}$.

At $x = \frac{1}{e}$, the power series reduces to

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^k \left(\frac{1}{e}\right)^k}{k!} = \sum_{k=1}^{\infty} \frac{k^k}{k!\, e^k.}$$

The nth term of this series satisfies

$$a_n = \frac{n^n}{n!\, e^n} = \frac{\left(\frac{n}{e}\right)^n}{n!} = \frac{\left(\frac{n}{e}\right)^n}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = \frac{1}{\sqrt{2\pi n}} > \frac{1}{\sqrt{3\pi n}} = b_n$$

for large $n > N$, where Stirling's approximation was again used. Since the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{3\pi k}} = \frac{1}{\sqrt{3\pi}} \sum\limits_{k=1}^{\infty} \frac{1}{k^{1/2}}$ is a constant multiple of a divergent p-series (divergent since $0 < p = \frac{1}{2} < 1$), by the Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k$ also diverges. (Note that convergence or divergence of a series is not governed by the first N terms which are finite in number, and the sum of whose terms are finite in value.) So the power series diverges for $x = \frac{1}{e}$.

Concluding, the interval of convergence of the power series is $\boxed{-\frac{1}{e} \leq x < \frac{1}{e}.}$

44. The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{3^k (x-2)^k}{k!} = 1 + \sum\limits_{k=1}^{\infty} \frac{3^k (x-2)^k}{k!}$ is a series of nonzero terms for $x \neq 2$. (At $x = 2$, the series converges to 1.) We have

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{3^{n+1}(x-2)^{n+1}}{(n+1)!}}{\frac{3^n (x-2)^n}{n!}}\right| = 3|x-2| \lim_{n\to\infty} \left|\frac{n!}{(n+1)!}\right| \\ &= 3|x-2| \lim_{n\to\infty} \frac{n!}{(n+1)n!} = 3|x-2| \lim_{n\to\infty} \frac{1}{n+1} \\ &= 0\end{aligned}$$

for all values of x. Since the limit is less than 1, by the Ratio Test, the series converges for any value of x. This means the radius of convergence $\boxed{R=\infty}$ and the interval of convergence of the power series is $\boxed{-\infty<x<\infty}$.

45. [In the problems 45-48, we shall extensively use the following result: If $|r|<1$, then the geometric series $\sum\limits_{k'=0}^{\infty} ar^{k'} = \sum\limits_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$, where $k'=k-1$. This allows us to start a geometric series at $k'=0$ and use the standard results of Section 8.2.]

(a) The domain of $f(x)=\sum\limits_{k=0}^{\infty}\frac{x^k}{3^k}=\sum\limits_{k=0}^{\infty}\left(\frac{x}{3}\right)^k$ is the interval of convergence of the power series that defines f. The series is a geometric series of the form $\sum\limits_{k=0}^{\infty} r^k$, so it converges if $|r|=\left|\frac{x}{3}\right|<1$, or $|x|<3$, or $-3<x<3$. So the domain of f is $\boxed{-3<x<3.}$

(b) $x=2$ and $x=-1$ are in the domain $-3<x<3$ of f. So we evaluate:

$$f(2)=\sum_{k=0}^{\infty}\frac{2^k}{3^k}=\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k=\frac{1}{1-\frac{2}{3}}=\frac{1}{\frac{1}{3}}=\boxed{3.}$$

$$f(-1)=\sum_{k=0}^{\infty}\frac{(-1)^k}{3^k}=\sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k=\frac{1}{1-\left(-\frac{1}{3}\right)}=\frac{1}{1+\frac{1}{3}}=\frac{1}{\frac{4}{3}}=\boxed{\frac{3}{4}.}$$

(c) Since f is defined by a geometric series, we can find its sum in its domain of convergence:

$$f(x)=\sum_{k=0}^{\infty}\frac{x^k}{3^k}=\sum_{k=0}^{\infty}\left(\frac{x}{3}\right)^k=\frac{1}{1-\frac{x}{3}}=\boxed{\frac{3}{3-x}\ \text{for}\ -3<x<3.}$$

46. (a) The domain of $f(x)=\sum\limits_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^k=\sum\limits_{k=0}^{\infty}\left(-\frac{x}{2}\right)^k=1+\sum\limits_{k=1}^{\infty}\left(-\frac{x}{2}\right)^k$ is the interval of convergence of the power series that defines f. The series is geometric of the form $\sum\limits_{k=0}^{\infty} r^k$, so it converges if $|r|=\left|-\frac{x}{2}\right|=\left|\frac{x}{2}\right|<1$, or $|x|<2$, or $-2<x<2$. So the domain of f is $\boxed{-2<x<2.}$

(b) $x=0$ and $x=1$ are in the domain $-2<x<2$ of f. So we evaluate:

$$f(0)=1+\sum_{k=1}^{\infty}\left(-\frac{0}{2}\right)^k=1+0=\boxed{1.}$$

$$f(1)=\sum_{k=0}^{\infty}(-1)^k\frac{1^k}{2^k}=\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k=\frac{1}{1+\frac{1}{2}}=\frac{1}{\frac{3}{2}}=\boxed{\frac{2}{3}.}$$

(c) Since f is defined by a geometric series, we can find its sum in the domain of convergence:

$$f(x)=\sum_{k=0}^{\infty}(-1)^k\frac{x^k}{2^k}=\sum_{k=0}^{\infty}\left(-\frac{x}{2}\right)^k=\frac{1}{1-\left(-\frac{x}{2}\right)}=\frac{1}{1+\frac{x}{2}}=\boxed{\frac{2}{2+x}\ \text{for}\ -2<x<2.}$$

47. (a) The domain of $f(x)=\sum\limits_{k=0}^{\infty}\frac{(x-2)^k}{2^k}=1+\sum\limits_{k=1}^{\infty}\frac{(x-2)^k}{2^k}$ is the interval of convergence of the power series that defines f. The series is geometric of the form $\sum\limits_{k=0}^{\infty} r^k$, so it converges if

$|r| = \left|\frac{x-2}{2}\right| < 1$, or $|x-2| < 2$, or $-2 < x-2 < 2$, or $0 < x < 4$. So the domain of f is $\boxed{0 < x < 4.}$

(b) $x = 1$ and $x = 2$ are in the domain $0 < x < 4$ of f. So we evaluate:

$$f(1) = \sum_{k=0}^{\infty} \frac{(1-2)^k}{2^k} = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k = \frac{1}{1-(-\frac{1}{2})} = \frac{1}{1+\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \boxed{\frac{2}{3}.}$$

$$f(2) = 1 + \sum_{k=1}^{\infty} \frac{(2-2)^k}{2^k} = 1 + 0 = \boxed{1.}$$

(c) Since f is defined by a geometric series, we can find its sum in the domain of convergence:

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{x-2}{2}\right)^k = \frac{1}{1-(\frac{x-2}{2})} = \frac{2}{2-x+2} = \boxed{\frac{2}{4-x} \text{ for } 0 < x < 4.}$$

48. (a) The domain of $f = \sum_{k=0}^{\infty} (-1)^k (x+3)^k = \sum_{k=0}^{\infty} [-(x+3)]^k = 1 + \sum_{k=1}^{\infty} [-(x+3)]^k$ is the interval of convergence of the power series that defines f. The series is geometric of the form $\sum_{k=0}^{\infty} r^k$, so it converges if $|r| = |-(x+3)| < 1$, or $-1 < x+3 < 1$, or $-4 < x < -2$. So the domain of f is $\boxed{-4 < x < -2.}$

(b) $x = -3$ and $x = -2.5 = -\frac{5}{2}$ are in the domain $-4 < x < 2$ of f. So we evaluate:

$$f(-3) = 1 + \sum_{k=1}^{\infty} [-(-3+3)]^k = \boxed{1.}$$

$$f(-2.5) = f\left(-\frac{5}{2}\right) = \sum_{k=0}^{\infty} \left[-\left(-\frac{5}{2}+3\right)\right]^k = \sum_{k=0}^{\infty} \left[-\left(\frac{1}{2}\right)\right]^k = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{\frac{3}{2}} = \boxed{\frac{2}{3}.}$$

(c) Since f is defined by a geometric series, we can find its sum in the domain of convergence:

$$f(x) = \sum_{k=0}^{\infty} [-(x+3)]^k = \frac{1}{1-[-(x+3)]} = \frac{1}{1+x+3} = \boxed{\frac{1}{x+4} \text{ for } -4 < x < -2.}$$

49. The series $\sum_{k=0}^{\infty} a_k x^k$ is a power series centered at $x = 0$. If the series is convergent for $x = 3$, then by the theorem on the convergence and divergence of power series, it is convergent for any x in a radius of convergence $R = 3 - 0 = 3$ about the point $x = 0$, that is, in the interval $-3 < x < 3$. Since convergence at $x = 3$ is given, we may extend the interval of convergence to $-3 < x \leq 3$. This means it is $\boxed{\textit{convergent at } \text{x=2}}$, since $x = 2$ is found in the interval $-3 < x \leq 3$. However, the series may or may not converge at $x = 5$. So no, $\boxed{\textit{nothing can be said about the convergence at } \text{x=5.}}$

50. The series $\sum_{k=0}^{\infty} a_k (x-2)^k$ is a power series centered at $x = 2$. If the series converges for $x = 6$, then it will necessarily converge for all other x in a radius of $R = 6 - 2 = 4$ about the point $x = 2$, that is, $-4 < x - 2 < 4$, or for all other x such that $\boxed{-2 < x < 6.}$

51. The series $\sum\limits_{k=0}^{\infty} a_k x^k$ is a power series centered at $x = 0$. If it converges for $x = 6$, then the theorem on convergence and divergence of power series ensures that it will converge absolutely for all x in a radius of $R = 6 - 0 = 6$ about the point $x = 0$, that is, $-6 < x < 6$. Since convergence at $x = 6$ is given, we can extend the range of convergence to at least $-6 < x \leq 6$. If the series diverges for $x = -8$, then it diverges for all $x \leq -8$ and also for all $x > 8$. Based on these observations:

(a) $\boxed{\textit{TRUE}:}$ The series will converge at $x = 2$ since $-6 < 2 \leq 6$.

(b) $\boxed{\textit{FALSE}:}$ The series is only guaranteed to converge from (and not necessarily including) $x = -6$, up to (and including) $x = 6$. It may or may not converge in the interval $6 < x \leq 8$, so it may or may not converge for $x = 7$.

(c) $\boxed{\textit{FALSE}:}$ It follows from the statement of the theorem on convergence/divergence of a power series, see p.602, that convergence of the series at $x = 6$ implies absolute convergence for $|x| < 6$. Nothing is known about absoute convergence at $x = 6$.

(d) $\boxed{\textit{FALSE}:}$ The theorem only guarantees convergence on an open interval. So the series is guaranteed to converge on $-6 < x < 6$. (Convergence at $x = 6$ is given to us, so this range may be extended to $-6 < x \leq 6$.) The series may or may not converge at $x = -6$.

(e) $\boxed{\textit{TRUE}:}$ The series diverges at $x = 10$ because it will diverge for all $x > 8$.

(f) $\boxed{\textit{TRUE}:}$ The theorem guarantees absolute convergence at $x = 4$ since it is in the interval $-6 < x \leq 6$.

52. If the radius of convergence of the power series $\sum\limits_{k=0}^{\infty} a_k (x-3)^k$ is 5, it means the series is absolutely convergent for $-5 < x - 3 < 5$, or $-2 < x < 8$, and divergent for $x < -2$ and $x > 8$. Based on this:

(a) $\boxed{\textit{TRUE}:}$ Since $x = 2$ falls in the interval of convergence $-2 < x < 8$.

(b) $\boxed{\textit{FALSE}:}$ Since $x = 7$ falls within the interval of convergence $-2 < x < 8$, the series will not diverge at $x = 7$.

(c) $\boxed{\textit{NOT NECESSARILY TRUE}:}$ At $x = 8$, the series may converge or it may diverge.

(d) $\boxed{\textit{FALSE}:}$ $x = -6$ is in the range $x < -2$, so it will diverge.

(e) $\boxed{\textit{NOT NECESSARILY TRUE}:}$ At $x = -2$, the series may converge or it may diverge.

53. (a) Using a geometric series we can represent the function as a power series centered at $x = 0$:

$$f(x) = \frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \boxed{\sum_{k=0}^{\infty} (-x^3)^k.}$$

(b) The geometric series converges if $|r| = |-x^3| < 1$, or $|x| < 1$. So the radius of convergence is $\boxed{R = 1}$ and the interval of convergence is $\boxed{-1 < x < 1.}$

54. (a) Using a geometric series we can represent the function as a power series centered at $x = 0$:

$$f(x) = \frac{1}{1-x^2} = \boxed{\sum_{k=0}^{\infty} (x^2)^k.}$$

(b) The geometric series converges if $|r| = \left|x^2\right| < 1$, or $|x| < 1$. So the radius of convergence is $\boxed{R = 1}$ and the interval of convergence is $\boxed{-1 < x < 1.}$

55. **(a)** Using a geometric series we can represent the function as a power series centered at $x = 0$:

$$f(x) = \frac{1}{6-2x} = \left(\frac{1}{6}\right)\frac{1}{1-\frac{2x}{6}} = \left(\frac{1}{6}\right)\frac{1}{1-\frac{x}{3}} = \boxed{\sum_{k=0}^{\infty}\frac{1}{6}\left(\frac{x}{3}\right)^k.}$$

(b) The geometric series converges if $|r| = \left|\frac{x}{3}\right| < 1$, or $|x| < 3$. So the radius of convergence is $\boxed{R = 3}$ and the interval of convergence is $\boxed{-3 < x < 3.}$

56. **(a)** Using a geometric series we can represent the function as a power series centered at $x = 0$:

$$f(x) = \frac{4}{x+2} = \frac{4}{2-(-x)} = \frac{2}{1-\left(-\frac{x}{2}\right)} = \boxed{\sum_{k=0}^{\infty}2\left(-\frac{x}{2}\right)^k.}$$

(b) The geometric series converges if $|r| = \left|-\frac{x}{2}\right| < 1$, or $|x| < 2$. So the radius of convergence is $\boxed{R = 2}$ and the interval of convergence is $\boxed{-2 < x < 2.}$

57. **(a)**Using a geometric series we can represent the function as a power series centered at $x = 0$:

$$f(x) = \frac{x}{1+x^3} = \frac{x}{1-(-x^3)} = \sum_{k=0}^{\infty}x\cdot(-x^3)^k = \sum_{k=0}^{\infty}(-1)^{3k}x^{3k+1} = \sum_{k=0}^{\infty}\left[(-1)^3\right]^k x^{3k+1} = \boxed{\sum_{k=0}^{\infty}(-1)^k x^{3k+1}.}$$

(b) The geometric series converges if $|r| = |(-x)^3| < 1$. So the radius of convergence is $\boxed{R = 1}$ and the interval of convergence is $\boxed{-1 < x < 1.}$

58. **(a)** Using a geometric series we can represent the function as a power series centered at $x = 0$:

$$f(x) = \frac{4x^2}{x+2} = \frac{2x^2}{1-\left(-\frac{x}{2}\right)} = \sum_{k=0}^{\infty}2x^2\left(-\frac{x}{2}\right)^k = \sum_{k=0}^{\infty}8\left(-\frac{x}{2}\right)^2\left(-\frac{x}{2}\right)^k = \boxed{\sum_{k=0}^{\infty}8\left(-\frac{x}{2}\right)^{k+2}.}$$

(b) The geometric series converges if $|r| = \left|-\frac{x}{2}\right| < 1$, or $|x| < 2$. So the radius of convergence is $\boxed{R = 2}$ and the interval of convergence is $\boxed{-2 < x < 2.}$

59. First, we check to see if the power series representing the function $f(x) = \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$ has a nonzero radius of convergence:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}}\right| = |x|^2\lim_{n\to\infty}\frac{(2n+1)!}{(2n+3)!}\\ &= |x|^2\lim_{n\to\infty}\frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} = |x|^2\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)}\\ &= 0 \text{ for any value of } x.\end{aligned}$$

So the radius of convergence is $R = \infty$, and the interval of convergence is $-\infty < x < \infty$.

(a) Using the differentiation property of power series we have

$$\begin{aligned}
f(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots \\
f'(x) &= 1 - \frac{3x^2}{3\cdot 2!} + \frac{5x^4}{5\cdot 4!} - \cdots + \frac{(-1)^k(2k+1)x^{2k+1-1}}{(2k+1)(2k)!} + \cdots \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots \\
&= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.}
\end{aligned}$$

(b) Using the integration property of power series we have

$$\begin{aligned}
f(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots \\
\int_0^x f(x)\,dx &= \frac{x^2}{2} - \frac{x^4}{4\cdot 3!} + \frac{x^6}{6\cdot 5!} - \cdots + \frac{(-1)^k x^{2k+2}}{(2k+2)(2k+1)!} + \cdots \\
&= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{(2k+2)!}.}
\end{aligned}$$

60. First, we check to see if the power series representing the function $f(x) = \sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$ has a nonzero radius of convergence:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!}}{\frac{(-1)^n x^{2n}}{(2n)!}}\right| = |x|^2 \lim_{n\to\infty}\frac{(2n)!}{(2n+2)!} \\
&= |x|^2 \lim_{n\to\infty}\frac{(2n)!}{(2n+2)(2n+1)(2n)!} = |x|^2 \lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} \\
&= 0, \text{ for any value of } x.
\end{aligned}$$

So the radius of convergence $R = \infty$, and the interval of convergence is $-\infty < x < \infty$.
(a) Using the differentiation property of power series we have

$$\begin{aligned}
f(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots \\
f'(x) &= 0 - \frac{2x}{2\cdot 1!} + \frac{4x^3}{4\cdot 3!} - \cdots + \frac{(-1)^k(2k)x^{2k-1}}{(2k)(2k-1)!} + \cdots \\
&= -x + \frac{x^3}{3!} - \cdots + \frac{(-1)^k x^{2k-1}}{(2k-1)!} + \cdots \\
&= \boxed{\sum_{k=1}^{\infty} \frac{(-1)^k x^{2k-1}}{(2k-1)!}.}
\end{aligned}$$

(b) Using the integration property of power series we have

$$\begin{aligned} f(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots \\ \int_0^x f(x)\,dx &= x - \frac{x^3}{3\cdot 2!} + \frac{x^5}{5\cdot 4!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)(2k)!} + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots \\ &= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.} \end{aligned}$$

61. Let's check to see if the power series representing the function $f(x) = \sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!}$ has a nonzero radius of convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = |x| \lim_{n\to\infty}\frac{n!}{(n+1)!} = |x| \lim_{n\to\infty}\frac{n!}{(n+1)n!} = |x| \lim_{n\to\infty}\frac{1}{n+1} = 0$$

for all values of x. So the radius of convergence $R = \infty$, and the interval of convergence is $-\infty < x < \infty$.

(a) Using the differentiation property of power series we have

$$\begin{aligned} f(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots \\ f'(x) &= 0 + 1 + \frac{2x}{2\cdot 1!} + \frac{3x^2}{3\cdot 2!} + \cdots + \frac{kx^{k-1}}{k(k-1)!} + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{k-1}}{(k-1)!} + \cdots \\ &= \boxed{\sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}.} \end{aligned}$$

(b) Using the integration property of power series we have

$$\begin{aligned} f(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots \\ \int_0^x f(x)\,dx &= x + \frac{x^2}{2\cdot 1!} + \frac{x^3}{3\cdot 2!} + \frac{x^4}{4\cdot 3!} + \cdots + \frac{x^{k+1}}{(k+1)k!} + \cdots \\ &= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{k+1}}{(k+1)!} + \cdots \\ &= \boxed{\sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!}.} \end{aligned}$$

62. Let's check to see if the power series representing the function $f(x) = \sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$ has a nonzero radius of convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{n+1}}{(n+1)!}}{\frac{(-1)^n x^n}{n!}}\right| = |x| \lim_{n\to\infty}\frac{n!}{(n+1)!} = |x| \lim_{n\to\infty}\frac{n!}{(n+1)n!} = |x| \lim_{n\to\infty}\frac{1}{n+1} = 0$$

for all values of x. So the radius of convergence $R = \infty$, and the interval of convergence is $-\infty < x < \infty$.

(a) Using the differentiation property of power series we have

$$\begin{aligned} f(x) &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^k x^k}{k!} + \cdots \\ f'(x) &= 0 - 1 + \frac{2x}{2 \cdot 1!} - \frac{3x^2}{3 \cdot 2!} + \cdots + \frac{(-1)^k k x^{k-1}}{k(k-1)!} + \cdots \\ &= -1 + x - \frac{x^2}{2!} + \cdots + \frac{(-1)^k x^{k-1}}{(k-1)!} + \cdots \\ &= \boxed{\sum_{k=1}^{\infty} \frac{(-1)^k x^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^k}{k!}.} \end{aligned}$$

(b) Using the integration property of power series we have

$$\begin{aligned} f(x) &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^k x^k}{k!} + \cdots \\ \int_0^x f(x)\,dx &= x - \frac{x^2}{2 \cdot 1!} + \frac{x^3}{3 \cdot 2!} - \frac{x^4}{4 \cdot 3!} + \cdots + \frac{(-1)^k x^{k+1}}{(k+1)k!} + \cdots \\ &= x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots + \frac{(-1)^k x^{k+1}}{(k+1)!} + \cdots \\ &= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{(k+1)!}.} \end{aligned}$$

63. We wish to find a power series representation of $f(x) = \frac{1}{(1+x)^2}$. The function $g(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum\limits_{k=0}^{\infty} (-x)^k$ has a geometric power series representation which converges for $|r| = |-x| = |x| < 1$. Using the differentiation property of power series, we get

$$\begin{aligned} f(x) &= -\frac{dg}{dx} = \frac{1}{(1+x)^2} \\ &= -\frac{d}{dx}\left[\sum_{k=0}^{\infty}(-x)^k\right] \\ &= -\frac{d}{dx}\left[(-x)^0 + (-x)^1 + (-x)^2 + (-x)^3 + \cdots + (-x)^k + \cdots\right] \\ &= -\frac{d}{dx}\left[1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots\right] \\ &= -\left[0 - 1 + 2x - 3x^2 + \cdots + (-1)^k k x^{k-1} + \cdots\right] \\ &= 1 - 2x + 3x^2 - \cdots + (-1)^{k+1} k x^{k-1} + \cdots \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} k x^{k-1} = \sum_{k=0}^{\infty} (-1)^k (k+1) x^k. \end{aligned}$$

The interval of convergence of the power series representation is at least $|x| < 1$ or $-1 < x < 1$. Note that at the left endpoint $x = -1$ the series representation reduces to $\sum\limits_{k=0}^{\infty} (-1)^{2k}(k+1)$,

which is divergent by the Divergence Test. At the right endpoint $x = 1$, the series representation reduces to $\sum\limits_{k=0}^{\infty}(-1)^k(k+1)$, which again diverges by the Divergence Test.
So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{\sum_{k=0}^{\infty}(-1)^k(k+1)x^k, \quad -1 < x < 1.}$$

64. We wish to find a power series representation of $f(x) = \frac{1}{(1-x)^3}$. The function

$$h(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots + x^k + \cdots$$

has a geometric power series representation which converges for $|r| = |x| < 1$. Using the differentiation property of power series, we have

$$h'(x) = \frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \cdots + kx^{k-1} + \cdots = \sum_{k=1}^{\infty} kx^{k-1}.$$

To see that this power series has a nonzero radius of convergence, with $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} kx^{k-1}$ being a series of nonzero terms, we compute

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^n}{nx^{n-1}}\right| = |x|\lim_{n\to\infty}\frac{n+1}{n} = |x|\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = |x|\cdot 1 < 1$$

for convergence by the Ratio Test. Since the condition $|x| < 1$ was already satisfied for the series representation of the function $h(x)$ to be convergent, we see that the series representation of $h'(x)$ is convergent as well. So using the differentiation property of power series again, we have

$$h''(x) = \frac{2}{(1-x)^3} = 2 + 6x + \cdots + k(k-1)x^{k-2} + \cdots = \sum_{k=2}^{\infty} k(k-1)x^{k-2} = \sum_{k'=0}^{\infty}(k'+2)(k'+1)x^{k'},$$

where we set $k' = k - 2$ or $k = k' + 2$. So finally

$$f(x) = \frac{1}{(1-x)^3} = \frac{1}{2}h''(x) = \frac{1}{2}\sum_{k=2}^{\infty} k(k-1)x^{k-2} = \frac{1}{2}\sum_{k=0}^{\infty}(k+2)(k+1)x^k.$$

The interval of convergence is at least $|x| < 1$ or $-1 < x < 1$. At the left endpoint $x = -1$ the series representation reduces to $\frac{1}{2}\sum\limits_{k=0}^{\infty}(-1)^k(k+2)(k+1)$, which is divergent by the Divergence Test. At the right endpoint $x = 1$, the series representation reduces to $\frac{1}{2}\sum\limits_{k=0}^{\infty}(k+2)(k+1)$, which again diverges by the Divergence Test.
So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{\frac{1}{2}\sum_{k=0}^{\infty}(k+2)(k+1)x^k, \quad -1 < x < 1.}$$

65. We wish to find a power series representation of $f(x) = \frac{2}{3(1-x)^2}$. From the result of Problem 64 above, we have

$$f(x) = \frac{2}{3(1-x)^2} = \frac{2}{3}h'(x) = \frac{2}{3}\sum_{k=1}^{\infty} kx^{k-1}.$$

The interval of convergence of the power series representation is at least $|x| < 1$ or $-1 < x < 1$. At the left endpoint $x = -1$ the series representation reduces to $\frac{2}{3}\sum_{k=0}^{\infty}(-1)^{k-1}k$, which is divergent by the Divergence Test. At the right endpoint $x = 1$, the series representation reduces to $\frac{2}{3}\sum_{k=0}^{\infty} k$ which again diverges by the Divergence Test.

So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{\frac{2}{3}\sum_{k=1}^{\infty} kx^{k-1}, \quad -1 < x < 1.}$$

66. We wish to find a power series representation of $f(x) = \frac{1}{(1-x)^4}$. From the result of Problem 64 above, we have that $h''(x) = \frac{2}{(1-x)^3} = \sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} k(k-1)x^{k-2}$. Let us compute the radius of convergence of this series:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)nx^{n-1}}{n(n-1)x^{n-2}}\right| = |x| \lim_{n\to\infty}\frac{n+1}{n-1} = |x|\frac{1+\frac{1}{n}}{1-\frac{1}{n}} = |x|\cdot 1 < 1$$

for convergence by the Ratio Test. Since the radius of convergence is nonzero, we apply the differentiation property of power series to this function to obtain

$$\begin{aligned} h'''(x) &= \frac{2\cdot 3}{(1-x)^4} = \frac{d}{dx}\left[2 + 6x + 12x^2 + \cdots + k(k-1)x^{k-2} + \cdots\right] \\ &= 0 + 6 + 24x + \cdots + k(k-1)(k-2)x^{k-3} + \cdots \\ &= \sum_{k=3}^{\infty} k(k-1)(k-2)x^{k-3} = \sum_{k=0}^{\infty}(k+3)(k+2)(k+1)x^k. \end{aligned}$$

Finally, we obtain

$$f(x) = \frac{1}{(1-x)^4} = \frac{1}{6}h'''(x) = \frac{1}{6}\sum_{k=3}^{\infty} k(k-1)(k-2)x^{k-3} = \frac{1}{6}\sum_{k=0}^{\infty}(k+3)(k+2)(k+1)x^k.$$

The interval of convergence of the power series representation is at least $|x| < 1$ or $-1 < x < 1$. At the left endpoint $x = -1$ the series representation reduces to $\frac{1}{6}\sum_{k=0}^{\infty}(-1)^k(k+3)(k+2)(k+1)$, which is divergent by the Divergence Test. At the right endpoint $x = 1$, the series representation reduces to $\frac{1}{6}\sum_{k=0}^{\infty}(k+3)(k+2)(k+1)$, which again diverges by the Divergence Test.

So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{\frac{1}{6}\sum_{k=0}^{\infty}(k+3)(k+2)(k+1)x^k, \quad -1 < x < 1.}$$

67. We wish to find a power series repesentation of $f(x) = \ln\left(\frac{1}{1+x}\right)$. Since

$$g(x) = \frac{1}{1+x} - \frac{1}{1-(-x)} = \sum_{k=0}^{\infty}(-x)^k = 1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots$$

is a geometric series that converges if $|r| = |-x| = |x| < 1$, we can use the integration property of power series and integrate term by term:

$$\begin{aligned}\int_0^x g(x)\,dx &= \int_0^x t\frac{dx}{1+x} = \ln(1+x)\\ &= \int_0^x \left[1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots\right] dx\\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^k\frac{x^{k+1}}{k+1} + \cdots\end{aligned}$$

or, $$\ln(1+x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{k+1}}{(k+1)}.$$

The function f can be written:

$$f(x) = \ln\left(\frac{1}{1+x}\right) = \ln 1 - \ln(1+x) = 0 - \ln(1+x).$$

So we have

$$f(x) = -\sum_{k=0}^{\infty}\frac{(-1)^k x^{k+1}}{k+1} = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}x^{k+1}}{k+1}$$

The interval of convergence is at least $|x| < 1$, or $-1 < x < 1$. At the left endpoint $x = -1$, the series representation reduces to $\sum\limits_{k=0}^{\infty}\frac{1}{k+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ which is the divergent harmonic series. At the right endpoint $x = 1$, the series representation reduces to $\sum\limits_{k=0}^{\infty}\frac{(-1)^{k+1}}{k+1} = -1 + \frac{1}{2} - \frac{1}{3} + \cdots$ which is a constant multiple of the convergent alternating harmonic series.

So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{\sum_{k=0}^{\infty}\frac{(-1)^{k+1}x^{k+1}}{k+1}, \quad -1 < x \le 1.}$$

68. We wish to find a power series representation of $f(x) = \ln(1-2x)$. Consider the function

$$g(x) = \frac{1}{1-2x} = \sum_{k=0}^{\infty}(2x)^k = 1 + 2x + 4x^2 + \cdots + 2^k x^k + \cdots$$

which is a function with a geometric power series representation that converges if $|r| = |2x| < 1$, or $|x| < \frac{1}{2}$. Since the interval of convergence is nonzero, we use the integration property of power series to integrate term by term:

$$\begin{aligned}\int_0^x g(x)\,dx &= \int_0^x \frac{dx}{1-2x} = \frac{\ln|1-2x|}{-2}\\ &= \int_0^x \left[1 + 2x + 4x^2 + \cdots + 2^k x^k + \cdots\right] dx\\ &= x + x^2 + \frac{4}{3}x^3 + \cdots + 2^k\frac{x^{k+1}}{k+1} + \cdots\end{aligned}$$

or, $$\begin{aligned}\ln|1-2x| &= -2\left[x + x^2 + \frac{4}{3}x^3 + \cdots + \frac{2^k x^{k+1}}{k+1} + \cdots\right] = \sum_{k=0}^{\infty}(-1)\frac{2^{k+1}x^{k+1}}{k+1}\\ &= \sum_{k=0}^{\infty}(-1)\frac{(2x)^{k+1}}{k+1}.\end{aligned}$$

The interval of convergence is at least $|x| < \frac{1}{2}$. At the left endpoint $x = -\frac{1}{2}$, the series representation reduces to $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} (-1)^k \frac{1}{k+1}$ which is the convergent alternating harmonic series. At the right endpoint, $x = \frac{1}{2}$, the series representation reduces to $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} (-1) \frac{1}{k+1}$ which is a constant multiple of the divergent harmonic series.
So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{\sum_{k=0}^{\infty} (-1) \frac{(2x)^{k+1}}{k+1}, \quad -\frac{1}{2} \leq x < \frac{1}{2}.}$$

69. We wish to find a power series representation of $f(x) = \ln(1 - x^2)$. Consider the function

$$g(x) = \frac{-2x}{1-x^2} = (-2x) \sum_{k=0}^{\infty} (x^2)^k = (-2x) \sum_{k=0}^{\infty} x^{2k} = \sum_{k=0}^{\infty} (-2) x^{2k+1} = -2x - 2x^3 - \cdots - 2x^{2k+1} - \cdots .$$

This is a geometric series that converges if $|r| = |x^2| < 1$, or $|x| < 1$, so the power series representation of the function g has a nonzero radius of convergence. Using the integration property of power series, we get

$$\begin{aligned}
\int_0^x g(x)\, dx &= \int_0^x \frac{-2x\, dx}{1-x^2} = \ln|1 - x^2| \\
&= \int_0^x \left[-2x - 2x^3 - \cdots - 2x^{2k+1} - \cdots\right] dx \\
&= -x^2 - \frac{2}{4}x^4 - \cdots - \frac{2}{2k+2}x^{2k+2} - \cdots \\
&= -x^2 - \frac{x^4}{2} - \cdots - \frac{x^{2k+2}}{k+1} - \cdots \\
\text{or,} \qquad \ln|1 - x^2| &= -\sum_{k=0}^{\infty} \frac{x^{2k+2}}{k+1}.
\end{aligned}$$

The interval of convergence of the power series representation is at least $|x| < 1$ or $-1 < x < 1$. At the left endpoint $x = -1$, the series representation reduces to $-\sum\limits_{k=0}^{\infty} \frac{1}{k+1}$, which, being a constant multiple of the harmonic series, diverges. At the right endpoint $x = 1$, the series representation reduces to the same expression as it did at $x = -1$, so once again it diverges.
So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{-\sum_{k=0}^{\infty} \frac{x^{2k+2}}{k+1}, \quad -1 < x < 1.}$$

70. We wish to find a power series representation of $f(x) = \ln(1 + x^2)$. Consider the function

$$g(x) = \frac{2x}{1+x^2} = \frac{2x}{1-(-x^2)} = 2x \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} 2(-1)^k x^{2k+1} = 2x - 2x^3 + 2x^5 - \cdots + 2(-1)^k x^{2k+1} + \cdots .$$

This is a geometric series that converges if $|r| = |x^2| < 1$, or $|x| < 1$, so the power series representation of the function g has a nonzero radius of convergence. Using the integration property of power series, we get

$$\begin{aligned}\int_0^x g(x)\,dx = &= \int_0^x \frac{2x\,dx}{1+x^2} = \ln|1+x^2| \\ &= \int_0^x \left[2x - 2x^3 + 2x^5 - \cdots + 2(-1)^k x^{2k+1} + \cdots\right] dx \\ &= x^2 - \frac{2x^4}{4} + \frac{2x^6}{6} - \cdots + (-1)^k \frac{2x^{2k+2}}{2k+2} + \cdots \\ &= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots + (-1)^k \frac{x^{2k+2}}{k+1} + \cdots \\ \text{or,} \qquad \ln(1+x^2) &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{k+1}.\end{aligned}$$

The interval of convergence of the power series representation is at least $|x| < 1$ or $-1 < x < 1$. At the left endpoint $x = -1$, the power series representation reduces to $\sum\limits_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ which is the convergent alternating harmonic series. At the right endpoint $x = 1$, we obtain the same series as we had for $x = -1$, so once again the series converges.
So we conclude that the power series representation of the given function $f(x)$ is

$$\boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{k+1}, \quad -1 \le x \le 1.}$$

Applications and Extensions

71. The power series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{x^k}{k}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to zero.) We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x| \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = |x| \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the power series converges for $|x| < 1$, or $-1 < x < 1$.
At $x = -1$, the series becomes $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k}$, which is the convergent alternating harmonic series. So the power series converges for $x = -1$.
At $x = 1$, the series becomes $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k}$ which is the divergent harmonic series. So the power series diverges for $x = 1$.
Concluding, the interval of convergence of the power series is $\boxed{-1 \le x < 1.}$

72. The power series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(x-4)^k}{k}$ is a series of nonzero terms for $x \neq 4$. (At $x = 4$, the series converges to 0.) We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(x-4)^{n+1}}{n+1}}{\frac{(x-4)^n}{n}}\right| = |x-4| \lim_{n\to\infty} \frac{n}{n+1} = |x-4| \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = |x-4| \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the series converges for $-1 < x - 4 < 1$, or $3 < x < 5$.
At $x = 3$, the series becomes $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(3-4)^k}{k} = \sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k}$, which is the convergent alternating harmonic series. So the power series converges for $x = 3$.
At $x = 5$, the series becomes $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(5-4)^k}{k} = \sum\limits_{k=1}^{\infty} \frac{1}{k}$ which is the divergent harmonic series. So the power series diverges for $x = 5$.
Concluding, the interval of convergence of the power series is $\boxed{3 \leq x < 5.}$

73. The power series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{x^k}{2k+1}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{k+1}}{2(n+1)+1}}{\frac{x^n}{2n+1}} \right| = |x| \lim_{n\to\infty} \frac{2n+1}{2n+3} = |x| \lim_{n\to\infty} \frac{2+\frac{1}{n}}{2+\frac{3}{n}} = |x| \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the power series converges for $|x| < 1$, or $-1 < x < 1$.
At $x = -1$, the series reduces to $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(-1)^k}{2k+1}$. The nth term of this series satisfies $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{2n+1} = 0$. By the Algebraic Ratio Test, we see

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1}{2(n+1)+1}}{\frac{1}{2n+1}} \right| = \frac{2n+1}{2n+3} < 1$$

for $n \geq 1$. So the terms of the series are nonincreasing. By the Alternating Series Test, the series converges. So the power series converges for $x = -1$.
At $x = 1$, the series reduces to $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{2k+1}$. The nth term of this series satisfies

$$a_n = \frac{1}{2n+1} > \frac{1}{2n+2} = b_n.$$

Since the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{2(k+1)}$ is a constant multiple of the series $\sum\limits_{k=1}^{\infty} \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which is the divergent harmonic series from the second term forward, by the Comparison Test, the series $\sum\limits_{k=1}^{\infty} a_k$ diverges. So the power series diverges for $x = 1$.
Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x < 1.}$

74. The power series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{x^k}{k^2}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)^2}}{\frac{x^n}{n^2}} \right| = |x| \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^2 = |x| \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}} \right)^2 = |x| \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the power series converges for $|x| < 1$ or $-1 < x < 1$.

At $x = -1$, the series reduces to $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k^2}$. The series of absolute values is $\sum\limits_{k=1}^{\infty} |a_k| = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$, which is a convergent p-series, since $p = 2 > 1$. Since the original series converges absolutely, it converges. So the power series is convergent for $x = -1$.
At $x = 1$, the series becomes $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$, which is again a convergent p-series for $p = 2 > 1$. So the power series converges for $x = 1$.
Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x \leq 1.}$

75. The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} x^{k^2} = 1 + \sum\limits_{k=1}^{\infty} x^{k^2}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have, applying the Root Test,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|x^{n^2}|} = \lim_{n\to\infty} |x^{n^2/n}| = \lim_{n\to\infty} |x^n|.$$

Now, since $\lim\limits_{n\to\infty} r^n = \infty$ if $|r| > 1$ and $\lim\limits_{n\to\infty} r^n = 0$ if $|r| < 1$, we require for convergence by the Root Test, $|x| < 1$, or $-1 < x < 1$.
At $x = -1$, the series becomes $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} (-1)^{k^2} = 1 - 1 + 1 - 1 + \cdots$. Since the sequence of partial sums $\{S_n\} = \{1, 0, 1, 0, \cdots\}$ is not convergent, the power series diverges for $x = -1$.
At $x = 1$, the series becomes $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} 1 = \infty$, so the power series diverges for $x = 1$.
Concluding, the interval of convergence of the power series is $\boxed{-1 < x < 1.}$

76. The power series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k^a}{a^k}(x-a)^k$. This is a series of nonzero terms when $x \neq a$. (At $x = a$, the series converges to 0.) We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^a (x-a)^{n+1}}{a^{n+1}}}{\frac{n^a (x-a)^n}{a^n}}\right| = \left|\frac{x-a}{a}\right| \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^a = \left|\frac{x-a}{a}\right| \cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the power series converges for $|x - a| < |a|$, or $-|a| < x - a < |a|$, or $-|a| + a < x < |a| + a$. If we assume $a > 0$, then $|a| = a$, and the power series converges for $0 < x < 2a$. Since $a \neq 0$ has been given, we have an interval of convergence of nonzero length. If we assume $a < 0$, then $|a| = -a$, and the power series converges for $2a < x < 0$.
At $x = 0$, the series reduces to

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^a}{a^k}(-a)^k = \sum_{k=1}^{\infty} (-1)^k k^a.$$

If $a = -1$, the series becomes $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k}$, which, being an alternating harmonic series, converges.
For $-1 < a < 0$, the series becomes an alternating p-series with $0 < p < 1$, which converges by the alternating series test; with $p = -a$, we have : $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \frac{1}{k^p} = 0$; also the terms of the series are nonincreasing, as can be seen by the Algebraic Ratio Test, since

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n^{p+1}}}{\frac{1}{n^p}} = \frac{n^p}{n^{p+1}} < 1$$

for $n \geq 1$. So by the Alternating Series Test, the series converges for $-1 \leq a < 0$ at $x = 0$. For $a > 0$, the series becomes $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} (-1)^k k^a$. By the Divergence Test,

$\lim\limits_{n\to\infty}|a_n| = \lim\limits_{n\to\infty} n^a = \infty$, so the series diverges. Finally, if $a < -1$, then $p = -a > 1$, so the series becomes $\sum\limits_{k=1}^{\infty}(-1)^k\frac{1}{k^p}$, which is a convergent p-series since $p > 1$. Summarizing, at $x = 0$, the power series converges for $a < -1$, $-1 \le a < 0$, and diverges for $a > 0$.
At $x = 2a$, the series reduces to

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^a}{a^k}(2a-a)^k = \sum_{k=1}^{\infty} k^a.$$

If $a < -1$, then with $p = -a > 1$, the series becomes $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty}\frac{1}{k^p}$, which is a convergent p-series since $p > 1$. If $-1 \le a < 0$, then using $p = -a$, we have $0 < p \le 1$, which makes the series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty}\frac{1}{k^p}$ a divergent p-series. For $a > 0$, we use the Divergence Test to see $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^a = \infty$, so the series diverges. Summarizing, for $x = 2a$, the series converges when $a < -1$, and diverges for $-1 \le a < 0$ and $a > 0$.
Concluding, we see the power series converges for the following ranges of the parameter a:

$$\boxed{a < -1 : 2a \le x \le 0;\ \ -1 \le a < 0 : 2a < x \le 0;\ a > 0 : 0 < x < 2a.}$$

77. The power series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty}\frac{(k!)^2}{(2k)!}(x-1)^k = 1 + \sum\limits_{k=1}^{\infty}\frac{(k!)^2}{(2k)!}(x-1)^k$ is a series of nonzero terms if $x \ne 1$. (At $x = 1$, the series converges to 1.) We have

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{((n+1)!)^2(x-1)^{n+1}}{(2(n+1))!}}{\frac{(n!)^2(x-1)^n}{(2n)!}}\right| = |x-1|\lim_{n\to\infty}\left\{\left[\frac{(n+1)!}{n!}\right]^2\cdot\frac{(2n)!}{(2n+2)!}\right\}\\
&= |x-1|\lim_{n\to\infty}\left\{\left[\frac{(n+1)n!}{n!}\right]^2\cdot\frac{(2n)!}{(2n+2)(2n+1)(2n)!}\right\}\\
&= |x-1|\lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+2)(2n+1)} = |x-1|\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}\\
&= |x-1|\cdot\frac{1}{4} < 1
\end{aligned}$$

for convergence by the Ratio Test if $|x-1| < 4$, or $-4 < x-1 < 4$, or $-3 < x < 5$.
At $x = -3$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty}\frac{(k!)^2}{(2k)!}(-3-1)^k = \sum_{k=0}^{\infty}\frac{(k!)^2}{(2k)!}(-4)^k = \sum_{k=0}^{\infty}(-1)^k\frac{(k!)^2}{(2k)!}2^{2k}.$$

As in Problem 43, we have to use Stirling's Approximation,

$$n! \approx \left(\frac{n}{e}\right)^n\sqrt{2\pi n}.$$

We see the absolute value of the nth term of the series is

$$\begin{aligned}
|a_n| = \frac{n!\,n!\,2^{2n}}{(2n)!} &\approx \frac{\left(\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right)\left(\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right)2^{2n}}{\left(\frac{2n}{e}\right)^{2n}\sqrt{2\pi\cdot 2n}}\\
&= \frac{n^{2n}\cdot 2^{2n}(2\pi n)}{2^{2n}n^{2n}\sqrt{4\pi n}} = \sqrt{n}\sqrt{\pi}\to\infty,\ \text{as } n\to\infty
\end{aligned}$$

which means the series diverges by the Divergence Test at $x = -3$.

At $x = 5$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}(5-1)^k = \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}4^k = \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}2^{2k}$$

which is identical to the series of absolute values of the series at $x = -3$: as such, it too diverges by the same calculation used to establish the divergence of the absolute value of the nth term above. So the series diverges by the Divergence Test at $x = 5$.
Concluding, the interval of convergence of the power series is $\boxed{-3 < x < 5.}$

78. The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{\sqrt{k!}}{(2k)!}x^k = 1 + \sum_{k=1}^{\infty} \frac{\sqrt{k!}}{(2k)!}x^k$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) We have

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{\sqrt{(n+1)!}x^{n+1}}{(2(n+1))!}}{\frac{\sqrt{n!}x^n}{(2n)!}}\right| = |x|\lim_{n\to\infty}\left\{\sqrt{\frac{(n+1)!}{n!}}\cdot\frac{(2n)!}{(2n+2)!}\right\} \\
&= |x|\lim_{n\to\infty}\left\{\sqrt{\frac{(n+1)n!}{n!}}\cdot\frac{(2n)!}{(2n+2)(2n+1)(2n)!}\right\} = |x|\lim_{n\to\infty}\frac{\sqrt{n+1}}{(2n+2)(2n+1)} \\
&= |x|\lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}}}{\left(2\sqrt{n}+\frac{2}{\sqrt{n}}\right)\left(2\sqrt{n}+\frac{1}{\sqrt{n}}\right)} \\
&= 0
\end{aligned}$$

for any value of x. So the interval of convergence is $\boxed{-\infty < x < \infty.}$

79. **(a)** We are given

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad -1 < x < 1.$$

Replacing x with x^2, we have

$$\frac{1}{1-x^2} = \sum_{k=0}^{\infty}(x^2)^k = \boxed{\sum_{k=0}^{\infty} x^{2k}.}$$

(b) The interval of convergence is $|x^2| < 1$, as it is a geometric series, so it is $\boxed{-1 < x < 1.}$

80. **(a)** Using partial fractions, we can write

$$\begin{aligned}
\frac{1}{1-x^2} &= \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right] \\
&= \sum_{k=0}^{\infty} x^{2k} = 1 + x^2 + x^4 + \cdots + x^{2k} + \cdots
\end{aligned}$$

Since the radius of convergence of the geometric series is $|x| < 1$, it is nonzero, so we may integrate the series term by term to obtain:

$$\int_0^x \frac{dx}{1-x^2} = \frac{1}{2}\left[\int_0^x \frac{dx}{1-x} + \int_0^x \frac{dx}{1+x}\right] = \int_0^x \left[1 + x^2 + x^4 + \cdots + x^{2n} + \cdots\right] dx$$

$$\text{or,} \qquad \frac{1}{2}\left[-\ln|1-x| + \ln|1+x|\right] = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2k+1}}{2k+1} + \cdots$$

$$\text{or,} \qquad \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = \boxed{\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}.}$$

(b) Since the series summed is geometric, it converges at least for $|x| < 1$, or $-1 < x < 1$. At $x = -1$, the series becomes $\sum\limits_{k=0}^{\infty} a_n = \sum\limits_{k=0}^{\infty} \frac{(-1)^{2k+1}}{2k+1} = -\sum\limits_{k=0}^{\infty} \frac{1}{2k+1}$. The absolute value of the nth term of the series satisfies

$$|a_n| = \frac{1}{2n+1} > \frac{1}{2n+2} = \frac{1}{2} \cdot \frac{1}{n+1} = b_n.$$

Since $\sum\limits_{k=0}^{\infty} b_k = \sum\limits_{k=0}^{\infty} \frac{1}{2} \cdot \frac{1}{k+1}$ is a constant multiple of the divergent harmonic series, by the Comparison Test, the series $\sum\limits_{k=0}^{\infty} a_n$ diverges as well. At $x = 1$, the series becomes $\sum\limits_{k=0}^{\infty} a_n = \sum\limits_{k=0}^{\infty} \frac{1}{2k+1}$. We can use the same Comparison Test as above to conclude that the series diverges.

So the interval of convergence of the power series is $\boxed{-1 < x < 1.}$

81. We require $\ln\left|\frac{1+x}{1-x}\right| = \ln 2$, which happens when $\frac{1+x}{1-x} = 2$, or $1 + x = 2(1-x)$, or $1 + x = 2 - 2x$, or $3x = 1$, or $x = \frac{1}{3}$. So using the results of Problem 80, we have

$$\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

$$\frac{1}{2}\ln\left|\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right| = \frac{1}{2}\ln\left|\frac{\frac{4}{3}}{\frac{2}{3}}\right| = \frac{1}{2}\ln 2 = \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{3^5} + \cdots$$

$$\begin{aligned} \text{or,} \qquad \ln 2 &= \frac{2}{3}\left[1 + \frac{1}{3^3} + \frac{1}{5 \cdot 3^4} + \cdots\right] \\ &= \frac{2}{3}\left[1 + \frac{1}{27} + \frac{1}{405} + \cdots\right] \\ &\approx \frac{2}{3}[1 + 0.037 + 0.002 + \cdots] \\ &\approx \frac{2}{3}[1.039] \approx 0.6927 \approx \boxed{0.693.} \end{aligned}$$

82. Gregory's series is (see p.708)

$$\begin{aligned} \tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \end{aligned}$$

$$\text{So,} \qquad \tan^{-1} 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^n \frac{1}{2n+1} + \cdots$$
$$\approx \sum_{k=0}^{1000} \frac{(-1)^k}{2k+1}$$
$$\approx \boxed{0.7856479}$$

using a CAS. This means

$$\pi \approx 4(0.7856479) \approx \boxed{3.142592.}$$

83. If R is the radius of convergence of the power series $\sum\limits_{k=1}^{\infty} a_k x^k$, then the power sereis converges for $|x| < R$. By the Ratio Test, which is applicable if the terms of the series are nonzero (it suffices to assume $a_n \neq 0$ as $n \to \infty$) for $x \neq 0$ (at $x = 0$, the series converges to 0), we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = |x| \cdot \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < R \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

for convergence by the Ratio Test. So we have $\lim\limits_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R}$, which was to be shown.

84. If the power series $\sum\limits_{k=1}^{\infty} a_k x^k$ has a radius of convergence R, then the power series $\sum\limits_{k=1}^{\infty} a_x x^{2k} = \sum\limits_{k=1}^{\infty} a_k (x^2)^k = \sum\limits_{k=1}^{\infty} a_k y^k$ also has a radius of convergence R, where $y = x^2$. That is, the series converges for $|y| < R$, or $|x^2| = |x|^2 < R$, or for $|x| < \sqrt{R}$. This means the radius of convergence of the power series $\sum\limits_{k=1}^{\infty} a_k x^{2k}$ is $\sqrt{R}$.

85. Let $\sum\limits_{k=0}^{\infty} a_k x^k$ be a power series centered at $x = 0$, with a radius if convergence R and an interval of convergence $-R < x < R$. If the power series is absolutely convergent at $x = R$, then the series of absolute values $\sum\limits_{k=0}^{\infty} |a_k||x|^k$ is convergent for all $0 \leq |x| \leq R$, by the theorem on convergence and divergence of a power series. This means it is absolutely convergent for $-R \leq x \leq R$, or, it is also absolutely convergent at $x = -R$, which is the other endpoint of the interval of convergence. Similarly, consider a power series $\sum\limits_{k=0}^{\infty} a_k (x - c)^k$ centered at $x = c$ with a radius of convergence R, and an interval of convergence $-R + c < x < R + c$. If this series is absolutely convergent at, say, the right endpoint $x = R + c$, then it is convergent for all x such that $0 \leq |x - c| \leq R$, or $-R \leq x - c \leq R$, or $-R + c \leq x \leq R + c$. That is, it is convergent at $x = -R + c$, which is the other endpoint of the interval of convergence. Conversely, suppose the series is absolutely convergent at the left endpoint, $x = -R + c$, then it is convergent for all x such that $0 \leq |x - c| \leq R$, or $-R \leq x - c \leq R$, or $-R + c \leq x \leq R + c$. In other words, it is also convergent at $x = R + c$. So we have shown that if a power series is absolutely convergent at one of its endpoints, it is absolutely convergent at the other endpoint.

86. From the result of Problem 83, $\lim\limits_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R}$ if $\lim\limits_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists for a convergent power sereis $\sum\limits_{k=0}^{\infty} a_k x^k$, which converges for $|x| < R$. We will use this result in the calculations to follow.

Let series $\sum_{k=1}^{\infty} ka_kx^{k-1}$ be a series of nonzero terms for $x \neq 0$ for at least large values of k. (That is, we assume that $a_n \neq 0$ as $n \to \infty$.) At $x = 0$, note that the series converges to 0. Taking the limit of the absolute value of the ratio of the $n+1$st and the nth term of the series, we have

$$\lim_{n\to\infty} \left| \frac{(n+1)a_{n+1}x^n}{na_nx^{n-1}} \right| = |x| \cdot \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot \lim_{n\to\infty} \left(\frac{n+1}{n} \right) = |x| \cdot \frac{1}{R} \cdot \lim_{n\to\infty} \left(1 + \frac{1}{n} \right) = \frac{|x|}{R} \cdot 1 < 1$$

for convergence of the series by the Ratio Test. That is, the series converges for $|x| < R$, as was to be proved.

The series $\sum_{k=0}^{\infty} \frac{a_k}{k+1}x^{k+1} = a_0 + \sum_{k=1}^{\infty} \frac{a_k}{k+1}x^{k+1}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to a_0.) Taking the limit of the absolute value of the ratio of the $n+1$st and the nth term of the series, we have

$$\lim_{n\to\infty} \left| \frac{\frac{a_{n+1}x^{n+2}}{n+2}}{\frac{a_nx^{n+1}}{n+1}} \right| = |x| \cdot \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot \lim_{n\to\infty} \left(\frac{n+1}{n+2} \right) = |x| \cdot \frac{1}{R} \cdot \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}} \right) = \frac{|x|}{R} \cdot 1 < 1$$

for convergence of the series by the Ratio Test. That is, the series converges for $|x| < R$, as was to be proved.

Challenge Problems

87. The differential equation is $(1+x^2)y'' - 4xy' + 6y = 0$. Let the solution be represented by a power series of the form $y(x) = \sum_{k=0}^{\infty} a_kx^k$, with $a_0 \neq 0$ and $a_1 \neq 0$. We have then:

$$\begin{aligned} y(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \\ y' &= a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots \\ y'' &= 2a_2 + 6a_3x + \cdots + n(n-1)a_nx^{n-2} + (n+1)na_{n+1}x^{n-1} + (n+2)(n+1)a_{n+2}x^n + \cdots \\ x^2y'' &= 2a_2x^2 + 6a_3x^3 + \cdots + n(n-1)a_nx^n + \cdots \\ xy' &= a_1x + 2a_2x^2 + 3a_3x^3 + \cdots + na_nx^n + \cdots . \end{aligned}$$

The coefficient of the x^n term of the series $(1+x^2)y'' - 4xy' + 6y = y'' + x^2y'' - 4xy' + 6y$ is given by

$$[(n+2)(n+1)a_{n+2} + n(n-1)a_n] - 4na_n + 6a_n = 0$$

in order that the differential equation $(1+x^2)y'' - 4xy' + 6y = 0$ be satisfied. That is,

$$\begin{aligned} (n+2)(n+1)a_{n+2} + n(n-1)a_n - 4na_n + 6a_n &= 0 \\ (n+2)(n+1)a_{n+2} + (n^2 - n - 4n + 6)a_n &= 0 \\ (n+2)(n+1)a_{n+2} + (n^2 - 5n + 6)a_n &= 0 \\ (n+2)(n+1)a_{n+2} + (n-2)(n-3)a_n &= 0. \end{aligned}$$

Solving for a_{n+2}, we get

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+2)(n+1)}a_n.$$

We have assumed $a_0 \neq 0$ and $a_1 \neq 0$. Substituting values for n in the recursion relation above, we can determine the coefficients a_n in terms of a_0 and a_1. So, setting $n = 0, 1, 2, \cdots$, we have

$$\begin{aligned}
a_2 &= a_{0+2} = -\frac{(0-2)(0-3)}{(0+2)(0+1)}a_0 = -\frac{(-2)(-3)}{2\cdot 1}a_0 = -\frac{6}{2}a_0 = \boxed{-3a_0} \\
a_3 &= a_{1+2} = -\frac{(1-2)(1-3)}{(1+2)(1+1)}a_1 = -\frac{(-1)(-2)}{3\cdot 2}a_1 = -\frac{2}{6}a_1 = \boxed{-\frac{1}{3}a_1} \\
a_4 &= a_{2+2} = -\frac{(2-2)(2-3)}{(2+2)(2+1)}a_2 = 0 \\
a_5 &= a_{3+2} = -\frac{(3-2)(3-3)}{(3+2)(3+1)}a_2 = 0.
\end{aligned}$$

All the higher even coefficients are 0 because a_{2n} for $n \geq 2$ will be related via the recursion relation to a_{2n-2}, and this to a_{2n-4}, and so forth until $a_4 = 0$ is reached. Similarly, all the higher odd coefficients will be 0 because a_{2n+1} for $n \geq 2$ will be related via the recursion relation to a_{2n-1} and this to a_{2n-3}, and so forth until $a_5 = 0$ is reached.

So the only nonzero coefficients of the series $y(x) = \sum\limits_{k=0}^{\infty} a_k x^k$ are $a_0 \neq 0$, $a_1 \neq 0$ (by assumption) and $a_2 = -3a_0$ and $a_3 = -\frac{1}{3}a_1$.

So finally the solution to the differential equation is

$$\begin{aligned}
y(x) &= \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\
&= a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3}a_1 x^3
\end{aligned}$$

or,
$$\boxed{y(x) = a_0(1-3x^2) + a_1\left(x - \frac{1}{3}x^3\right).}$$

88. If the power series $\sum\limits_{k=0}^{\infty} a_k 3^k$ converges, we have (if the terms of the series $a_n \neq 0$ as $n \to \infty$) applying the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}3^{n+1}}{a_n 3^n}\right| = 3\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

for convergence of the series. This means

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < \frac{1}{3}.$$

To check the convergence of $\sum\limits_{k=1}^{\infty} k a_k 2^k$ (which is also a series of nonzero terms), use the Ratio Test again:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{(n+1)a_{n+1}2^{n+1}}{n\, a_n\, 2^n}\right| &= 2\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\cdot\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = 1\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right) \\
&= 2\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\cdot 1 < 2\cdot\frac{1}{3} = \frac{2}{3} < 1
\end{aligned}$$

so this series will be convergent.

89. The power series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(x-2)^k}{k\,3^k}$ is a series of nonzero terms at $x \neq 2$. (At $x = 2$, the series converges to 0.) We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{(n+1)\,3^{n+1}}}{\frac{(x-2)^n}{n\,3^n}}\right| = \frac{1}{3}|x-2|\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = \frac{|x-2|}{3}\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = \frac{|x-2|}{3}\cdot 1 < 1$$

for convergence of the series by the Ratio Test. So the series converges if $|x-2| < 3$, or $-3 < x-2 < 3$, or $-1 < x < 5$.
At $x = -1$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\frac{(-1-2)^k}{k\,3^k} = \sum_{k=1}^{\infty}\frac{(-3)^k}{k\,3^k} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k}$$

which, being the alternating harmonic series, is convergent. So the power series converges for $x = -1$.
At $x = 5$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\frac{(5-2)^k}{k\,3^k} = \sum_{k=1}^{\infty}\frac{3^k}{k\,3^k} = \sum_{k=1}^{\infty}\frac{1}{k}$$

which, being the harmonic series, is divergent. So the power series diverges for $x = 5$.
Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x < 5.}$

90. Let $-R < x_0 < R$. To prove continuity of $S(x)$ at $x = x_0$, we have to show that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $|x - x_0| < \delta$, $|S(x) - S(x_0)| < \varepsilon$. From the inequality given to us, we know there is a number $N > 0$ such that for $n > N$, $|S(x) - S_n(x)| < \frac{\varepsilon}{3}$ for $|x| < R$. We also have that for $n > N$, $|S(x_0) - S_n(x_0)| < \frac{\varepsilon}{3}$, since $-R < x_0 < R$. Then, for $n = N+1$, there exists a $\delta > 0$ such that for all $|x - x_0| < \delta$, $|S_{N+1}(x) - S_{N+1}(x_0)| < \frac{\varepsilon}{3}$ due to the continuity of $S_n(x)$ at x_0, since the $S_n(x)$ being polynomials in x are continuous in the stated interval, $|x| < R$. We have, for $n = N+1$,

$$\begin{aligned}
|S(x) - S(x_0)| &= |S(x) - S_{N+1}(x) + S_{N+1}(x) - S_{N+1}(x_0) + S_{N+1}(x_0) - S(x_0)| \\
&\leq |S(x) - S_{N+1}(x)| + |S_{N+1}(x) - S_{N+1}(x_0)| + |S_{N+1}(x_0) - S(x_0)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

where we have used the triangle inequality. So we have shown that $\lim\limits_{x\to x_0} S(x) = S(x_0)$, which means $S(x)$ is continuous at $x = x_0$. Since $x = x_0$ was a general point in $|x| < R$, we have proved that $S(x)$ is continuous on $(-R, R)$.

91. Let the power series solution to the differential equation $f''(x) + f(x) = 0$ with the boundary conditions $f(0) = 0$ and $f'(0) = 1$ be given by the power series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots .$$

Since $f(0) = a_0 = 0$, we have that $\boxed{a_0 = 0.}$
Differentiating the series twice, assuming nonzero radius of convergence each time, we obtain

$$\begin{aligned}
f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots \\
f''(x) &= 2a_2 + 6a_3 x + \cdots + n(n-1)x^{n-2} + (n+1)n x^{n-1} + (n+2)(n+1)x^n + \cdots .
\end{aligned}$$

Since $f'(0) = a_1 = 1$, we have $\boxed{a_1 = 1.}$
The coefficient of the x^n term of the series $f''(x) + f(x)$ must satisfy

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

for the differential equation $f''(x) + f(x) = 0$ to be satisfied. Solving, and substituting various values of n, we have

$$\begin{aligned}
a_{n+2} &= -\frac{a_n}{(n+2)(n+1)} \\
a_2 = a_{0+2} &= -\frac{a_0}{(0+2)(0+1)} = 0, \text{ since } a_0 = 0; \\
a_3 = a_{1+2} &= -\frac{a_1}{(1+2)(1+1)} = -\frac{a_1}{3 \cdot 2} = -\frac{a_1}{3!} = -\frac{1}{3!}, \text{ since } a_1 = 1; \\
a_4 = a_{2+2} &= -\frac{a_2}{(2+2)(2+1)} = 0, \text{ since } a_2 = 0; \\
a_5 = a_{3+2} &= -\frac{a_3}{(3+2)(3+1)} = -\frac{a_3}{5 \cdot 4} = (-1)(-1)\frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{a_1}{5!} = \frac{1}{5!} \text{ since } a_1 = 1,
\end{aligned}$$

and so on. Only the odd coefficients are nonzero, with the most general term being of the form

$$a_{2k+1} = (-1)^k \frac{a_1}{(2k+1)!} = \frac{(-1)^k}{(2k+1)!}.$$

So, the solution of the differential equation is

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.}$$

To find the radius of convergence of $f(x)$, we use the Ratio Test (noting that the terms of the series are nonzero if $x \neq 0$, and the series converges to 0 if $x = 0$):

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}} \right| &= |x|^2 \lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)!} = |x|^2 \lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} \\
&= |x|^2 \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = 0 < 1
\end{aligned}$$

for all values of x. So the radius of convergence of the series $\boxed{R = \infty.}$

92. The Bessel function of the first kind of order $m \geq 0$ is

$$J_m(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+m)!\, k!} \left(\frac{x}{2}\right)^{2k+m}.$$

(a) We have, for $m = 0$ and $m = 1$,

$$\begin{aligned}
J_0(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\, k!} \left(\frac{x}{2}\right)^{2k} \\
J_1(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+1)!\, k!} \left(\frac{x}{2}\right)^{2k+1}.
\end{aligned}$$

Using the product rule of differentiation, and noting that the series has a nonzero radius of convergence (seen using the Ratio Test) we get

$$\begin{aligned}
\frac{d}{dx}(xJ_1(x)) &= J_1(x) + xJ_1'(x)\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+1)!\,k!}\left(\frac{x}{2}\right)^{2k+1} + x\cdot\sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+1)!\,k!}(2k+1)\cdot\left(\frac{x}{2}\right)^{2k}\cdot\frac{1}{2}\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+1)!\,k!}[1+(2k+1)]\left(\frac{x}{2}\right)^{2k+1};\\
x^{-1}\frac{d}{dx}(xJ_1(x)) &= \sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+1)!\,k!}\left(\frac{2k+2}{2}\right)\left(\frac{x}{2}\right)^{2k}\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+1)k!\,k!}(k+1)\left(\frac{x}{2}\right)^{2k}\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\,k!}\left(\frac{x}{2}\right)^{2k}\\
&= J_0(x),
\end{aligned}$$

that is, $J_0(x) = x^{-1}\frac{d}{dx}(xJ_1(x))$, as was to be shown.

(b) Note that for $m = 2$,

$$J_2(x) = \sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+2)!\,k!}\left(\frac{x}{2}\right)^{2k+2}.$$

Using the product rule of differentiation, and noting the series has a nonzero radius of convergence (seen using the Ratio Test) we get

$$\begin{aligned}
\frac{d}{dx}(x^2J_2(x)) &= 2xJ_2(x) + x^2J_2'(x)\\
&= 2x\sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+2)!\,k!}\left(\frac{x}{2}\right)^{2k+2} + x^2\cdot\sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+2)!\,k!}(2k+2)\left(\frac{x}{2}\right)^{2k+1}\cdot\frac{1}{2}\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+2)!\,k!}\left[\frac{x^{2k+3}}{2^{2k+1}}\right] + \sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+2)!\,k!}(k+1)\left[\frac{x^{2k+3}}{2^{2k+1}}\right];\\
x^{-2}\frac{d}{dx}(x^2J_2(x)) &= \sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+2)!\,k!}\left(\frac{x}{2}\right)^{2k+1} + \sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+2)!\,k!}(k+1)\left(\frac{x}{2}\right)^{2k+1}\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+2)!\,k!}[1+(k+1)]\left(\frac{x}{2}\right)^{2k+1}\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+2)(k+1)!\,k!}(k+2)\left(\frac{x}{2}\right)^{2k+1}\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+1)!\,k!}\left(\frac{x}{2}\right)^{2k+1}\\
&= J_1(x),
\end{aligned}$$

that is, $J_1(x) = x^{-2}\frac{d}{dx}(x^2J_2(x))$, as was to be shown.

AP$^{®}$ Practice Problems

1. If the power series $\sum_{k=0}^{\infty} a_k x^k$ converges for a number $x_0 \neq 0$, then it converges absolutely for all numbers x for which $|x| < |x_0|$.

   Here, II and III satisfy that requirement.

   CHOICE C

2. For $f = \sum_{k=0}^{\infty} a_k(x-3)^k$ the domain equals the interval of convergence of the power series. The radius of convergence = 2 means

   $$\begin{aligned}|x-3| &< 2\\ -2 < x-3 &< 2\\ 1 < x &< 5.\end{aligned}$$

   $x = 2$ (Choice B) falls in this domain.

   Whether the series converges at the endpoints, $x = 1$ and $x = 5$, depends on what a_k is, as a function of k.

   The series might or might not converge at either one of the endpoints.

   Convergence is not guaranteed there.

   CHOICE B

3. $\sum_{k=1}^{\infty} \frac{(x+3)^k}{k}$ is a power series centered at -3.

   We use the Ratio Test with $a_n = \frac{(x+3)^n}{n}$ and $a_{n+1} = \frac{(x+3)^{n+1}}{n+1}$.

   Then

   $$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(x+3)^{n+1}}{n+1}}{\frac{(x+3)^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{(x+3)^{n+1}n}{(x+3)^n(n+1)}\right| = \lim_{n\to\infty}\left|\frac{(x+3)^n(x+3)n}{(x+3)^n(n+1)}\right|\\ &= |x+3|\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |x+3|\lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right| = |x+3|.\end{aligned}$$

   The series converges absolutely if $|x+3| < 1$, or equivalently if $-4 < x < -2$.

   The radius of convergence is $R = 1$.

   To find the interval of convergence, test the endpoints:

   For $x = -4$:

   $$\sum_{k=1}^{\infty}\frac{(x+3)^k}{k} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k} = \frac{-1}{1} + \frac{1}{2} + \frac{-1}{3} + \frac{1}{4} + \ldots + \frac{(-1)^n}{n} + \ldots,$$

   which is a convergent alternating series.

   The endpoint $x = -4$ is included in the interval of convergence.

   For $x = -2$:

   $$\sum_{k=1}^{\infty}\frac{(x+3)^k}{k} = \sum_{k=1}^{\infty}\frac{(1)^k}{k} = \sum_{k=1}^{\infty}\frac{1}{k},$$

which is the divergent harmonic series.

The endpoint $x = -2$ is not included in the interval of convergence.

The interval of convergence is $\boxed{[-4, -2)}$.

CHOICE B

4. I. $\sum\limits_{k=1}^{\infty} \frac{x^k}{k!}$ is a power series centered at 0.

We use the Ratio Test with $a_n = \frac{x^n}{n!}$ and $a_{n+1} = \frac{(x)^{n+1}}{(n+1)!}$.

Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x)^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{(x)^{n+1}n!}{(n+1)!x^n}\right| = \lim_{n\to\infty}\left|\frac{x^n \cdot x \cdot n!}{(n+1)n!x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = |x|\lim_{n\to\infty}\left|\frac{1}{n+1}\right| = 0 \text{ for any value } x.$$

Since the limit is 0, by the Ratio Test, the series converges for any value of x. The radius of convergence is $R = \infty$ and the interval of convergence of the power series is $\boxed{-\infty < x < \infty}$.

II. $\sum\limits_{k=1}^{\infty} \frac{x^k}{k}$ is a power series centered at 0. We use the Ratio Test with $a_n = \frac{x^n}{n}$ and $a_{n+1} = \frac{(x)^{n+1}}{(n+1)}$.

Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x)^{n+1}}{(n+1)}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{(x)^{n+1}n}{(n+1)x^n}\right| = \lim_{n\to\infty}\left|\frac{x^n \cdot x \cdot n}{(n+1)x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x \cdot n}{n+1}\right| = |x|\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |x|\lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right| = |x|$$

The series converges absolutely if $|x| < 1$, or equivalently if $-1 < x < 1$.
The radius of convergence is $R = 1$. To find the interval of convergence, test the endpoints:
For $x = -1$:

$$\sum_{k=1}^{\infty}\frac{x^k}{k} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k} = \frac{-1}{1} + \frac{1}{2} + \frac{-1}{3} + \frac{1}{4} + \ldots + \frac{(-1)^n}{n} + \ldots$$

which is a convergent alternating series. So $x = -1$ is included in the interval of convergence.
For $x = 1$:

$$\sum_{k=1}^{\infty}\frac{x^k}{k} = \sum_{k=1}^{\infty}\frac{(1)^k}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots,$$

which is the divergent harmonic series. So $x = 1$ is not included in the interval of convergence.

The interval of convergence is $\boxed{-1 \le x < 1}$.

III. $\sum_{k=1}^{\infty} \frac{x^k}{k^3}$ is a power series centered at 0. We use the Ratio Test with $a_n = \frac{x^n}{n^3}$ and $a_{n+1} = \frac{x^{n+1}}{(n+1)^3}$.

Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)^3}}{\frac{x^n}{n^3}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}n^3}{(n+1)^3x^n}\right| = \lim_{n\to\infty}\left|\frac{x^n\cdot x\cdot n^3}{(n+1)^3x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x\cdot n^3}{(n+1)^3}\right| = |x|\lim_{n\to\infty}\left|\frac{n^3}{(n+1)^3}\right| = |x|\lim_{n\to\infty}\left|\left(\frac{1}{1+\frac{1}{n}}\right)^3\right| = |x|$$

The series converges absolutely if $|x| < 1$, or equivalently if $-1 < x < 1$.
The radius of convergence is $R = 1$. To find the interval of convergence, test the endpoints:
For $x = -1$:

$$\sum_{k=1}^{\infty}\frac{x^k}{k^3} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k^3} = \frac{-1}{1} + \frac{1}{2^3} + \frac{-1}{3^3} + \frac{1}{4^3} + \ldots + \frac{(-1)^n}{n^3} + \ldots$$

which is a convergent alternating series. So $x = -1$ is included in the interval of convergence.
For $x = 1$:

$$\sum_{k=1}^{\infty}\frac{x^k}{k^3} = \sum_{k=1}^{\infty}\frac{(1)^k}{k^3} = \frac{1}{1} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots + \frac{1}{n^3} + \ldots$$

which is a convergent p-series. So $x = 1$ is included in the interval of convergence.

The interval of convergence is $\boxed{-1 \le x \le 1}$.

CHOICE B

5. $\sum_{k=1}^{\infty} \frac{x^k}{3^k}$ is a power series centered at 0. We use the Ratio Test with $a_n = \frac{x^n}{3^n}$ and $a_{n+1} = \frac{x^{n+1}}{3^{n+1}}$.

Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{3^{n+1}}}{\frac{x^n}{3^n}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}\cdot 3^n}{3^{n+1}x^n}\right| = \lim_{n\to\infty}\left|\frac{3^n\cdot x^n\cdot x}{3^n\cdot 3\cdot x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{3}\right| = \left|\frac{x}{3}\right|$$

The series converges absolutely if $\left|\frac{x}{3}\right| < 1$, or equivalently if $-3 < x < 3$.

The radius of convergence is $R = 3$. To find the interval of convergence, test the endpoints:

For $x = -3$:

$$\sum_{k=1}^{\infty}\frac{x^k}{3^k} = \sum_{k=1}^{\infty}\frac{(-3)^k}{3^k} = \sum_{k=1}^{\infty}\frac{(-1)^k3^k}{3^k} = \sum_{k=1}^{\infty}(-1)^k = -1+1-1+1-1+\ldots+(-1)^n + \ldots$$

which is a divergent alternating series. So $x = -3$ is not included in the interval of convergence.

For $x = 3$:

$$\sum_{k=1}^{\infty} \frac{x^k}{3^k} = \sum_{k=1}^{\infty} \frac{(3)^k}{3^k} = \sum_{k=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \ldots \text{ which diverges.}$$

So $x = 1$ is not included in the interval of convergence.

The interval of convergence is $\boxed{(-3, 3)}$.

CHOICE C

6. $$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^k}{k!} + \ldots = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$$

$$f'(x) = 0 + \sum_{k=1}^{\infty} \frac{kx^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{kx^{k-1}}{k(k-1)!} = \boxed{\sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}}$$

CHOICE A

7. (a) $\sum\limits_{k=1}^{\infty} \frac{(x-2)^k}{3^k}$ is a power series centered at 0.

We use the Ratio Test with $a_n = \frac{(x-2)^n}{3^n}$ and $a_{n+1} = \frac{(x-2)^{n+1}}{3^{n+1}}$.
Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x)^{n+1}}{(n+1)}}{\frac{x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{3^{n+1}}}{\frac{(x-2)^n}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1} 3^n}{3^{n+1} (x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-2)^n (x-2) 3^n}{3^n (3) (x-2)^n} \right| = \lim_{n\to\infty} \left| \frac{x-2}{3} \right| = \left| \frac{x-2}{3} \right|$$

The series converges absolutely if

$$\left| \frac{x-2}{3} \right| < 1$$
$$-3 < x - 2 < 3,$$
$$-1 < x < 5.$$

The radius of convergence is $\boxed{R = 3}$.

(b) To find the interval of convergence, test the endpoints:
For $x = -1$:

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{3^k} = \sum_{k=1}^{\infty} \frac{(-3)^k}{3^k} = \sum_{k=1}^{\infty} \frac{(-1)^k \cdot 3^k}{3^k} = \sum_{k=1}^{\infty} (-1)^k$$
$$= -1 + 1 - 1 + 1 + \ldots + (-1)^n + \ldots,$$

which is a divergent alternating series.
So $x = -1$ is not included in the interval of convergence.
For $x = 5$:

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{3^k} = \sum_{k=1}^{\infty} \frac{3^k}{3^k} = \sum_{k=1}^{\infty} 1 = 1 + 1 + 1 + \ldots + 1 + \ldots,$$

which is a divergent series.

So $x = 5$ is not included in the interval of convergence.

The interval of convergence is $\boxed{-1 < x < 5}$.

8. (a) Utilizing the power series $\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$,

$$f(x) = \ln\left(1+x^2\right) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(x^2\right)^{k+1}}{k+1}$$

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+2}}{k+1} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots}$$

(b) $f(x) = \ln\left(1+x^2\right) = (-1)^k \sum_{k=0}^{\infty} \frac{\left(x^{2k+2}\right)}{k+1}$

$$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots + (-1)\frac{{}^k\left(x^{2k+2}\right)}{k+1} + \cdots$$

$$f'(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(2k+2)\cdot x^{2k+1}}{k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2(k+1)\cdot x^{2k+1}}{k+1}$$

$$= \boxed{2\sum_{k=0}^{\infty} (-1)^k x^{2k+1}} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^k\left(2x^{2k+1}\right) + \cdots$$

(c) $f'(x) = 2\sum_{k=0}^{\infty} (-1)^k x^{2k+1}$ is a power series centered at 0.

We use the Ratio Test with $a_n = (-1)^k x^{2k+1}$ and $a_{n+1} = (-1)^k x^{2k+3}$.
The constant multiplier 2 does not affect the radius of convergence.

Then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{k+1}x^{2k+3}}{(-1)^k x^{2k+1}}\right| = \lim_{n\to\infty}\left|x^2\right|$.

The series converges absolutely if

$\left|x^2\right| < 1$
$-1 < x < 1$.

The radius of convergence is $R = 1$.
To find the interval of convergence, test the endpoints:
If $x = -1$,

$$\sum_{k=0}^{\infty} (-1)^k x^{2k+1} = \sum_{k=0}^{\infty} (-1)^k \cdot (-1)^{2k+1} = \sum_{k=0}^{\infty} (-1)^{3k+1} = -1+1-1+1-1+\ldots,$$

which is a divergent alternating series.
The endpoint $x = -1$ is not included in the interval of convergence.
If $x = 1$,

$$\sum_{k=0}^{\infty} (-1)^k x^{2k+1} = \sum_{k=0}^{\infty} (-1)^k \cdot (1)^{2k+1} = \sum_{k=0}^{\infty} (-1)^k = 1-1+1-1+\cdots,$$

which is a divergent alternating series.
So $x = 1$ is not included in the interval of convergence.

The interval of convergence is $\boxed{(-1, 1)}$.

8.9 Taylor Series; Maclaurin Series

Concepts and Vocabulary

1. $\boxed{\underline{\textit{Taylor Series}}}$: See p.613 for the full definition.

2. $\boxed{\underline{\textit{Maclaurin}}}$: See p.613 for the full definition.

3. $f(x) = \dfrac{2x^0}{0!} + \dfrac{-4x^1}{1} + \dfrac{3x^2}{2!} + \dfrac{-2x^3}{3!} = \boxed{2 - 4x + \dfrac{3}{2}x^2 - \dfrac{2}{3}x^3}$

4.
$$\begin{aligned} f(x) &= \frac{3(x-5)^0}{0!} + \frac{2(x-5)^1}{1!} + \frac{-1(x-5)^2}{2!} + \frac{-4(x-5)^3}{3!} \\ &= \boxed{3 + 2(x-5) - \frac{1}{2}(x-5)^2 - \frac{2}{3}(x-5)^3} \end{aligned}$$

Skill Building

5. We use the Maclaurin expansion for $e^x = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots$ and substitute $-x^2$ for x:

$$\begin{aligned} f(x) = e^{-x^2} &= 1 + \left(-x^2\right) + \frac{\left(-x^2\right)^2}{2!} + \frac{\left(-x^2\right)^3}{3!} + \frac{\left(-x^2\right)^4}{4!} + \ldots + \frac{\left(-x^2\right)^n}{n!} + \ldots \\ &= 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \ldots + (-1)^n \frac{x^{2n}}{n!} + \ldots \\ &= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}} \end{aligned}$$

6. We use the Maclaurin expansion for $e^x = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots$ and substitute $2x$ for x:

$$\begin{aligned} f(x) = e^{2x} &= 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \ldots \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots \frac{(2x)^n}{n!} + \ldots \\ &= \boxed{\sum_{k=0}^{\infty} \frac{(2x)^k}{k!}} \end{aligned}$$

7. To express $f(x) = \frac{1}{(1+x)^2}$ as a Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{aligned} f(x) &= \frac{1}{(1+x)^2} & f(0) &= \frac{1}{(1+0)^2} = 1 \\ f'(x) &= -\frac{2}{(1+x)^3} & f''(0) &= -\frac{2}{(1+0)^3} = -2 = -2! \\ f''(x) &= \frac{3 \cdot 2}{(1+x)^4} & f''(0) &= \frac{3 \cdot 2}{(1+0)^4} = 3! \\ f'''(x) &= -\frac{4 \cdot 3 \cdot 2}{(1+x)^5} & f'''(0) &= -\frac{4 \cdot 3 \cdot 2}{(1+0)^5} = -4! \\ &\vdots & &\vdots \end{aligned}$$

The Maclaurin series is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= 1 - \frac{2!}{1!}x + \frac{3!}{2!}x^2 - \frac{4!}{3!}x^3 + \cdot + (-1)^n \frac{(n+1)!}{n!}x^n + \cdots \\ &= 1 - 2x + 3x^2 - 4x^3 + \cdots + (-1)^n (n+1)x^n + \cdots \\ &= \boxed{\sum_{k=0}^{\infty} (-1)^k (k+1) x^k.} \end{aligned}$$

8. To express $f(x) = (1+x)^{-3}$ as a Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{aligned} f(x) &= (1+x)^{-3} & f(0) &= (1+0)^{-3} = 1 \\ f'(x) &= (-3)(1+x)^{-4} & f'(0) &= (-3)(1+0)^{-4} = -3 \\ f''(x) &= (-3)(-4)(1+x)^{-5} & f''(0) &= (-3)(-4)(1+0)^{-5} = 3 \cdot 4 \\ f'''(x) &= (-3)(-4)(-5)(1+x)^{-6} & f'''(0) &= (-3)(-4)(-5)(1+0)^{-6} = -3 \cdot 4 \cdot 5 \\ &\vdots & &\vdots \end{aligned}$$

The Maclaurin series is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= 1 - 3x + \frac{3 \cdot 4}{2!}x^2 - \frac{3 \cdot 4 \cdot 5}{3!}x^3 + \cdots + (-1)^n \frac{3 \cdot 4 \cdot 5 \cdots (n+1)(n+2)}{n!}x^n + \cdots \\ &= 1 - 3x + \frac{3 \cdot 4}{2}x^2 - \frac{4 \cdot 5}{2}x^3 + \cdots + (-1)^n \frac{(n+1)(n+2)}{2}x^n + \cdots \\ &= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{(k+1)(k+2)}{2} x^k.} \end{aligned}$$

9. To express $f(x) = \frac{1}{1+x^2}$ as a Maclaurin series, first we find the Maclaurin series for the simpler function $g(x) = \frac{1}{1+x}$ by evaluating it and its derivatives at $x = 0$.

$$\begin{aligned} g(x) &= \frac{1}{1+x} & g(0) &= \frac{1}{1+0} = 1 \\ g'(x) &= -\frac{1}{(1+x)^2} & g'(0) &= -\frac{1}{(1+0)^2} = -1 \\ g''(x) &= \frac{2}{(1+x)^3} & g''(0) &= \frac{2}{(1+0)^3} = 2 = 2! \\ g'''(x) &= -\frac{3\cdot 2}{(1+x)^4} & g'''(0) &= -\frac{3\cdot 2}{(1+0)^4} = -3\cdot 2 = -3! \\ &\vdots & &\vdots \end{aligned}$$

The Maclaurin series of $g(x)$ is

$$\begin{aligned} g(x) &= \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k \\ &= 1 - \frac{1}{1!}x + \frac{2!}{2!}x^2 - \frac{3!}{3!}x^3 + \cdots + (-1)^n \frac{n!}{n!}x^n + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots \end{aligned}$$

Then the Maclaurin explansion of $f(x) = \frac{1}{1+x^2} = g(x^2)$ is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} (x^2)^k \\ &= 1 - x^2 + (x^2)^2 - (x^2)^3 + \cdots + (-1)(x^2)^n + \cdots \\ &= 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots \\ &= \boxed{\sum_{k=0}^{\infty} (-1)^k x^{2k}.} \end{aligned}$$

10. To express $f(x) = \frac{1}{1+2x^3}$ as a Maclaurin series, first we find the Maclaurin series for the simpler function $g(x) = \frac{1}{1+x}$ by evaluating it and its derivatives at $x = 0$.

$$\begin{aligned} g(x) &= \frac{1}{1+x} & g(0) &= \frac{1}{1+0} = 1 \\ g'(x) &= -\frac{1}{(1+x)^2} & g'(0) &= -\frac{1}{(1+0)^2} = -1 \\ g''(x) &= \frac{2}{(1+x)^3} & g''(0) &= \frac{2}{(1+0)^3} = 2 = 2! \\ g'''(x) &= -\frac{3\cdot 2}{(1+x)^4} & g'''(0) &= -\frac{3\cdot 2}{(1+0)^4} = -3\cdot 2 = -3! \\ &\vdots & &\vdots \end{aligned}$$

The Maclaurin series of $g(x)$ is

$$\begin{aligned} g(x) &= \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k \\ &= 1 - \frac{1}{1!}x + \frac{2!}{2!}x^2 - \frac{3!}{3!}x^3 + \cdots + (-1)^n \frac{n!}{n!}x^n + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots \end{aligned}$$

Then the Maclaurin expansion of $f(x) = \frac{1}{1+2x^3} = g(2x^3)$ is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} (2x^3)^k \\ &= 1 - 2x^3 + (2x^3)^2 - (2x^3)^3 + \cdots + (-1)^n (2x^3)^n + \cdots \\ &= 1 - 2x^3 + 4x^6 - 8x^9 + \cdots + (-1)^n 2^n x^{3n} + \cdots \\ &= \boxed{\sum_{k=0}^{\infty} (-1)^k 2^k x^{3k}.} \end{aligned}$$

11. To express $f(x) = \sin(\pi x)$ as a Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{aligned} f(x) &= \sin(\pi x) & f(0) &= 0 \\ f'(x) &= \pi \cos(\pi x) & f'(0) &= \pi \cos(0) = \pi \\ f''(x) &= -\pi^2 \sin(\pi x) & f''(0) &= 0 \\ f'''(x) &= -\pi^3 \cos(\pi x) & f'''(0) &= -\pi^3 \cos(0) = -\pi^3 \\ f^{(4)}(x) &= \pi^4 \sin(\pi x) & f^{(4)}(0) &= 0 \\ &\vdots & &\vdots \end{aligned}$$

For derivatives of odd order, $f^{(2n+1)}(0) = (-1)^n \pi^{2n+1}$. For derivatives of even order, $f^{(2n)}(0) = 0$. So the Maclaurin series is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots + \frac{(-1)^n (\pi x)^{2n+1}}{(2n+1)!} + \cdots \\ &= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1} x^{2k+1}}{(2k+1)!}.} \end{aligned}$$

12. To express $f(x) = \cos(\pi x)$ as a Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$:

$$\begin{aligned} f(x) &= \cos(\pi x) & f(0) &= 1 \\ f'(x) &= -\pi \sin(\pi x) & f'(0) &= 0 \\ f''(x) &= -\pi^2 \cos(\pi x) & f'(0) &= -\pi^2 \\ f'''(x) &= \pi^3 \sin(\pi x) & f'''(0) &= 0 \\ f^{(4)}(x) &= \pi^4 \cos(\pi x) & f^{(4)}(0) &= \pi^4 \\ &\vdots \end{aligned}$$

For derivatives of even order, $f^{(2n)}(0) = (-1)^n \pi^{2n}$.
For derivatives of odd order, $f^{(2n+1)}(0) = 0$.
So the Maclaurin series is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} x^k = 1 - \frac{\pi^2 x^2}{2!} + \frac{\pi^4 x^4}{4!} - \ldots + \frac{(-1)^n \pi^{2n} x^{2n}}{(2n)!} + \ldots \\ &= 1 - \frac{\pi^2}{2} x^2 + \frac{\pi^4}{24} x^4 - \ldots + \frac{(-1)^n \pi^{2n} x^{2n}}{(2n)!} + \ldots \\ &= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} x^{2k}}{(2k)!}} \end{aligned}$$

13. Since $\sinh x = \frac{d}{dx} \cosh x$ and $\cosh x = \sum\limits_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots$, then

$$\begin{aligned} \sinh x &= \frac{d}{dx}\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots + \frac{x^{2n}}{(2n)!} + \frac{x^{2n+2}}{(2n+2)!} + \ldots\right] \\ &= 0 + \frac{2x}{2!} + \frac{4x^3}{4!} + \frac{6x^5}{6!} + \ldots + \frac{2nx^{2n-1}}{(2n)!} + \frac{(2n+2)x^{2n+1}}{(2n+2)!} + \ldots \\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + \ldots \end{aligned}$$

and substitute x^2 for x:

$$\begin{aligned} f(x) = \sinh x^2 &= \left(x^2\right) + \frac{\left(x^2\right)^3}{3!} + \frac{\left(x^2\right)^5}{5!} + \frac{\left(x^2\right)^7}{7!} + \ldots + \frac{\left(x^2\right)^{2n+1}}{(2n+1)!} + \ldots \\ &= x^2 + \frac{x^6}{3!} + \frac{x^{10}}{5!} + \frac{x^{14}}{7!} + \ldots + \frac{x^{2(2n+1)}}{(2n+1)!} + \ldots \\ &= \boxed{\sum_{k=0}^{\infty} \frac{x^{2(2k+1)}}{(2k+1)!}} \end{aligned}$$

14. We use the Maclaurin expansion for

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots$$

and substitute $4x$ for x:

$$\begin{aligned} f(x) = \cosh 4x &= 1 + \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} + \frac{(4x)^6}{6!} + \ldots + \frac{(4x)^{2n}}{(2n)!} + \ldots \\ &= \boxed{\sum_{k=0}^{\infty} \frac{(4x)^{2k}}{(2k)!}} \end{aligned}$$

15. To find the Taylor expansion of $f(x) = \sqrt{x} = x^{1/2}$ about $c = 1$, we begin by evaluating $f(x)$ and its derivatives at $x = 1$.

$$
\begin{aligned}
f(x) &= x^{1/2} & f(1) &= 1 \\
f'(x) &= \frac{1}{2}x^{-1/2} & f'(1) &= \frac{1}{2} \\
f''(x) &= \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^{-3/2} & f''(1) &= -\frac{1}{2^2} \\
f'''(x) &= \left(\frac{1}{2}\right)^2\left(\frac{3}{2}\right)x^{-5/2} & f'''(1) &= \frac{1\cdot 3}{2^3} \\
f^{(4)}(x) &= \left(\frac{1}{2}\right)^2\left(\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-5/2} & f^{(4)}(1) &= -\frac{1\cdot 3\cdot 5}{2^4} \\
&\vdots & &\vdots
\end{aligned}
$$

The Taylor expansion about $c = 1$ is

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\
&= 1 + \frac{\frac{1}{2}}{1!}(x-1) + \frac{-\frac{1}{2^2}}{2!}(x-1)^2 + \frac{\frac{1\cdot 3}{2^3}}{3!}(x-1)^3 + \frac{-\frac{1\cdot 3\cdot 5}{2^4}}{4!}(x-1)^4 + \cdots \\
&\qquad + \frac{(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}}{n!}(x-1)^n + \cdots \\
&= \boxed{1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 + \cdots.}
\end{aligned}
$$

16. To express $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ as a Taylor expansion about $c = 1$, we evaluate f and its derivatives at $c = 1$.

$$
\begin{aligned}
f(x) &= x^{1/3} & f(1) &= 1 \\
f'(x) &= \frac{1}{3}x^{-2/3} & f'(1) &= \frac{1}{3} \\
f''(x) &= \frac{-2}{9}x^{-5/3} & f''(1) &= \frac{-2}{9} \\
f'''(x) &= \frac{10}{27}x^{-8/3} & f'''(1) &= \frac{10}{27} \\
f^{(4)}(x) &= \frac{-80}{81}x^{-11/3} & f^{(4)}(1) &= \frac{-80}{81}
\end{aligned}
$$

The Taylor Expansion of f centered at c is

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^k(c)}{k!}(x-c)^k \\
&= 1 + \frac{x-1}{3} - \frac{2}{9}\cdot\frac{(x-1)^2}{2!} + \frac{10}{27}\cdot\frac{5(x-1)^3}{3!} - \frac{80}{81}\cdot\frac{(x-1)^4}{4!} + \ldots \\
&= \boxed{1 + \frac{x-1}{3} - \frac{(x-1)^2}{9} + \frac{5(x-1)^3}{81} - \frac{10(x-1)^4}{243} + \ldots}
\end{aligned}
$$

17. To find the Taylor expansion of $f(x) = \ln x$ about $c = 1$, we begin by evaluating $f(x)$ and its derivatives at $x = 1$.

$$\begin{aligned} f(x) &= \ln x & f(1) &= \ln 1 = 0 \\ f'(x) &= \frac{1}{x} & f'(1) &= \frac{1}{1} = 1 \\ f''(x) &= -\frac{1}{x^2} & f''(1) &= -\frac{1}{1^2} = -1 \\ f'''(x) &= \frac{2}{x^3} & f'''(1) &= \frac{2}{1^3} = 2 = 2! \\ f^{(4)}(x) &= -\frac{3\cdot 2}{x^4} & f^{(4)}(1) &= -\frac{3\cdot 2}{1^4} = -3\cdot 2 = -3! \\ &\vdots & &\vdots \end{aligned}$$

The Taylor expansion about $c = 1$ is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\ &= \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 - \frac{1}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 \\ &\qquad - \frac{3!}{4!}(x-1)^4 + \cdots + (-1)^{n+1}\frac{(n-1)!}{n!}(x-1)^n + \cdots \\ &= x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + (-1)^{n+1}\frac{(x-1)^n}{n} + \cdots \\ &= \boxed{\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x-1)^k}{k}.} \end{aligned}$$

18. To express $f(x) = \ln(x-1)$ as a Taylor expansion about $c = 2$, we evaluate f and its derivatives at $c = 2$.

$$\begin{aligned} f(x) &= \ln(x-1) & f(2) &= 0 \\ f'(x) &= \frac{1}{x-1} & f'(2) &= 1 \\ f''(x) &= \frac{-1}{(x-1)^2} & f''(2) &= -1 \\ f'''(x) &= \frac{2}{(x-1)^3} & f'''(2) &= 2 \\ f^{iv}(x) &= \frac{-6}{(x-1)^3} & f^{iv}(2) &= -6 \end{aligned}$$

The Taylor Expansion of f centered at c is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^k(c)}{k!}(x-c)^k = 0 + \frac{1\cdot(x-2)}{1!} + \frac{-1\cdot(x-2)^2}{2!} + \frac{2(x-2)^3}{3!} - \frac{6(x-2)^4}{4!} + \ldots \\ &= \boxed{(x-2) - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} - \frac{(x-2)^4}{4} + \ldots + (-1)^{n-1}\frac{(x-2)^n}{n} + \ldots} \\ &= \boxed{\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(x-2)^k}{k}} \end{aligned}$$

19. To find the Taylor expansion of $f(x)=\frac{1}{x}$ about $c=1$, we begin by evaluating $f(x)$ and its derivatives at $x=1$.

$$\begin{aligned} f(x)&=\frac{1}{x} & f(1)&=\frac{1}{1}=1\\ f'(x)&=-\frac{1}{x^2} & f'(1)&=-\frac{1}{1^2}=-1\\ f''(x)&=\frac{2}{x^3} & f''(1)&=\frac{2}{1^3}=2=2!\\ f'''(x)&=-\frac{3\cdot 2}{x^4} & f'''(1)&=-\frac{3\cdot 2}{1^4}=-3!\\ &\vdots & &\vdots \end{aligned}$$

The Taylor expansion about $c=1$ is

$$\begin{aligned} f(x)&=\sum_{k=0}^{\infty}\frac{f^{(k)}(c)}{k!}(x-c)^k\\ &=1-\frac{1}{1!}(x-1)^1+\frac{2!}{2!}(x-1)^2-\frac{3!}{3!}(x-1)^3+\cdots+(-1)^n\frac{n!}{n!}(x-1)^n+\cdots\\ &=1-(x-1)+(x-1)^2-(x-1)^3+\cdots+(-1)^n(x-1)^n+\cdots\\ &=\boxed{\sum_{k=0}^{\infty}(-1)^k(x-1)^k.} \end{aligned}$$

20. To find the Taylor expansion of $f(x)=\frac{1}{\sqrt{x}}=x^{-1/2}$ about $c=4$, we begin by evaluating $f(x)$ and its derivatives at $x=4$.

$$\begin{aligned} f(x)&=x^{-1/2} & f(4)&=4^{-1/2}=\left(2^2\right)^{-1/2}=2^{-1}\\ f'(x)&=-\frac{1}{2}x^{-3/2} & f'(4)&=-\frac{1}{2}\left(2^2\right)^{-3/2}=-\frac{1}{2}\cdot 2^{-3}\\ f''(x)&=\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2} & f''(4)&=\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(2^2\right)^{-5/2}=\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)2^{-5}\\ f'''(x)&=\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2} & f'''(4)&=\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(2^2\right)^{-7/2}=\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)2^{-7}\\ &\vdots & &\vdots \end{aligned}$$

The Taylor expansion about $c=4$ is

$$\begin{aligned} f(x)&=\sum_{k=0}^{\infty}\frac{f^{(k)}(c)}{k!}(x-c)^k\\ &=2^{-1}-1\frac{(1)\left(2^{-3}\right)}{2}\frac{(x-4)}{1!}+\frac{(1\cdot 3)\left(2^{-5}\right)}{2^2}\frac{(x-4)^2}{2!}-\frac{(1\cdot 3\cdot 5)\left(2^{-7}\right)}{2^3}\frac{(x-4)^3}{3!}+\cdots\\ &\quad+(-1)^n\frac{(1\cdot 3\cdot 5\cdots(2n-1))\left(2^{-(2n+1)}\right)}{2^n}\frac{(x-4)^n}{n!}+\cdots\\ &=\boxed{\frac{1}{2}-\frac{x-4}{16}+\frac{3(x-4)^2}{256}-\frac{5(x-4)^3}{2048}+\cdots.} \end{aligned}$$

21. To find the Taylor expansion of $f(x) = \sin x$ at $c = \frac{\pi}{6}$, we begin by evaluating $f(x)$ and its derivatives at $x = \frac{\pi}{6}$.

$$\begin{aligned}
f(x) &= \sin x & f\left(\frac{\pi}{6}\right) &= \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \\
f'(x) &= \cos x & f'\left(\frac{\pi}{6}\right) &= \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \\
f''(x) &= -\sin x & f''\left(\frac{\pi}{6}\right) &= -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2} \\
f'''(x) &= -\cos x & f'''\left(\frac{\pi}{6}\right) &= -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2} \\
f^{(4)}(x) &= \sin x & f^{(4)}\left(\frac{\pi}{6}\right) &= \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \\
&\vdots & &\vdots
\end{aligned}$$

For the even derivatives $f^{(2k)}\left(\frac{\pi}{6}\right) = (-1)^k \frac{1}{2}$. For the odd derivatives, $f^{(2k+1)}\left(\frac{\pi}{6}\right) = (-1)^k \frac{\sqrt{3}}{2}$. The Taylor expansion about $c = \frac{\pi}{6}$ is

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\
&= \frac{1}{2} + \frac{\frac{\sqrt{3}}{2}}{1!}\left(x - \frac{\pi}{6}\right) + \frac{-\frac{1}{2}}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{-\frac{\sqrt{3}}{2}}{3!}\left(x - \frac{\pi}{6}\right)^3 + \frac{\frac{1}{2}}{4!}\left(x - \frac{\pi}{6}\right)^4 + \cdots \\
&\qquad + \frac{(-1)^n \frac{1}{2}}{(2n)!}\left(x - \frac{\pi}{6}\right)^{2n} + \frac{(-1)^n \frac{\sqrt{3}}{2}}{(2n+1)!}\left(x - \frac{\pi}{6}\right)^{2n+1} + \cdots \\
&= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{6}\right)^4 + \cdots \\
&= \boxed{\sum_{k=0}^{\infty} \frac{\sin\left(\frac{1}{6}(\pi + 3k\pi)\right)\left(x - \frac{\pi}{6}\right)^k}{k!}.}
\end{aligned}$$

22. To find the Taylor expansion of $f(x) = \cos x$ about $c = -\frac{\pi}{2}$, we begin by evaluating $f(x)$ and its derivatives at $x = -\frac{\pi}{2}$.

$$\begin{aligned}
f(x) &= \cos x & f\left(-\frac{\pi}{2}\right) &= \cos\left(-\frac{\pi}{2}\right) = 0 \\
f'(x) &= -\sin x & f'\left(-\frac{\pi}{2}\right) &= -\sin\left(-\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1 \\
f''(x) &= -\cos x & f''\left(-\frac{\pi}{2}\right) &= -\cos\left(-\frac{\pi}{2}\right) = 0 \\
f'''(x) &= \sin x & f'''\left(-\frac{\pi}{2}\right) &= \sin\left(-\frac{\pi}{2}\right) = -1 \\
f^{(4)}(x) &= \cos x & f^{(4)}\left(-\frac{\pi}{2}\right) &= \cos\left(-\frac{\pi}{2}\right) = 0 \\
&\vdots & &\vdots
\end{aligned}$$

For the even derivatives $f^{(2n)}\left(-\frac{\pi}{2}\right) = 0$. For the odd derivatives $f^{(2n+1)}\left(-\frac{\pi}{2}\right) = (-1)^n$. The Taylor expansion about $c = -\frac{\pi}{2}$ is

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\
&= 0 + \frac{1}{1!}\left(x+\frac{\pi}{2}\right) + \frac{0}{2!}\left(x+\frac{\pi}{2}\right)^2 + \frac{(-1)}{3!}\left(x+\frac{\pi}{2}\right)^3 + \frac{0}{4!}\left(x+\frac{\pi}{2}\right)^4 + \cdots \\
&\quad + \frac{(-1)^n}{(2n+1)!}\left(x+\frac{\pi}{2}\right)^{2n+1} + \cdots \\
&= \boxed{\sum_{k=0}^{\infty}(-1)^k \frac{\left(x+\frac{\pi}{2}\right)^{2k+1}}{(2k+1)!}.}
\end{aligned}$$

23. To find the Taylor expansion of $f(x) = 3x^2 + 2x^2 + 5x - 6$ about $c = 0$, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{aligned}
f(x) &= 3x^2 + 2x^2 + 5x - 6 & f(0) &= 3(0)^2 + 2(0)^2 + 5(0) - 6 = -6 \\
f'(x) &= 9x^2 + 4x + 5 & f'(0) &= 9(0)^2 + 4(0) + 5 = 5 \\
f''(x) &= 18x + 4 & f''(0) &= 18(0) + 4 = 4 \\
f'''(x) &= 18 & f'''(0) &= 18 \\
f^{(4)}(x) &= 0 & f^{(4)}(0) &= 0.
\end{aligned}$$

All the higher order derivatives will be zero, so their evaluation at $x = 0$ will also be zero. The Taylor expansion about $c = 0$ is

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\
&= -6 + \frac{5}{1!}x + \frac{4}{2!}x^2 + \frac{18}{3!}x^3 \\
&= -6 + 5x + 2x^2 + 3x^3 \\
&= \boxed{3x^3 + 2x^2 + 5x - 6.}
\end{aligned}$$

We see that the Taylor expansion of the function about $c = 0$ is identical to the function itself.

24. To find the Taylor expansion of $f(x) = 4x^4 - 2x^3 - x$ about $c = 0$, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{aligned}
f(x) &= 4x^4 - 2x^3 - x & f(0) &= 4(0)^4 - 2(0)^3 - 0 = 0 \\
f'(x) &= 16x^3 - 6x^2 - 1 & f'(0) &= 16(0)^3 - 6(0)^2 - 1 = -1 \\
f''(x) &= 48x^2 - 12x & f''(0) &= 48(0)^2 - 12(0) = 0 \\
f'''(x) &= 96x - 12 & f'''(0) &= 96(0) - 12 = -12 \\
f^{(4)}(x) &= 96 & f^{(4)}(0) &= 96 \\
f^{(5)}(x) &= 0 & f^{(5)}(0) &= 0.
\end{aligned}$$

All the higher order derivatives will be zero, so their evaluation at $x = 0$ will also be zero. The Taylor expansion about $c = 0$ is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\ &= 0 + \frac{-1}{1!}x + \frac{0}{2!}x^2 + \frac{-12}{3!}x^3 + \frac{96}{4!}x^4 \\ &= -x - 2x^3 + 4x^4 \\ &= \boxed{4x^4 - 2x^3 - x.} \end{aligned}$$

We see that the Taylor expansion of the function about $c = 0$ is identical to the function itself.

25. To find the Taylor expansion of $f(x) = 3x^3 + 2x^2 + 5x - 6$ about $c = 1$, we begin by evaluating $f(x)$ and its derivatives at $x = 1$.

$$\begin{aligned} f(x) &= 3x^3 + 2x^2 + 5x - 6 & f(1) &= 3(1)^3 + 2(1)^2 + 5(1) - 6 = 3 + 2 + 5 - 6 = 4 \\ f'(x) &= 9x^2 + 4x + 5 & f'(1) &= 9(1)^2 + 4(1) + 5 = 9 + 4 + 5 = 18 \\ f''(x) &= 18x + 4 & f''(1) &= 18(1) + 4 = 22 \\ f'''(x) &= 18 & f'''(1) &= 18 \\ f^{(4)}(x) &= 0 & f^{(4)}(1) &= 0. \end{aligned}$$

All the higher order derivatives will be zero, so their evaluation at $x = 1$ will also be zero. The Taylor expansion about $c = 1$ is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\ &= 4 + \frac{18}{1!}(x-1) + \frac{22}{2!}(x-1)^2 + \frac{18}{3!}(x-1)^3 \\ &= \boxed{4 + 18(x-1) + 11(x-1)^2 + 3(x-1)^3} \\ &= 4 + 18x - 18 + 11(x^2 - 2x + 1) + 3(x^3 - 3x^2 + 3x - 1) \\ &= 4 + 18x - 18 + 11x^2 - 22x + 11 + 3x^3 - 9x^2 + 9x - 3 \\ &= (4 - 18 + 11 - 3) + (18x - 22x + 9x) + (11x^2 - 9x^2) + 3x^3 \\ &= -6 + 5x + 2x^2 + 3x^3 \\ &= 3x^3 + 2x^2 + 5x - 6. \end{aligned}$$

We see that the Taylor expansion of the function about $c = 1$ is identical to the function itself.

26. To find the Taylor expansion of $f(x) = 4x^4 - 2x^3 + x$ about $c = 1$, we begin by evaluating $f(x)$ and its derivatives at $x = 1$.

$$\begin{aligned} f(x) &= 4x^4 - 2x^3 + x & f(1) &= 4(1)^4 - 2(1)^3 + 1 = 4 - 2 + 1 = 3 \\ f'(x) &= 16x^3 - 6x^2 + 1 & f'(1) &= 16(1)^3 - 6(1)^2 + 1 = 16 - 6 + 1 = 11 \\ f''(x) &= 48x^2 - 12x & f''(1) &= 48(1)^2 - 12(1) = 48 - 12 = 36 \\ f'''(x) &= 96x - 12 & f'''(1) &= 96(1) - 12 = 96 - 12 = 84 \\ f^{(4)}(x) &= 96 & f^{(4)}(1) &= 96 \\ f^{(5)}(x) &= 0 & f^{(5)}(1) &= 0. \end{aligned}$$

All the higher order derivatives will be zero, so their evaluation at $x = 1$ will also be zero. The Taylor expansion about $c = 1$ is

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\
&= 3 + \frac{11}{1!}(x-1) + \frac{36}{2!}(x-1)^2 + \frac{84}{3!}(x-1)^3 + \frac{96}{4!}(x-1)^4 \\
&= \boxed{3 + 11(x-1) + 18(x-1)^2 + 14(x-1)^3 + 4(x-1)^4} \\
&= 3 + 11x - 11 + 18(x^2 - 2x + 1) + 14(x^3 - 3x^2 + 3x - 1) + 4(x^4 - 4x^3 + 6x^2 - 4x + 1) \\
&= (3 - 11 + 18 - 14 + 4) + (11x - 36x + 42x - 16x) + (18x^2 - 42x^2 + 24x^2) + (14x^3 - 16x^3) + 4x^4 \\
&= 0 + x + 0 - 2x^3 + 4x^4 \\
&= 4x^4 - 2x^3 + x.
\end{aligned}$$

We see that the Taylor expansion of the function about $c = 1$ is identical to the function itself.

27. The Maclaurin expansion for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and the Maclaurin expansion for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

The the first five terms of the Maclaurin expansion for $e^x + \sin x$ are

$$\begin{aligned}
e^x + \sin x &= (1 + x) + \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^2}{2!} + \frac{x^5}{5!}\right) + \left(\frac{x^3}{3!} - \frac{x^7}{7!}\right) + \left(\frac{x^4}{4!} + \frac{x^9}{9!}\right) + \left(\frac{x^5}{5!} + \ldots\right) + \ldots \\
&= 1 + (x + x) + \frac{x^2}{2!} + \left(-\frac{x^3}{3!} + \frac{x^3}{3!}\right) + \frac{x^4}{4!} + \left(\frac{x^5}{5!} + \frac{x^5}{5!}\right) + \ldots \\
&= \boxed{1 + 2x + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^5}{60}} + \ldots
\end{aligned}$$

28. In Example 8 of Section 8.8 we found that the Maclaurin expansion for $\ln \frac{1}{1-x}$ is

$$\ln \frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \ldots + \frac{x^{n+1}}{n+1} + \ldots = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

and the Maclaurin expansion for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

Therefore the first five terms of the Maclaurin expansion for $\ln \frac{1}{1-x} + \cos x$ are

$$\begin{aligned}
\ln \frac{1}{1-x} + \cos x &= (x + 1) + \left(\frac{x^2}{2} - \frac{x^2}{2!}\right) + \left(\frac{x^3}{3} + \frac{x^4}{4!}\right) + \left(\frac{x^4}{4} - \frac{x^6}{6!}\right) + \left(\frac{x^5}{5} + \frac{x^8}{8!}\right) + \ldots \\
&= 1 + x + \left(\frac{x^2}{2} - \frac{x^2}{2!}\right) + \frac{x^3}{3} + \left(\frac{x^4}{4!} + \frac{x^4}{4}\right) + \frac{x^5}{5} + \ldots \\
&= \boxed{1 + x + \frac{x^3}{3} + \frac{7x^4}{24} + \frac{x^5}{5}} + \ldots
\end{aligned}$$

29. The Maclaurin expansion of $f(x) = xe^x$ is obtained by multiplying the Maclaurin expansion of x, which is x, and that of $g(x) = e^x$ where (see Problem 27),

$$g(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots + \frac{x^n}{n!} + \cdots .$$

So the first five terms of the Maclaurin series of $f(x)$ are

$$\begin{aligned} f(x) &= xg(x) \\ &= x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots\right) \\ &= \boxed{x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots .} \end{aligned}$$

30. The Maclaurin expansion of $f(x) = xe^{-x}$ is obtained by multiplying the Maclaurin expansion of x, which is x, and that of $g(-x) = e^{-x}$ where (see Problem 27),

$$g(-x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^n}{n!} + \cdots .$$

So the first five terms of the Maclaurin series of $f(x)$ are

$$\begin{aligned} f(x) &= xg(-x) \\ &= x\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^n}{n!} + \cdots\right) \\ &= \boxed{x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \frac{x^5}{4!} - \cdots .} \end{aligned}$$

31. The Maclaurin expansion of $f(x) = e^{-x} \sin x$ is obtained by multiplying the Maclaurin expansions of e^{-x} and $\sin x$ together. From Problem 27, we have

$$g(-x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^n}{n!} + \cdots .$$

From p.717 of the book, we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots .$$

So the first five terms of the Maclaurin series of $f(x)$ are

$$\begin{aligned} f(x) &= e^{-x} \sin x \\ &= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots\right) \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right) \\ &= \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots\right) \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\ &= 1 \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) - x \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \frac{x^2}{2} \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\ &\quad - \frac{x^3}{6} \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \frac{x^4}{24} \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) - \frac{x^5}{120} \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \cdots \end{aligned}$$

$$\begin{aligned}
&= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \left(-x^2 + \frac{x^4}{6} - \frac{x^6}{120} + \cdots\right) + \left(\frac{x^3}{2} - \frac{x^5}{12} + \frac{x^7}{240} - \cdots\right) + \\
&\qquad \left(-\frac{x^4}{6} + \frac{x^6}{36} - \frac{x^8}{720} + \cdots\right) + \left(\frac{x^5}{24} - \frac{x^7}{144} + \frac{x^9}{2880} - \cdots\right) + \left(-\frac{x^6}{120} - \cdots\right) + \cdots \\
&= x - x^2 + x^3\left(\frac{1}{2} - \frac{1}{6}\right) + x^4\left(\frac{1}{6} - \frac{1}{6}\right) + x^5\left(\frac{1}{120} + \frac{1}{24} - \frac{1}{12}\right) + x^6\left(-\frac{1}{120} + \frac{1}{36} - \frac{1}{120}\right) - \cdots \\
&= \boxed{x - x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \frac{1}{90}x^6 - \cdots .}
\end{aligned}$$

32. The Maclaurin expansion of $f(x) = e^{-x}\cos x$ is obtained by multiplying the Maclaurin expansions of e^{-x} and $\cos x$ together. From Problem 27, we have

$$g(-x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^n}{n!} + \cdots .$$

From p.718 of the book, we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots .$$

So the first five terms of the Maclaurin series of $f(x)$ are

$$\begin{aligned}
f(x) &= e^{-x}\cos x \\
&= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots + (-1)^n\frac{x^n}{n!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots\right) \\
&= \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) \\
&= 1\cdot\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) - x\cdot\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) + \frac{x^2}{2}\cdot\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) \\
&\qquad - \frac{x^3}{6}\cdot\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) + \frac{x^4}{24}\cdot\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) - \frac{x^5}{120}\cdot\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) + \cdots \\
&= \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) + \left(-x + \frac{x^3}{2} - \frac{x^5}{24}\right) + \left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{48}\right) \\
&\qquad + \left(-\frac{x^3}{6} + \frac{x^5}{12}\right) + \left(\frac{x^4}{24} - \frac{x^6}{48}\right) + \left(-\frac{x^5}{120}\right) + \cdots \\
&= 1 - x + x^2\left(\frac{1}{2} - \frac{1}{2}\right) + x^3\left(\frac{1}{2} - \frac{1}{6}\right) + x^4\left(\frac{1}{24} - \frac{1}{4} + \frac{1}{24}\right) \\
&\qquad + x^5\left(-\frac{1}{24} + \frac{1}{12} - \frac{1}{120}\right) + \cdots \\
&= \boxed{1 - x + \frac{x^3}{3} - \frac{x^4}{6} + \frac{x^5}{30} + \cdots .}
\end{aligned}$$

33. To find a Maclaurin expansion for $f(x) = \frac{1}{\sqrt{1-x^2}}$, we first find the Maclaurin expansion for the function $h(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{-1/2}$, by evaluating the function and its derivatives at $x = 0$.

$$
\begin{aligned}
h(x) &= (1-x)^{-1/2} & h(0) &= 1 \\
h'(x) &= -\frac{1}{2}(1-x)^{-3/2} \cdot (-1) = \frac{1}{2}(1-x)^{-3/2} & h'(0) &= \frac{1}{2} \\
h''(x) &= \left(\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1-x)^{-5/2} \cdot (-1) = \left(\frac{1 \cdot 3}{2^2}\right)(1-x)^{-5/2} & h''(0) &= \frac{3}{4} \\
h'''(x) &= \left(\frac{1 \cdot 3 \cdot 5}{2^3}\right)(1-x)^{-7/2} & h'''(0) &= \frac{15}{8} \\
h^{(4)}(x) &= \left(\frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4}\right)(1-x)^{-9/2} & h^{(4)}(0) &= \frac{105}{16} \\
&\vdots & &\vdots
\end{aligned}
$$

So the Maclaurin expansion of $h(x) = \frac{1}{\sqrt{1-x}}$ is

$$
\begin{aligned}
h(x) &= \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} x^k \\
&= 1 + \frac{\frac{1}{2}}{1!}x + \frac{\frac{3}{4}}{2!}x^2 + \frac{\frac{15}{8}}{3!}x^3 + \frac{\frac{105}{16}}{4!}x^4 + \cdots \\
&= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 + \cdots .
\end{aligned}
$$

The Maclaurin expansion of $f(x)$ can now be found:

$$f(x) = \frac{1}{\sqrt{1-x^2}} = h(x^2) = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \cdots .$$

Integrating this series term by term gives us the required first four terms of the Maclaurin expansion for $g(x) = \sin^{-1} x$:

$$
\begin{aligned}
g(x) = \sin^{-1} x &= \int_0^x \frac{dx}{\sqrt{1-x^2}} = \int_0^x f(x)\,dx \\
&= \int_0^x \left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots\right) dx \\
&= \boxed{x + \frac{x^3}{6} + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots .}
\end{aligned}
$$

34. To find the Maclaurin series for $f(x) = \tan x$, we begin by evaluating the function and its derivatives at $x = 0$. Let $y = f(x)$, $y_0 = f(0)$, $y' = f'(x)$, $y_0' = f'(0)$, etc.

$$
\begin{aligned}
y &= \tan x & y_0 &= 0 \\
y' &= \sec^2 x = 1 + \tan^2 x = 1 + y^2 & y_0' &= 1 + y_0^2 = 1 + 0 = 1 \\
y'' &= 2yy' & y_0'' &= 2y_0y_0' = 2(0)(1) = 0 \\
y''' &= 2y'^2 + 2yy'' & y_0''' &= 2y_0'^2 + 2y_0y_0'' \\
& & &= 2(1)^2 + 2(0)(0) = 2 \\
y^{(4)} &= 4y'y'' + 2y'y'' + 2y'y''' & & \\
&= 6y'y'' + 2yy''' & y_0^{(4)} &= 6y_0'y_0'' + 2y_0y_0''' \\
& & &= 6(1)(0) + 2(0)(2) = 0
\end{aligned}
$$

$$\begin{aligned} y^{(5)} &= 6y''^2 + 6y'y''' + 2y'y''' + 2yy^{(4)} \\ &= 6y''^2 + 8y'y''' + 2yy^{(4)} \end{aligned}$$

$$\begin{aligned} y_0^{(5)} &= 6y_0''^2 + 8y_0'y_0''' + 2y_0y_0^{(4)} \\ &= 6(0) + 8(1)(2) + 2(0)(0) = 16 \end{aligned}$$

$$\begin{aligned} y^{(6)} &= 12y''y''' + 8y''y''' + 8y'y^{(4)} + 2y'y^{(4)} + 2yy^{(5)} \\ &= 20y''y''' + 10y'y^{(4)} + 2yy^{(5)} \end{aligned}$$

$$\begin{aligned} y_0^{(6)} &= 20y_0''y_0''' + 10y_0'y_0^{(4)} + 2y_0y_0^{(5)} \\ &= 20(0)(2) + 10(1)(0) + 2(0)(16) \\ &= 0 \end{aligned}$$

$$\begin{aligned} y^{(7)} &= 20y'''^2 + 20y''y^{(4)} + 10y''y^{(4)} + 10y'y^{(5)} + 2y'y^{(5)} + 2yy^{(6)} \\ &= 20y'''^2 + 30y''y^{(4)} + 12y'y^{(5)} + 2yy^{(6)} \end{aligned}$$

$$\begin{aligned} y_0^{(7)} &= 20y_0'''^2 + 30y_0''y_0^{(4)} + 12y_0'y_0^{(5)} + 2y_0y_0^{(6)} \\ &= 20(2^2) + 30(0)(0) + 12(1)(16) + 2(0)(0) \\ &= 80 + 192 = 272 \end{aligned}$$

$$\vdots \qquad\qquad \vdots$$

The Maclaurin series for $f(x) = \tan x$ to four nonzero terms is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= 0 + \frac{1}{1!}x + 0 + \frac{2}{3!}x^3 + 0 + \frac{16}{5!}x^5 + 0 + \frac{272}{7!}x^7 + \cdots \\ &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots . \end{aligned}$$

By integrating this term by term, one obtains the first four nonzero terms of the Maclaurin series for $g(x)$ because

$$\begin{aligned} g(x) &= \ln(\cos x) = -\ln\left(\frac{1}{\cos x}\right) \\ &= -\ln(\sec x) \\ &= -\int_0^x \tan x \, dx \\ &= -\int_0^x \left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots\right) dx \\ &= -\left(\frac{x^2}{2} + \frac{x^4}{3 \cdot 4} + \frac{2x^6}{15 \cdot 6} + \frac{17x^8}{315 \cdot 8} + \cdots\right) \\ &= \boxed{-\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \cdots .} \end{aligned}$$

35. $g'(x) = \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{x^2+1} = f(x)$.
Therefore

$$g(x) = \int_0^x f(t)\,dt = \int_0^x \left(\frac{1}{t^2+1}\right) dt.$$

First, find the Maclaurin series for $\frac{1}{t^2+1} = \left(t^2+1\right)^{-1}$:
Start with $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots$ with $m = -1$:

$$\begin{aligned} (1+x)^{-1} &= 1 + (-1)x + \frac{(-1)(-1-1)}{2!}x^2 + \frac{(-1)(-1-1)(-1-2)}{3!}x^3 + \ldots \\ &= 1 - x + x^2 - x^3. \end{aligned}$$

Then, substituting t^2 for x,

$$\frac{1}{t^2+1} = \left(1+t^2\right)^{-1} = 1 - t^2 + \left(t^2\right)^2 - \left(t^2\right)^3 + \dots = 1 - t^2 + t^4 - t^6 + \dots$$

So

$$\begin{aligned} g(x) &= \int_0^x \left(\frac{1}{t^2+1}\right) dt = \int_0^x \left(1 - t^2 + t^4 - t^6 + \dots\right) dt \\ &= \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{x^7}{7} + \dots\right]_0^x = \boxed{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}} + \dots \end{aligned}$$

36. $g'(x) = \frac{d}{dx}(\tan x) = \sec^2 x = f(x)$
Therefore

$$g(x) = \int_0^x f(t)\,dt = \int_0^x \sec^2 t\,dt$$

First, find the Maclaurin series for $\sec^2 t$:
Using $\cos t = \sum\limits_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots = 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720} + \dots$, we have
$\sec t = \frac{1}{\cos t} = \frac{1}{1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720} + \dots}$

$$\begin{array}{r r}
 & 1 + \frac{t^2}{2} + \frac{5t^4}{24} + \frac{61t^6}{720} + \dots \\
\cline{2-2}
1 + \frac{t^2}{2} + \frac{t^4}{24} + \frac{t^6}{720} + \dots \Big) & 1 \qquad\qquad\qquad\qquad \\
 & 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720} + \dots \\
\cline{2-2}
 & \frac{t^2}{2} - \frac{t^4}{24} + \frac{t^6}{720} - \dots \\
 & \frac{t^2}{2} - \frac{t^4}{4} + \frac{t^6}{48} - \dots \\
\cline{2-2}
 & \frac{5t^4}{24} - \frac{7t^6}{360} + \dots \\
 & \frac{5t^4}{24} - \frac{5t^6}{48} + \dots \\
\cline{2-2}
 & \frac{61t^6}{720} - \dots \\
 & \frac{61t^6}{720} - \dots \\
\cline{2-2}
 & + \dots
\end{array}$$

Then

$$\begin{aligned} \sec^2 t &= \left(1 + \frac{t^2}{2} + \frac{5t^4}{24} + \frac{61t^6}{720} + \dots\right)^2 \\ &= 1^2 + 2\cdot\left(1\cdot\frac{t^2}{2}\right) + \left[\left(\frac{t^2}{2}\right) + 2\cdot\left(1\cdot\frac{5t^4}{24}\right)\right] + \left[2\cdot\left(1\cdot\frac{61t^6}{720}\right) + 2\cdot\left(\frac{t^2}{2}\cdot\frac{5t^4}{24}\right)\right] \end{aligned}$$

+ terms in t^8, t^{10}, and t^{12}, plus more terms …

$$= 1 + \frac{t^2}{2} + \left(\frac{t^4}{4} + \frac{5t^4}{12}\right) + \left(\frac{61t^6}{360} + \frac{5t^6}{24}\right) + \dots = 1 + \frac{1}{2}t^2 + \frac{2}{3}t^4 + \frac{17}{45}t^6 + \dots$$

Then

$$\begin{aligned}\tan x &= \int_0^x \sec^2 t\,dt = \int_0^x \left(1+\frac{t^2}{2}+\frac{2t^4}{3}+\frac{17t^6}{45}+\dots\right)\\ &= \left[t+\frac{t^3}{6}+\frac{2t^5}{15}+\frac{17t^7}{315}+\dots\right]_0^x = \boxed{x+\frac{x^3}{6}+\frac{2x^5}{15}+\frac{17x^7}{315}+\dots}\end{aligned}$$

Notice that the Maclaurin series for $\tan x$ could have been found more quickly by simply dividing the series for $\sin x$ by the series for $\cos x$:

$$\begin{aligned}\tan x = \frac{\sin x}{\cos x} &= \frac{\sum\limits_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}}{\sum\limits_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}} = \frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots}{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots}\\ &= \frac{x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\dots}{1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots}\end{aligned}$$

$$\begin{array}{r|r}
 & \boxed{x+\frac{x^3}{6}+\frac{2x^5}{15}+\frac{17x^7}{315}+\dots} \\ \cline{2-2}
1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\dots & x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\dots \\
 & x-\frac{x^3}{2}+\frac{x^5}{24}-\frac{x^7}{720}+\dots \\ \cline{2-2}
 & \frac{x^3}{3}-\frac{x^5}{30}+\frac{x^7}{840}-\dots \\
 & \frac{x^3}{3}-\frac{x^5}{6}+\frac{x^7}{72}-\dots \\ \cline{2-2}
 & \frac{2x^5}{15}-\frac{4x^7}{315}+\dots \\
 & \frac{2x^5}{15}-\frac{x^7}{15}+\dots \\ \cline{2-2}
 & \frac{17x^7}{315}-\dots \\
 & \frac{17x^7}{315}-\dots \\ \cline{2-2}
 & +\dots
\end{array}$$

37. Since $(1+x)^m = \sum\limits_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $f(x)=\sqrt{1+x^2}=(1+x^2)^{1/2}$ is given by

$$\begin{aligned}f(x) &= \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^{2k}\\ &= \binom{\frac{1}{2}}{0}x^0+\binom{\frac{1}{2}}{1}x^2+\binom{\frac{1}{2}}{2}x^4+\binom{\frac{1}{2}}{3}x^6+\cdots\\ &= 1+\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}x^4+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^6+\cdots\\ &= \boxed{1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6-\cdots.}\end{aligned}$$

Since $m = \frac{1}{2}$ satisfies $m > 0$ but m is not an integer, by the conditions of the theorem on p.721, the series converges on the closed interval $\boxed{[-1, 1].}$

38. Since $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $f(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{-1/2}$ is given by

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-1)^k x^k \\ &= \binom{-\frac{1}{2}}{0}(-1)^0 x^0 + \binom{-\frac{1}{2}}{1}(-1)^1 x^1 + \binom{-\frac{1}{2}}{2}(-1)^2 x^2 + \binom{-\frac{1}{2}}{3}(-1)^3 x^3 + \cdots \\ &= 1 - \left(-\frac{1}{2}\right) x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} x^2 - \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} x^3 + \cdots \\ &= \boxed{1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots .} \end{aligned}$$

Since $m = -\frac{1}{2}$ satisfies $-1 < m < 0$, by the conditions of the theorem on p.721, the series converges on the half open interval $\boxed{(-1, 1].}$

39. Since $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $f(x) = (1+x)^{1/5}$ is given by

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \binom{\frac{1}{5}}{k} x^k \\ &= \binom{\frac{1}{5}}{0} x^0 + \binom{\frac{1}{5}}{1} x^1 + \binom{\frac{1}{5}}{2} x^2 + \binom{\frac{1}{5}}{3} x^3 + \cdots \\ &= 1 + \frac{1}{5}x + \frac{\left(\frac{1}{5}\right)\left(\frac{1}{5}-1\right)}{2!} x^2 + \frac{\left(\frac{1}{5}\right)\left(\frac{1}{5}-1\right)\left(\frac{1}{5}-2\right)}{3!} x^3 + \cdots \\ &= \boxed{1 + \frac{1}{5}x - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \cdots .} \end{aligned}$$

Since $m > 0$ but is not an integer, by the conditions of the theorem on p.721, the sereis converges on the closed interval $\boxed{[-1, 1].}$

40. Since $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $f(x) = (1-x)^{5/3}$ is given by

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \binom{\frac{5}{3}}{k} (-1)^k x^k \\ &= \binom{\frac{5}{3}}{0}(-1)^0 x^0 + \binom{\frac{5}{3}}{1}(-1)^1 x^1 + \binom{\frac{5}{3}}{2}(-1)^2 x^2 + \binom{\frac{5}{3}}{3}(-1)^3 x^3 + \cdots \\ &= 1 - \frac{5}{3}x + \frac{\left(\frac{5}{3}\right)\left(\frac{5}{3}-1\right)}{2!} x^2 - \frac{\left(\frac{5}{3}\right)\left(\frac{5}{3}-1\right)\left(\frac{5}{3}-2\right)}{3!} x^3 + \cdots \\ &= \boxed{1 - \frac{5}{3}x + \frac{5}{9}x^2 + \frac{5}{81}x^3 + \cdots .} \end{aligned}$$

Since $m > 0$ but is not an integer, by the conditions of the theorem on p.721, the sereis converges on the closed interval $\boxed{[-1, 1].}$

41. Since $(1+x)^m = \sum\limits_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $f(x) = \frac{1}{(1+x^2)^{1/2}} = (1+x^2)^{-1/2}$ is given by

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} x^{2k} \\ &= \binom{-\frac{1}{2}}{0} x^0 + \binom{-\frac{1}{2}}{1} x^2 + \binom{-\frac{1}{2}}{2} x^4 + \binom{-\frac{1}{2}}{3} x^6 + \cdots \\ &= 1 + \left(-\frac{1}{2}\right) x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} x^4 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} x^6 + \cdots \\ &= \boxed{1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \cdots .} \end{aligned}$$

Since $m = -\frac{1}{2}$ satisfies $-1 < m < 0$, by the conditions of the theorem on p.721, the series converges on the half open interval $\boxed{(-1, 1].}$

42. Since $(1+x)^m = \sum\limits_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $f(x) = \frac{1}{(1+x)^{3/4}} = (1+x)^{-3/4}$ is given by

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \binom{-\frac{3}{4}}{k} x^{k} \\ &= \binom{-\frac{3}{4}}{0} x^0 + \binom{-\frac{3}{4}}{1} x^1 + \binom{-\frac{3}{4}}{2} x^2 + \binom{-\frac{3}{4}}{3} x^3 + \cdots \\ &= 1 - \frac{3}{4}x + \frac{\left(-\frac{3}{4}\right)\left(-\frac{3}{4}-1\right)}{2!} x^2 + \frac{\left(-\frac{3}{4}\right)\left(-\frac{3}{4}-1\right)\left(-\frac{3}{4}-2\right)}{3!} x^3 + \cdots \\ &= \boxed{1 - \frac{3}{4}x + \frac{21}{32}x^2 - \frac{77}{128}x^3 + \cdots .} \end{aligned}$$

Since $m = -\frac{3}{4}$ satisfies $-1 < m < 0$, by the conditions of the theorem on p.721, the series converges on the half open interval $\boxed{(-1, 1].}$

43. $f(x) = \frac{2x}{\sqrt{1-x}}$. From Problem 38, use the binomial series for

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots .$$

So the expansion for $f(x)$ is

$$\begin{aligned} f(x) &= \frac{2x}{\sqrt{1-x}} \\ &= 2x\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots\right) \\ &= \boxed{2x + x^2 + \frac{3}{4}x^3 + \frac{5}{8}x^4 + \cdots .} \end{aligned}$$

Since $m = -\frac{1}{2}$ satisfies $-1 < m < 0$, by the conditions of the theorem on p.721, the binomial series $\frac{1}{\sqrt{x-1}}$ converges on the half open interval $(-1, 1]$. Since multiplying the binomial series $\frac{1}{\sqrt{x-1}}$ by $2x$ to obtain $f(x)$ will not change the interval of convergence, we conclude that the series for $f(x)$ converges in the interval $\boxed{(-1, 1].}$

44. Write $f(x) = \frac{x}{1+x^3} = x(1 + x^3)^{-1}$. Since $(1 + x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$, the binomial expansion of $(1 + x^3)^{-1}$ is given by

$$\begin{aligned}(1 + x^3)^{-1} &= \sum_{k=0}^{\infty} \binom{-1}{k} x^{3k}\\ &= \binom{-1}{0} x^0 + \binom{-1}{1} x^3 + \binom{-1}{2} x^6 + \binom{-1}{3} x^9 + \cdots\\ &= 1 - x^3 + \frac{(-1)(-1-1)}{2!} x^6 + \frac{(-1)(-1-1)(-1-2)}{3!} x^9 + \cdots\\ &= 1 - x^3 + \frac{2}{2!} x^6 - \frac{2 \cdot 3}{3!} x^9 + \cdots\\ &= 1 - x^3 + x^6 - x^9 + \cdots .\end{aligned}$$

So the expansion of $f(x)$ is

$$\begin{aligned}f(x) &= \frac{x}{1+x^3}\\ &= x(1 - x^3 + x^6 - x^9 + \cdots)\\ &= \boxed{x - x^4 + x^7 - x^{10} + \cdots .}\end{aligned}$$

Since $m = -1$ satisfies $m \leq -1$, by the conditions of the theorem on p.620, the binomial series $\frac{1}{(1+x^3)}$ converges on the open interval $(-1, 1)$. Since multiplying the binomial series $\frac{1}{(1+x^3)}$ by x to obtain $f(x)$ will not change the interval of convergence, we conclude that the series for $f(x)$ converges in the interval $\boxed{(-1, 1).}$

Applications and Extensions

45. The expansion for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Substitute x^2 for x to determine the Maclaurin expansion for e^{x^2}:

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \ldots$$

Then

$$\begin{aligned}\int e^{x^2} dx &= \int \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \ldots\right) dx\\ &= \boxed{x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \frac{x^9}{216} + \ldots}\end{aligned}$$

46. The expansion for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!}$.

Substitute $\sqrt{x} = x^{1/2}$ for x to determine the Maclaurin expansion for $e^{\sqrt{x}}$:

$$e^{\sqrt{x}} = 1 + x^{1/2} + \frac{x}{2!} + \frac{x^{3/2}}{3!} + \frac{x^2}{4!} + \frac{x^{5/2}}{5!} + \ldots$$

Then

$$\begin{aligned}\int e^{\sqrt{x}}\,dx &= \int \left(1 + x^{1/2} + \frac{x}{2!} + \frac{x^{3/2}}{3!} + \frac{x^2}{4!} + \frac{x^{5/2}}{5!} + \ldots\right)dx \\ &= \boxed{x + \frac{2x^{3/2}}{3} + \frac{x^2}{4} + \frac{x^{5/2}}{15} + \frac{x^3}{72} + \ldots}\end{aligned}$$

Notice that the result is *not* actually a Maclaurin series, because it involves fractional powers of x.

47. The Maclaurin expansion for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

To determine the expansion for $\sin x^2$, substitute x^2 for x:

$$\begin{aligned}\sin x^2 &= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots \\ \int \sin x^2 dx &= \int \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots\right)dx \\ &= \boxed{\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} - \frac{x^{15}}{75,600} + \ldots}\end{aligned}$$

48. The Maclaurin expansion for

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

To determine $\cos x^2$, substitute x^2 for x:

$$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} + \ldots$$

Then

$$\begin{aligned}\int \cos x^2\,dx &= \int \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} + \ldots\right)dx \\ &= \boxed{\frac{x^2}{2} - \frac{x^5}{10} + \frac{x^9}{216} - \frac{x^{13}}{9360} + \frac{x^{17}}{685,440} + \ldots}\end{aligned}$$

49. We write $f(x) = \sin^2 x = \frac{1-\cos(2x)}{2} = \frac{1}{2} - \frac{1}{2}\cos(2x)$. From p.718 of the book, we see that the Maclaurin series for $\cos x$ is given by $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$. So The Maclaurin series for $f(x)$ is given by

$$\begin{aligned}
f(x) = \sin^2 x &= \frac{1}{2} - \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} \\
&= \frac{1}{2} - \frac{1}{2}\left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots\right] \\
&= \frac{1}{2} - \frac{1}{2} + \frac{2x^2}{2} - \frac{2^3x^4}{24} + \frac{2^5x^6}{720} - \frac{2^7x^8}{40320} + \cdots + (-1)^{n+1}\frac{2^{2n-1}x^{2n}}{(2n)!} + \cdots \\
&= \boxed{x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 - \frac{1}{315}x^8 + \cdots .}
\end{aligned}$$

50. We write $f(x) = \cos^2 x = \frac{1+\cos(2x)}{2} = \frac{1}{2} + \frac{1}{2}\cos(2x)$. From p.718 of the book, we see that the Maclaurin series for $\cos x$ is given by $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$. So The Maclaurin series for $f(x)$ is given by

$$\begin{aligned}
f(x) = \cos^2 x &= \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k)!} \\
&= \frac{1}{2} + \frac{1}{2}\left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots\right] \\
&= \frac{1}{2} + \frac{1}{2} - \frac{2x^2}{2} + \frac{2^3x^4}{24} - \frac{2^5x^6}{720} + \frac{2^7x^8}{40320} - \cdots + (-1)^n\frac{2^{2n-1}x^{2n}}{(2n)!} + \cdots \\
&= \boxed{1 - x^2 + \frac{x^4}{3} - \frac{2}{45}x^6 + \frac{1}{315}x^8 - \cdots .}
\end{aligned}$$

<u>Check:</u> Since $\cos^2 x = 1 - \sin^2 x$, we could also get this series directly from Problem 49:

$$\begin{aligned}
\cos^2 x &= 1 - \sin^2 x \\
&= 1 - \left(x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 - \frac{1}{315}x^8 + \cdots\right) \\
&= 1 - x^2 + \frac{x^4}{3} - \frac{2}{45}x^6 + \frac{1}{315}x^8 - \cdots .
\end{aligned}$$

51. The Maclaurin expansion for $\sin x$, from p.717 of the book, is $\sin x = \sum\limits_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$. Integrating both sides of this, we have:

$$\begin{aligned}
\int_0^x \sin t\,dt &= \int_0^x \left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + (-1)^n \frac{t^{2n+1}}{(2n+1)!} + \cdots\right] dt \\
[\cos t]\Big|_0^x &= \left[\frac{t^2}{2} - \frac{t^4}{4\cdot 3!} + \frac{t^6}{6\cdot 5!} - \frac{t^8}{8\cdot 7!} + \cdots + (-1)^n \frac{t^{2n+2}}{(2n+2)\cdot(2n+1)!} + \cdots\right]_0^x \\
(\cos x - 1) &= \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots + (-1)^n \frac{x^{2n+2}}{(2n+2)!} + \cdots \\
\cos x &= \boxed{1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots + (-1)^n \frac{x^{2n+2}}{(2n+2)!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.}
\end{aligned}$$

52. We write $f(x) = \ln \frac{1}{1-x} = -\ln(1-x)$. To find the Maclaurin series, we begin by evaluating $f(x)$ and its derivatives at $x = 0$.

$$\begin{aligned}
f(x) &= -\ln(1-x) & f(0) &= -\ln(1-0) = -\ln 1 = 0 \\
f'(x) &= -\frac{1}{1-x}\cdot(-1) = (1-x)^{-1} = \frac{1}{1-x} & f'(0) &= \frac{1}{1-0} = 1 \\
f''(x) &= (-1)(1-x)^{-2}\cdot(-1) = (1-x)^{-2} & f''(0) &= (1-0)^{-2} = 1! \\
f'''(x) &= (-2)(1-x)^{-3}\cdot(-1) = (2)(1-x)^{-3} & f'''(0) &= 2(1-0)^{-3} = 2! \\
f^{(4)}(x) &= (-3)(2)(1-x)^{-4}\cdot(-1) = (3)(2)(1-x)^{-4} & f^{(4)}(0) &= 3\cdot 2(1-0)^{-4} = 3! \\
&\vdots & &\vdots \\
f^{(n)}(x) &= (2)(3)\cdots(-(n-1))(1-x)^{-n}\cdot(-1) = (n-1)!(1-x)^{-n} & f^{(n)}(0) &= (n-1)!(1-0)^{-n} = (n-1)! \\
&\vdots & &\vdots
\end{aligned}$$

So the Maclaurin series of $f(x)$ is

$$\begin{aligned}
f(x) &= \ln \frac{1}{1-x} \\
&= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\
&= 0 + \frac{1}{1!}x^1 + \frac{1!}{2!}x^2 + \frac{2!}{3!}x^3 + \frac{3!}{4!}x^4 + \cdots + \frac{(n-1)!}{n!}x^n + \cdots \\
&= \boxed{x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + \frac{x^n}{n} + \cdots.}
\end{aligned}$$

Comparing the Maclaurin series with the power series found in Section 8.8, Example 8, p.707, we see they are identical.

53. To find the Maclaurin series for $f(x) = \sec x$, we begin by evluating $f(x)$ and its derivatives at $x = 0$. To simplify notation, let $y = f(x) = \sec x$. Let $y_0 = f(0)$, $y' = f'(x)$,

$y_0' = f'(0)$, etc. Also note that $\frac{d}{dx} \tan x = \sec^2 x = y^2$. (We'll use this directly for the seventh and eighth derivatives to save writing.) We compute until we get five nonzero derivatives:

$$y = \sec x \qquad\qquad y_0 = \sec 0 = \boxed{1.}$$

$$\begin{aligned} y' &= \sec x \tan x \\ &= y \tan x \end{aligned}$$

$$\begin{aligned} y_0' &= y_0 \tan 0 \\ &= (1)(0) \\ &= \boxed{0.} \end{aligned}$$

$$\begin{aligned} y'' &= y' \tan x + y \sec^2 x \\ &= y' \tan x + yy^2 \\ &= y' \tan x + y^3 \end{aligned}$$

$$\begin{aligned} y_0'' &= y_0' \tan 0 + y_0^3 \\ &= (0)(0) + (1)^3 \\ &= \boxed{1.} \end{aligned}$$

$$\begin{aligned} y''' &= y'' \tan x + y' \sec^2 x + 3y^2 y' \\ &= y'' \tan x + y' y^2 + 3y^2 y' \\ &= y'' \tan x + 4y^2 y' \end{aligned}$$

$$\begin{aligned} y_0''' &= y_0'' \tan 0 + 4y_0^2 y_0' \\ &= (1)(0) + 4(1)^2(0) \\ &= \boxed{0.} \end{aligned}$$

$$\begin{aligned} y^{(4)} &= y''' \tan x + y'' \sec^2 x + 8yy'^2 + 4y^2 y'' \\ &= y''' \tan x + y'' y^2 + 8yy'^2 + 4y^2 y'' \\ &= y''' \tan x + 5y^2 y'' + 8yy'^2 \end{aligned}$$

$$\begin{aligned} y_0^{(4)} &= y_0''' \tan 0 + 5y_0^2 y_0'' + 8y_0 y_0'^2 \\ &= (0)(0) + 5(1)^2(1) + 8(1)(0) \\ &= \boxed{5.} \end{aligned}$$

$$\begin{aligned} y^{(5)} &= y^{(4)} \tan x + y''' \sec^2 x + 10yy'y'' + 5y^2 y''' + 8y'^3 + 16yy'y'' \\ &= y^{(4)} \tan x + y''' y^2 + 10yy'y'' + 5y^2 y''' + 8y'^3 + 16yy'y'' \\ &= y^{(4)} \tan x + 6y^2 y''' + 26yy'y'' + 8y'^3 \end{aligned}$$

$$\begin{aligned} y_0^{(5)} &= y_0^{(4)} \tan 0 + 6y_0^2 y_0''' + 26y_0 y_0' y_0'' + 8y_0'^3 \\ &= (5)(0) + 6(1)^2(0) + 26(1)(0)(1) + 8(0)^3 \\ &= \boxed{0.} \end{aligned}$$

$$\begin{aligned} y^{(6)} &= y^{(5)} \tan x + y^{(4)} \sec^2 x + 12yy'y''' + 6y^2 y^{(4)} \\ &\quad + 26y'^2 y'' + 26yy''^2 + 26yy'y''' + 24y'^2 y'' \\ &= y^{(5)} \tan x + y^{(4)} y^2 + 12yy'y''' + 6y^2 y^{(4)} \\ &\quad + 26y'^2 y'' + 26yy''^2 + 26yy'y''' + 24y'^2 y'' \\ &= y^{(5)} \tan x + 7y^{(4)} y^2 + 38yy'y''' + \\ &\quad + 50y'^2 y'' + 26yy''^2 \end{aligned}$$

$$\begin{aligned} y_0^{(6)} &= y_0^{(5)} \tan 0 + 7y_0^{(4)} y_0^2 + 38y_0 y_0' y_0''' + \\ &\quad + 50y_0'^2 y_0'' + 26y_0 y_0''^2 \\ &= (0)(0) + 7(5)(1)^2 + 38(1)(0)(0) + \\ &\quad + 50(0)^2(1) + 26(1)(1)^2 \\ &= 35 + 26 \\ &= \boxed{61.} \end{aligned}$$

$$\begin{aligned} y^{(7)} &= y^{(6)}\tan x + y^{(5)}y^2 + 7y^{(5)}y^2 + 14y^{(4)}yy' + \\ &\quad + 38y'^2y''' + 38yy''y''' + 38yy'y^{(4)} + \\ &\quad + 100y'y''^2 + 50y'^2y''' + 26y'y''^2 + 52yy''y''' \\ &= y^{(6)}\tan x + 8y^{(5)}y^2 + 52y^{(4)}yy' + \\ &\quad + 88y'''y'^2 + 90yy''y''' + 126y'y''^2 \end{aligned}$$

$$\begin{aligned} y_0^{(7)} &= y_0^{(6)}\tan 0 + 8y_0^{(5)}y_0^2 + 52y_0^{(4)}y_0y_0' + \\ &\quad + 88y_0'''y_0'^2 + 90y_0y_0''y_0''' + 126y_0'y_0''^2 \\ &= (61)(0) + 8(0)(1)^2 + 52(5)(1)(0) + \\ &\quad + 88(0)(0)^2 + 90(1)(1)(0) + 126(0)(1)^2 \\ &= \boxed{0.} \end{aligned}$$

$$\begin{aligned} y^{(8)} &= y^{(7)}\tan x + y^{(6)}y^2 + 8y^{(6)}y^2 + 16y^{(5)}yy' + \\ &\quad + 52y^{(5)}yy' + 52y^{(4)}y'^2 + 52y^{(4)}yy'' + \\ &\quad + 88y^{(4)}y'^2 + 176y'''y'y'' + 90y'y''y''' + \\ &\quad + 90yy'''^2 + 90yy''y^{(4)} + 126y''^3 + 252y'y''y''' \\ &= y^{(7)}\tan x + 9y^{(6)}y^2 + 68y^{(5)}yy' + \\ &\quad + 140y^{(4)}y'^2 + 142y^{(4)}yy'' + 518y'''y''y' + \\ &\quad + 90yy'''^2 + 126y''^3 \end{aligned}$$

$$\begin{aligned} y_0^{(8)} &= y_0^{(7)}\tan 0 + 9y_0^{(6)}y_0^2 + 68y_0^{(5)}y_0y_0' + \\ &\quad + 140y_0^{(4)}y_0'^2 + 142y_0^{(4)}y_0y_0'' + 518y_0'''y_0''y_0' + \\ &\quad + 90y_0y_0'''^2 + 126y_0''^3 \\ &= (0)(0) + 9(61)(1) + 68(0)(1)(0) + \\ &\quad + 140(5)(0)^2 + 142(5)(1)(1) + 518(0)(1)(0) + \\ &\quad + 90(1)(0)^2 + 126(1)^3 \\ &= 9\cdot 61 + 142\cdot 5 + 126\cdot 1 \\ &= 549 + 710 + 126 \\ &= \boxed{1385.} \end{aligned}$$

So the first five nonzero terms of the Maclaurin series are

$$\begin{aligned} f(x) &= \sec x \\ &= \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k \\ &= 1 + \frac{0}{1!}x + \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{5}{4!}x^4 + \frac{0}{5!}x^5 + \frac{61}{6!}x^6 + \frac{0}{7!}x^7 + \frac{1385}{8!}x^8 + \cdots \\ &= \boxed{1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots .} \end{aligned}$$

54. (a) The Maclaurin series for $g(x) = e^x$ (see Problem 27) is

$$g(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

So the Maclaurin expansion of $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is given by

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} e^{-x^2/2} &= \frac{1}{\sqrt{2\pi}} g\left(-\frac{x^2}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \left[1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2!}\left(-\frac{x^2}{2}\right)^2 + \frac{1}{3!}\left(-\frac{x^2}{2}\right)^3 + \cdots + \frac{1}{n!}\left(-\frac{x^2}{2}\right)^n + \cdots\right] \\
&= \frac{1}{\sqrt{2\pi}} \left[1 - \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n}}{2^n \cdot n!} + \cdots\right] \\
&= \boxed{\frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^k \cdot k!}.}
\end{aligned}$$

(b) Integrating the series found in part **(a)** term by term, we obtain

$$\begin{aligned}
P(a \le z \le b) &= \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_a^b \left[1 - \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n}}{2^n \cdot n!} + \cdots\right] dx \\
&= \frac{1}{\sqrt{2\pi}} \left[x - \frac{x^3}{3 \cdot 2} + \frac{x^5}{5 \cdot 2^2 \cdot 2!} - \frac{x^7}{7 \cdot 2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1) \cdot 2^n \cdot n!} + \cdots\right]_a^b \\
&= \boxed{\frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{[b^{2k+1} - a^{2k+1}]}{(2k+1) 2^k\, k!}.}
\end{aligned}$$

(c) Using the first four terms of the series found in part **(b)**, we get the desired approximation

$$\begin{aligned}
P(-0.5 \le z \le 0.3) &\approx \frac{1}{\sqrt{2\pi}} \left[x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336}\right]_{-0.5}^{0.3} \\
&= \frac{1}{\sqrt{2\pi}} \left[\left\{(0.3) - \frac{(0.3)^3}{6} + \frac{(0.3)^5}{40} - \frac{(0.3)^7}{336}\right\} - \left\{(-0.5) - \frac{(-0.5)^3}{6} + \frac{(-0.5)^5}{40} - \frac{(-0.5)^7}{336}\right\}\right] \\
&\approx \frac{1}{2.507} [\{0.3 - 0.0045 + 0.000061 - 0.0000006\} - \{-0.5 + 0.02083 - 0.00078 + 0.0000233\}] \\
&\approx \frac{1}{2.507} [0.7755] \\
&\approx \boxed{0.309.}
\end{aligned}$$

(d) Using a CAS we find the exact value to be approximately $P(-0.5 \le z \le 0.3) \approx \boxed{0.309.}$

55. If f is an even function, then $f(x) = f(-x)$. Taking derivatives of both sides with respect to x, $f'(x) = f'(-x) \cdot (-1) = -f(-x)$, using the Chain Rule. So the first derivative is an odd function of x. At $x = 0$, $f'(0) = -f'(0)$, which means $f'(0) = 0$.
If $g(x)$ is an odd function of x, then $g(x) = -g(-x)$. Taking the derivatives of both sides with respect to x, $g'(x) = -g'(-x) \cdot (-1) = g(-x)$, using the Chain Rule. So the derivative of an odd function is an even function. This means the second derivative of $f(x)$ will be an even function, and possibly evaluate to a nonzero value at $x = 0$. The third derivative will now be odd, and so forth, leading to $f^{(k)}(0) = 0$ for all odd k. This means the Maclaurin series $f(x) = \sum\limits_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ will only have even powers of x, because they are multiplied by (possibly nonzero) even order derivatives evaluated at $x = 0$.

56. If f is an odd function, then $f(x) = -f(-x)$. (Note that this rules out the x^0 term of the Maclaurin series for $f(x)$.) Taking derivatives of both sides with respect to x, $f'(x) = -f'(-x) \cdot (-1) = f(-x)$, using the Chain Rule. So the first derivative is an even function of x. At $x = 0$, the derivative possibly evaluates to a nonzero value.
If $g(x)$ is an even function of x, then $g(x) = g(-x)$. Taking the derivatives of both sides with respect to x, $g'(x) = g'(-x) \cdot (-1) = -g(-x)$, using the Chain Rule. Since $g(0) = -g(0)$, we have that $g(0) = 0$. So the derivative of an even function is an odd function. This means the second derivative of $f(x)$ will be an odd function, and it evaluates to zero at $x = 0$. The third derivative will now be even, possibly evaluating to a nonzero value, and so forth, leading to $f^{(k)}(0) = 0$ for all even k. This means the Maclaurin series $f(x) = \sum\limits_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ will only have odd powers of x, because they are multiplied by (possibly nonzero) odd order derivatives evaluated at $x = 0$.

57. The remainder term is given by

$$R_n(x) = \frac{f^{n+1}(u)}{(n+1)!}(x-c)^{n+1},$$

where u belongs to the interval (x, c). Since the function $f(x) = (1+x)^m$, we have

$$\begin{aligned}
f'(x) &= m(1+x)^{m-1} = 1! \cdot \binom{m}{1}(1+x)^{m-1} \\
f''(x) &= m(m-1)(1+x)^{m-2} = 2! \cdot \binom{m}{2}(1+x)^{m-2} \\
&\vdots \\
f^{(n+1)}(x) &= m(m-1)\cdots(m-n)(1+x)^{m-(n+1)} = (n+1)! \cdot \binom{m}{n+1}(1+x)^{m-(n+1)}
\end{aligned}$$

The remainder term becomes

$$\begin{aligned}
R_n(x) &= \frac{f^{n+1}(u)}{(n+1)!}(x-c)^{n+1} \\
&= \binom{m}{n+1}(1+u)^{m-(n+1)}(x-c)^{n+1} \\
&= \binom{m}{n+1}(1+u)^m \left(\frac{x-c}{1+u}\right)^{n+1}.
\end{aligned}$$

Now, if m is a nonnegative integer,

$$\binom{m}{n+1} = 0 \text{ for } n+1 > m.$$

So we have $R_n(x) = 0$ if $n > m - 1$. So, as $n \to \infty$, $R_n(x) \to 0$, as required. So we have shown that $(1+x)^m = \sum\limits_{k=0}^{\infty} \binom{m}{k} x^k$ when m is a nonnegative integer.

58. For $m < 0$ the series $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \binom{m}{k} x^k$ is a series of nonzero terms if $x \neq 0$. (If $x = 0$, the series converges to 1.) Applying the Ratio Test,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\binom{m}{n+1}x^{n+1}}{\binom{m}{n}x^n}\right| = |x|\lim_{n\to\infty}\left|\frac{\frac{m(m-1)(m-2)\cdots(m-n-1+1)}{(n+1)!}}{\frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}}\right| \\
&= |x|\lim_{n\to\infty}\left[\frac{n!}{(n+1)!}\cdot\left|\frac{m(m-1)(m-2)\cdots(m-n)}{m(m-1)(m-2)\cdots(m-n+1)}\right|\right] \\
&= |x|\lim_{n\to\infty}\left[\frac{n!}{(n+1)n!}\cdot|m-n|\right] \\
&= |x|\lim_{n\to\infty}\left|\frac{m-n}{n+1}\right| = |x|\lim_{n\to\infty}\left|\frac{\frac{m}{n}-1}{1+\frac{1}{n}}\right| \\
&= |x| < 1
\end{aligned}$$

for absolute convergence of the series, and $|x| > 1$ for absolute divergence of the series.

59. From p.721, Example 11 of the textbook, the Maclaurin series for $f(x) = \sin^{-1} x$ can be written as

$$f(x) = \sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \cdots + \frac{1\cdot 3\cdot 5\cdot\cdots(2n-1)}{2\cdot 4\cdot 6\cdot\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1} + \cdots.$$

The fractional factor multiplying the nth term can be expressed as follows:

$$\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots(2n)} = \frac{1\cdot 2\cdot 3\cdot\cdots(2n-1)\cdot(2n)}{(2\cdot 4\cdot 6\cdot\cdots(2n))^2} = \frac{(2n)!}{(2^n n!)^2} = \frac{(2n)!}{4^n (n!)^2}.$$

So the series can be written as

$$\sin^{-1} x = \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2}\frac{x^{2k+1}}{(2k+1)}.$$

Since the terms of the series are nonzero for $x \neq 0$, by the Ratio Test,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(2(n+1))!x^{2(n+1)+1}}{4^{n+1}((n+1)!)^2(2(n+1)+1)}}{\frac{(2n)!x^{2n+1}}{4^n(n!)^2(2n+1)}}\right| \\
&= |x^2|\lim_{n\to\infty}\left[\frac{(2n+2)!}{(2n)!}\cdot\frac{4^n}{4^{n+1}}\cdot\frac{2n+1}{2n+3}\cdot\frac{(n!)^2}{((n+1)!)^2}\right] \\
&= \frac{|x|^2}{4}\lim_{n\to\infty}\left[\frac{(2n+2)(2n+1)(2n)!}{(2n)!}\cdot\frac{2+\frac{1}{n}}{2+\frac{3}{n}}\cdot\frac{(n!)^2}{(n+1)^2(n!)^2}\right] \\
&= \frac{|x|^2}{4}\lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2}\cdot\lim_{n\to\infty}\frac{2+\frac{1}{n}}{2+\frac{3}{n}} \\
&= \frac{|x|^2}{4}\cdot\lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^2}\cdot\left(\frac{2+0}{2+0}\right) \\
&= \frac{|x|^2}{4}\cdot\frac{(2+0)(2+0)}{(1+0)^2}\cdot 1 \\
&= \frac{|x|^2}{4}\cdot 4 = |x|^2 < 1
\end{aligned}$$

for convergence of the series. So the series for $\sin^{-1} x$ converges for $-1 < x < 1$. For $x = 1$, the series becomes

$$\begin{aligned}\lim_{x\to 1}\sin^{-1} x &= 1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{7} + \cdots \\ &< 1 + \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \cdots \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \\ &= \frac{1}{1-\frac{1}{2}} = 2,\end{aligned}$$

using the formula for the sum of a geometric series. That is to say, the sum of the series for $\sin^{-1} 1$ is bounded from above by 2; also, since the terms of the series are all positive, the sequence of partial sums is increasing. Since the sequence of partial sums is increasing and bounded from above, it converges, so the series converges as well. (The actual sum is $\sin^{-1} 1 = \frac{\pi}{2} < 2$.)
At $x = -1$, the series becomes an alternating version of the series at $x = 1$, namely

$$\sin^{-1}(-1) = 1 - \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{7} + \cdots .$$

The series of absolute values of this alternating series is precisely the series at $x = 1$, which we have shown converges. Since the alternating series converges absolutely, it converges.
So we have shown that the interval of convergence for the Maclaurin expansion of $f(x) = \sin^{-1} x$ is $[-1, 1]$.

60. (a)

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty}(-x)^k = \sum_{k=0}^{\infty}(-1)^k x^k,$$

using the sum of a geometric series of the form $\sum\limits_{k=0}^{\infty} r^k$, with convergent sum $\frac{1}{1-r}$ whenever $|r| = |-x| = |x| < 1$.
(b) Setting $x = 1$ in the above formula, we obtain

$$\frac{1}{1+1} = \frac{1}{2} = \sum_{k=0}^{\infty}(-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + \cdots ,$$

which is the erroneous formula arrived at by Euler.
(c) One of the conditions in writing the series representation of $\frac{1}{1+x}$ was that $|x| < 1$. So we cannot put $x = 1$ in the series representation, as it is divergent there.

Challenge Problems

61. The series given to us can be written as

$$\frac{x^3}{1(3)} - \frac{x^5}{3(5)} + \frac{x^7}{5(7)} - \frac{x^9}{7(9)} - \cdots = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k+1}}{(2k-1)(2k+1)} = \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k+1}x^{2k+1}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right),$$

using partial fractions. At $x = 1$, the series becomes

$$\frac{1}{1(3)} - \frac{1}{3(5)} + \frac{1}{5(7)} - \frac{1}{7(9)} + \cdots = \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k}\left(\frac{1}{2k+1} - \frac{1}{2k-1}\right).$$

Before proceeding further, let us prove that the series at $x = 1$ converges absolutely, which would allow us to rearrange the terms at will. Since the series of absolute values of the series at $x = 1$ satisfies

$$\begin{aligned}\frac{1}{1(3)} + \frac{1}{3(5)} + \frac{1}{5(7)} + \frac{1}{7(9)} + \cdots &< \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \\ &< \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots \\ &= \sum_{k=1}^{\infty} \frac{1}{k^2},\end{aligned}$$

and the series $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series (since $p = 2 > 1$), by the Comparison Test for convergence, the series of absolute values at $x = 1$ converges, establishing absolute convergence for the original series at $x = 1$.

Now, Gregory's series for $\tan^{-1} x$ (see p.708, Section 8.8) is given by

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}.$$

At $x = 1$, we have

$$\tan^{-1} 1 = \frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1}.$$

So,

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4} - 1.$$

Consider now the sum

$$-\sum_{k=1}^{\infty} (-1)^k \frac{1}{2k-1} = -\sum_{k'=0}^{\infty} (-1)^{k'+1} \frac{1}{2k'+1} = \sum_{k'=0}^{\infty} (-1)^{k'} \frac{1}{2k'+1} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} = \frac{\pi}{4}.$$

(Here, the index change $k = k' + 1$ was made, and then the index was changed back to k in the final step as it is a dummy summation index.) So, the sum of the original series is

$$\frac{1}{2}\left[\sum_{k=1}^{\infty} (-1)^k \frac{1}{2k+1} - \sum_{k=1}^{\infty} (-1)^k \frac{1}{2k-1}\right] = \frac{1}{2}\left[\frac{\pi}{4} - 1 + \frac{\pi}{4}\right] = \boxed{\frac{\pi}{4} - \frac{1}{2}.}$$

62. The Maclaurin expansion for $\ln(1 - x)$ is

$$\ln(1 - x) = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = -\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

So $\ln \frac{1}{1-x} = -\ln(1 - x)$ has a Maclaurin expansion

$$\ln \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{x^k}{k} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots.$$

Integrating term by term gives us

$$\begin{aligned}
\int_0^x \ln\frac{1}{1-t}\,dt &= \int_0^x \left(t+\frac{t^2}{2}+\frac{t^3}{3}+\cdots+\frac{t^n}{n}+\cdots\right)dt\\
-\int_0^x \ln(1-t)\,dt &= \left[\frac{t^2}{1\cdot 2}+\frac{t^3}{2\cdot 3}+\frac{t^4}{3\cdot 4}+\cdots+\frac{t^n}{n(n+1)}+\cdots\right]_0^x\\
\int_1^{1-x} \ln y\,dy &= \frac{x^2}{1\cdot 2}+\frac{x^3}{2\cdot 3}+\frac{x^4}{3\cdot 4}+\cdots+\frac{x^n}{n(n+1)}+\cdots\\
[y\ln y-y]\Big|_1^{1-x} &= \sum_{k=1}^{\infty}\frac{x^{k+1}}{k(k+1)}\\
\{(1-x)\ln(1-x)-(1-x)\}-\{1\ln 1-1\} &= \sum_{k=1}^{\infty}\frac{x^{k+1}}{k(k+1)}\\
\text{or,}\qquad \sum_{k=1}^{\infty}\frac{x^{k+1}}{k(k+1)} &= \boxed{(1-x)\ln(1-x)+x}
\end{aligned}$$

where the substitution $y=1-x$ was made, and the result $\int \ln y\,dy = x\ln y - y + C$ used. (This can be verified by differentiating the result or directly checked using integration by parts. (Use $u=\ln x$ and $dv=dx$.))

63. The partial sums of the series $\sum\limits_{k=1}^{\infty}\frac{k}{(k+1)!}$ are

$$\begin{aligned}
S_1 &= \frac{1}{2!}=\frac{1}{2}=\frac{2-1}{2}=\frac{2!-1}{2!}\\
S_2 &= \frac{1}{2}+\frac{2}{3!}=\frac{1}{2}+\frac{2}{6}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}=\frac{6-1}{6}=\frac{3!-1}{3!}\\
S_3 &= \frac{5}{6}+\frac{3}{4!}=\frac{5}{6}+\frac{3}{24}=\frac{20}{24}+\frac{3}{24}=\frac{23}{24}=\frac{24-1}{24}=\frac{4!-1}{4!}\\
S_4 &= \frac{23}{24}+\frac{4}{5!}=\frac{23}{24}+\frac{4}{120}=\frac{115}{120}+\frac{4}{120}=\frac{119}{120}=\frac{120-1}{120}=\frac{5!-1}{5!},\cdots.
\end{aligned}$$

In general, the nth partial sum is

$$S_n=\frac{(n+1)!-1}{(n+1)!}.$$

The sum of the series is the limit of the nth partial sum:

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{(n+1)!-1}{(n+1)!}=\lim_{n\to\infty}\left(1-\frac{1}{(n+1)!}\right)=\boxed{1.}$$

64. (a) To show $n!\ge 2^{n-1}$, we write

$$n!=n(n-1)(n-2)\cdots 4\cdot 3\cdot 2\cdot 1=n(n-1)(n-2)\cdots 4\cdot 3\cdot 2\ge 2\cdot 2\cdot 2\cdots 2\cdot 2\cdot 2=2^{n-1}.$$

(b) From (a), we have

$$\frac{1}{n!}\le\frac{1}{2^{n-1}}.$$

So

$$\begin{aligned}
0<s_n=\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} &\le \frac{1}{2^{1-1}}+\frac{1}{2^{2-1}}+\frac{1}{2^{3-1}}+\cdots+\frac{1}{2^{n-1}}\\
&=1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots+\left(\frac{1}{2}\right)^{n-1}.
\end{aligned}$$

(c) Since from the definition of s_n we have

$$s_{n+1} = s_n + \frac{1}{(n+1)!},$$

we can write $0 < s_n < s_{n+1}$. Now, following the inequality used in part (b), we have

$$\begin{aligned} s_{n+1} &= \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n+1)!} \\ &\le 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^n \\ \text{so that} \quad \frac{1}{2}s_{n+1} &\le \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{n+1}. \\ \text{Now,} \quad s_{n+1} - \frac{1}{2}s_{n+1} = \frac{1}{2}s_{n+1} &\le 1 - \left(\frac{1}{2}\right)^{n+1} \\ \text{or,} \quad s_{n+1} &\le 2\left(1 - \left(\frac{1}{2}\right)^{n+1}\right). \end{aligned}$$

Since

$$\lim_{n\to\infty} s_{n+1} \le \lim_{n\to\infty} 2\left(1 - \left(\frac{1}{2}\right)^{n+1}\right) = 2,$$

we have $s_{n+1} < 2$ for any n, since s_n is an increasing function of n (due to the fact that $s_n < s_{n+1}$). So finally, we have

$$0 < s_n < s_{n+1} < 2.$$

(d) $t_n = \left[1 + \frac{1}{n}\right]^n$. Applying the binomial expansion to the function t_n we get

$$\begin{aligned} \left[1 + \frac{1}{n}\right]^n &= \binom{n}{0}\left(\frac{1}{n}\right)^0 + \binom{n}{1}\left(\frac{1}{n}\right)^1 + \binom{n}{2}\left(\frac{1}{n}\right)^2 + \binom{n}{3}\left(\frac{1}{n}\right)^3 + \cdots + \binom{n}{n}\left(\frac{1}{n}\right)^n \\ &= 1 + \frac{n}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3} + \cdots + \frac{n(n-1)(n-2)\cdots(n-(n-1))}{n!}\cdot\frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!}\left[1 - \frac{1}{n}\right] + \frac{1}{3!}\left[1 - \frac{1}{n}\right]\left[1 - \frac{2}{n}\right] + \cdots + \frac{1}{n!}\left[1 - \frac{1}{n}\right]\left[1 - \frac{2}{n}\right]\cdots\left[1 - \frac{n-1}{n}\right]. \end{aligned}$$

We have for large n,

$$t_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = s_n + 1.$$

So $t_n < s_n + 1$ for finite n.

(e) Let $t(x) = \left(1 + \frac{1}{x}\right)^x$. Then $\ln t = x \ln\left(1 + \frac{1}{x}\right)$. Differentiating, we get

$$\begin{aligned} \frac{1}{t}t' &= \ln\left(1 + \frac{1}{x}\right) + x \cdot \frac{1}{1 + \frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) \\ &= \ln\left(1 + \frac{1}{x}\right) - \frac{1}{1+x} > 0 \end{aligned}$$

for $x \ge 1$. To see this is the case, proceed as follows. We require

$$\begin{aligned}\ln\left(1+\frac{1}{x}\right) &> \frac{1}{1+x}\\ 1+\frac{1}{x} &> e^{\frac{1}{1+x}}\\ \text{or,}\quad e^{\frac{1}{1+x}} &< 1+\frac{1}{x}.\end{aligned}$$

The following computation proves this last inequality:

$$\begin{aligned}e^{\frac{1}{1+x}} &= 1+\frac{1}{1+x}+\frac{1}{2!}\frac{1}{(1+x)^2}+\frac{1}{3!}\frac{1}{(1+x)^3}+\cdots+\frac{1}{n!}\frac{1}{(1+x)^n}+\cdots\\ &< 1+\frac{1}{1+x}+\frac{1}{(1+x)^2}+\frac{1}{(1+x)^3}+\cdots+\frac{1}{(1+x)^n}+\cdots\\ &= \sum_{k=0}^{\infty}\left(\frac{1}{1+x}\right)^n.\end{aligned}$$

Now, the series $\sum\limits_{k=0}^{\infty} r^n = \sum\limits_{k=0}^{\infty}\left(\frac{1}{1+x}\right)^n$ is a geometric series which converges to the sum of $\frac{1}{1-r}$ since $|r| = \left|\frac{1}{1+x}\right| < 1$ for $x \geq 1$. So we have

$$\begin{aligned}e^{\frac{1}{1+x}} &< \sum_{k=0}^{\infty}\left(\frac{1}{1+x}\right)^n\\ &= \frac{1}{1-\frac{1}{1+x}}\\ &= \frac{x+1}{1+x-1}\\ &= \frac{x+1}{x}\\ &= 1+\frac{1}{x}\end{aligned}$$

that is,

$$e^{\frac{1}{1+x}} < 1+\frac{1}{x}$$

for $x \geq 1$ as was to be shown. So we have shown that the function $t(x)$ is an increasing function for $x \geq 1$, which means the function $t_n = t(n)$ is increasing for integers $n \geq 1$. This means

$$0 < t_n < t_{n+1} < s_{n+1} + 1 < 2 + 1 = 3.$$

Since $e = \lim\limits_{n\to\infty}\left[1+\frac{1}{n}\right]^n = \lim\limits_{n\to\infty} t_n$, we see that by the Squeeze theorem,

$$0 \leq \lim_{n\to\infty} t_n = \lim_{n\to\infty} t_{n+1} \leq 3$$

or $e = \lim\limits_{n\to\infty} t_n \leq 3$.

65. From the results of Problem 64, since t_n is an increasing function of n and $\lim\limits_{n\to\infty} t_n = e$, for any finite value of n, we must have $t_n < e$ for $n > 0$.

66. Let $x \geq 0$. We are given that $\sin t \leq t$. By integrating both sides repeatedly we have:

$$\int_0^x \sin t\,dt \leq \int_0^x t\,dt$$

$$[-\cos t]\Big|_0^x \leq \frac{x^2}{2}$$

$$-\cos x + 1 \leq \frac{x^2}{2}$$

or, $$1 - \frac{x^2}{2} \leq \cos x.$$

$$\int_0^x \left(1 - \frac{t^2}{2}\right) dt \leq \int_0^x \cos t\,dt$$

$$\left[t - \frac{t^3}{6}\right]_0^x \leq [\sin t]\Big|_0^x$$

$$x - \frac{x^3}{6} \leq \sin x.$$

$$\int_0^x \left(t - \frac{t^3}{6}\right) dt \leq \int_0^x \sin t\,dt$$

$$\left[\frac{t^2}{2} - \frac{t^4}{24}\right]_0^x \leq [-\cos t]\Big|_0^x = -\cos x + 1$$

or, $$\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$

So we conclude that

$$\boxed{1 - \frac{x^2}{2!} \leq \cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \text{ for all } x \geq 0.}$$

67. $f(x) = (1+x)^x$. Let $y = f(x)$, $y' = f'(x)$, etc, and let $y_0 = f(0)$, $y_0' = f'(0)$, etc. Also, since $\ln y = x\ln(1+x)$, we can differentiate both sides to get $\frac{y'}{y} = \ln(1+x) + \frac{x}{1+x}$, or $y' = y\left[\ln(1+x) + \frac{x}{1+x}\right]$. To find the Maclaurin expansion of $f(x)$ to four terms we begin by evaluating the function and its derivatives until the third nonzero derivative is reached.

$$y = (1+x)^x$$

$$y_0 = (1)^0 = \boxed{1.}$$

$$y' = y\left[\ln(1+x) + \frac{x}{1+x}\right]$$

$$y_0' = y_0\left[\ln 1 + \frac{0}{1+0}\right] = (1)[0+0] = \boxed{0.}$$

$$y'' = y'\left[\ln(1+x) + \frac{x}{1+x}\right] + y\left[\frac{1}{1+x} + \frac{(1+x)\cdot 1 - x}{(1+x)^2}\right]$$
$$= y'\left[\ln(1+x) + \frac{x}{1+x}\right] + y\left[\frac{1}{1+x} + \frac{1}{(1+x)^2}\right]$$

$$y_0'' = y_0'\left[\ln 1 + \frac{0}{1+0}\right] + y_0\left[\frac{1}{1+0} + \frac{1}{(1+0)^2}\right] = (0)[0+0] + (1)[1+1] = \boxed{2.}$$

$$y''' = y''\left[\ln(1+x) + \frac{x}{1+x}\right] + y'\left[\frac{1}{1+x} + \frac{1}{(1+x)^2}\right] + y'\left[\frac{1}{1+x} + \frac{1}{(1+x)^2}\right] + y\left[-\frac{1}{(1+x)^2} - \frac{2}{(1+x)^3}\right]$$
$$= y''\left[\ln(1+x) + \frac{x}{1+x}\right] + 2y'\left[\frac{1}{1+x} + \frac{1}{(1+x)^2}\right] + y\left[-\frac{1}{(1+x)^2} - \frac{2}{(1+x)^3}\right]$$

$$y_0''' = y_0''\left[\ln 1 + \frac{0}{1+0}\right] + 2y_0'\left[\frac{1}{1+0} + \frac{1}{(1+0)^2}\right] + y_0\left[-\frac{1}{(1+0)^2} - \frac{2}{(1+0)^3}\right]$$
$$= (2)[0+0] + 2(0)[1+1] + (1)[-1-2] = \boxed{-3.}$$

$$\begin{aligned} y^{(4)} &= y''' \left[\ln(1+x) + \frac{x}{1+x}\right] + y'' \left[\frac{1}{1+x} + \frac{1}{(1+x)^2}\right] \\ &\quad + 2y'' \left[\frac{1}{1+x} + \frac{1}{(1+x)^2}\right] + 2y' \left[-\frac{1}{(1+x)^2} - \frac{2}{(1+x)^3}\right] \\ &\quad + y' \left[-\frac{1}{(1+x)^2} - \frac{2}{(1+x)^3}\right] + y \left[\frac{2}{(1+x)^3} + \frac{6}{(1+x)^4}\right] \\ &= y''' \left[\ln(1+x) + \frac{x}{1+x}\right] + 3y'' \left[\frac{1}{1+x} + \frac{1}{(1+x)^2}\right] \\ &\quad + 3y' \left[-\frac{1}{(1+x)^2} - \frac{2}{(1+x)^3}\right] + y \left[\frac{2}{(1+x)^3} + \frac{6}{(1+x)^4}\right] \end{aligned}$$

$$\begin{aligned} y_0^{(4)} &= y_0''' \left[\ln 1 + \frac{0}{1+0}\right] + 3y_0'' \left[\frac{1}{1+0} + \frac{1}{(1+0)^2}\right] \\ &\quad + 3y_0' \left[-\frac{1}{(1+0)^2} - \frac{2}{(1+0)^3}\right] \\ &\quad + y_0 \left[\frac{2}{(1+0)^3} + \frac{6}{(1+0)^4}\right] \\ &= (-3)[0+0] + 3(2)[1+1] \\ &\quad + 3(0)[-1-2] + (1)[2+6] = \boxed{20.} \end{aligned}$$

The first four nonzero terms of the Maclaurin series are

$$\begin{aligned} f(x) &= (1+x)^x \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= 1 + \frac{0}{1!}x^1 + \frac{2}{2!}x^2 + \frac{-3}{3!}x^3 + \frac{20}{4!}x^4 + \cdots \\ &= 1 + 0 + \frac{2}{2}x^2 + \frac{-3}{6}x^3 + \frac{20}{24}x^4 + \cdots \\ &= \boxed{1 + x^2 - \frac{x^3}{2} + \frac{5x^4}{6} + \cdots.} \end{aligned}$$

68. Let us prove first the following result: For any positive integer n,

$$\lim_{x \to 0} \frac{1}{x^n} e^{-1/x^2} = 0.$$

Using $y = \frac{1}{x}$ and applying L'Hôpital's rule repeatedly, we have

$$\begin{aligned} \lim_{x \to 0^+} \frac{1}{x^n} e^{-1/x^2} &= \lim_{y \to \infty} y^n e^{-y^2} = \lim_{y \to \infty} \frac{y^n}{e^{y^2}} \\ &= \lim_{y \to \infty} \frac{ny^{n-1}}{2y\, e^{y^2}} = \lim_{y \to \infty} \frac{ny^{n-2}}{2\, e^{y^2}} \\ &= \lim_{y \to \infty} \frac{n(n-2)y^{n-3}}{2 \cdot 2y\, e^{y^2}} = \lim_{y \to \infty} \frac{n(n-2)y^{n-4}}{2 \cdot 2\, e^{y^2}} \\ &= \cdots = \lim_{y \to \infty} \frac{n(n-2)(n-4) \cdots 4 \cdot 2}{2 \cdot 2 \cdots 2\, e^{y^2}} \\ &= 0. \end{aligned}$$

We also have

$$\lim_{x \to 0^-} \frac{1}{x^n} e^{-1/x^2} = \lim_{y \to -\infty} y^n e^{-y^2} = \lim_{y' \to \infty} (-y')^n e^{-y'^2} = (-1)^n \lim_{y' \to \infty} y'^n e^{-y'^2} = 0,$$

where $y' = -y$, using the result just proved above. Since we have shown the two sided limit is zero, we have shown that the limit is zero. The function $f(x) = e^{-1/x^2}$ is continuous at $x = 0$, infinitely differentiable at $x = 0$, and all its derivatives at $x = 0$ are equal to zero, due to the result proved above. For example, the first derivative evaluation at $x = 0$ is:

$$f'(x) = \frac{d}{dx}\left(-\frac{1}{x^2}\right)e^{-1/x^2} = \frac{2}{x^3}e^{-1/x^2}$$
$$f'(0) = 2 \cdot \lim_{x\to 0}\frac{1}{x^3}e^{-1/x^2} = 2(0) = 0,$$

because it is an instance of the limit result for $n = 3$. The second derivative evaluated at $x = 0$ is:

$$f''(x) = -\frac{2\cdot 3}{x^4}e^{-1/x^2} + \frac{2\cdot 2}{x^6}e^{-1/x^2}$$
$$f''(0) = -6 \cdot \lim_{x\to 0}\frac{1}{x^4}e^{-1/x^2} + 4 \cdot \lim_{x\to 0}\frac{1}{x^6}e^{-1/x^2} = -6(0) + 4(0) = 0,$$

because we have instances of the limit result for $n = 4$ and $n = 6$. Since all the derivatives $f^{(n)}(0) = 0$, the Maclaurin series of the function, is $\sum\limits_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = 0$. However, the function itself is not the zero function. So we have shown that the Maclaurin series of $f(x)$ does not converge to the function $f(x)$.

AP® Practice Problems

1. $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots$$
$$(3x)^2 \cos x = (3x)^2\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots\right]$$
$$= 9x^2 - \frac{9x^4}{2!} + \frac{9x^6}{4!} - \frac{9x^8}{6!} + \ldots \frac{(-1)^n 9x^{2n+2}}{(2n)!} + \ldots$$

The coefficient of x^8 is $-\frac{9}{6!} = \boxed{-\frac{1}{80}}$

CHOICE B

2. Utilize the Maclaurin expansion for

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots$$
$$f(x) = e^{x/3} = \sum_{k=0}^{\infty}\frac{\left(\frac{x}{3}\right)^k}{k!} = 1 + \frac{x}{3} + \frac{\left(\frac{x}{3}\right)^2}{2!} + \frac{\left(\frac{x}{3}\right)^3}{3!} + \frac{\left(\frac{x}{3}\right)^4}{4!} + \ldots + \frac{\left(\frac{x}{3}\right)^n}{n!} + \ldots$$
$$= \boxed{\sum_{k=0}^{\infty}\frac{x^k}{3^k k!}}$$

CHOICE C

3.
$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots \\
e^{2x} &= 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \ldots + \frac{(2x)^n}{n!} + \ldots \\
e^{2x} - 1 &= 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \ldots + \frac{(2x)^{n+1}}{(n+1)!} + \ldots \\
\frac{e^{2x} - 1}{x} &= \frac{1}{x}\left[2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \ldots + \frac{(2x)^{n+1}}{(n+1)!} + \ldots\right] \\
&= \frac{1}{x}\left[2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \ldots + \frac{2^{n+1}x^{n+1}}{(n+1)!} + \ldots\right] \\
&= 2 + \frac{4x}{2!} + \frac{8x^2}{3!} + \frac{16x^3}{4!} + \ldots + \frac{2^{n+1}x^n}{(n+1)!} + \ldots \\
&= \boxed{\sum_{k=0}^{\infty} \frac{2^{k+1}x^k}{(k+1)!}}
\end{aligned}$$

CHOICE B

4.
$$\begin{aligned}
\int_0^x \cos t\,dt &= \int_0^x \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots + \frac{(-1)^n t^{2n}}{(2n)!} + \ldots\right) dt \\
&= \left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots + \frac{(-1)^n t^{2n+1}}{(2n+1)!} + \ldots\right]_0^x \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots - 0
\end{aligned}$$

The Taylor expansion for $\int_0^x \cos t\,dt$ centered at $\frac{\pi}{2}$ is

$$\left(x - \frac{\pi}{2}\right) - \frac{\left(x - \frac{\pi}{2}\right)^3}{3!} + \frac{\left(x - \frac{\pi}{2}\right)^5}{5!} - \frac{\left(x - \frac{\pi}{2}\right)^7}{7!} + \ldots + \frac{(-1)^n\left(x - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} + \ldots$$
$$= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k\left(x - \frac{\pi}{2}\right)^{2k+1}}{(2k+1)!}}$$

CHOICE D

5. $f(x) = \ln(2x^3 + 1)$

The Maclaurin Expansion for $\ln(x+1)$ is

$$\ln(x+1) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

Substituting $2x^3$ for x,

$$\begin{aligned}
\ln(2x^3 + 1) &= 2x^3 - \frac{(2x^3)^2}{2} + \frac{(2x^3)^3}{3} + \ldots \\
&= 2x^3 - 2x^6 + \frac{8x^9}{3} - \ldots
\end{aligned}$$

The first non-zero term is $\boxed{2x^3}$

CHOICE C

8.10 Approximations Using Taylor/Maclaurin Expansions

Skill Building

1. To express $f(x) = \ln x^2$ as a Taylor expansion about $c = 1$, we evaluate f and its derivatives at $c = 1$:

$$\begin{aligned}
f(x) &= \ln x^2 = 2\ln x & f(1) &= 0 \\
f'(x) &= \frac{2}{x} & f'(1) &= 2 \\
f''(x) &= -\frac{2}{x^2} & f''(1) &= -2 \\
f'''(x) &= \frac{4}{x^3} & f'''(1) &= 4 \\
f^{(4)}(x) &= -\frac{12}{x^4} & f^{(4)}(2) &= -12 \\
f^{(5)}(x) &= \frac{48}{x^5} & f^{(5)}(1) &= 48
\end{aligned}$$

The Taylor expansion of f centered at c is

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\
&= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(c-x)^n + \ldots
\end{aligned}$$

Here,

$$\begin{aligned}
f(x) &= 0 + 2(x-1) - \frac{2}{2!}(x-1)^2 + \frac{4}{3!}(x-1)^3 - \frac{12}{4!}(x-1)^4 + \frac{48}{5!}(x-5)^5 + \ldots \\
P_5(x) &= \boxed{2(x-1) - (x-1)^2 + \frac{2}{3}(x-1)^3 - \frac{1}{2}(x-1)^4 + \frac{2}{5}(x-1)^5}
\end{aligned}$$

2. The Taylor Polynomial for $f(x) = \ln(1+x) = \sum\limits_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1}$

$$P_5(x) = \boxed{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}}$$

3. To express $f(x) = \frac{1}{x}$ as a Taylor expansion about $c = 1$, we evaluate f and its derivatives at $c = 1$:

$$\begin{aligned} f(x) &= \frac{1}{x} & f(1) &= 1 \\ f'(x) &= -\frac{1}{x^2} & f'(1) &= -1 \\ f''(x) &= \frac{2}{x^3} & f''(1) &= 2 \\ f'''(x) &= -\frac{6}{x^4} & f'''(1) &= -6 \\ f^{(4)}(x) &= \frac{24}{x^5} & f^{(4)}(1) &= 24 \\ f^{(5)}(x) &= -\frac{120}{x^6} & f^{(5)}(1) &= -120 \end{aligned}$$

The Taylor expansion of f centered at c is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(c-x)^n + \ldots \end{aligned}$$

Here,

$$\begin{aligned} f(x) &= 1 - 1 \cdot (x-1)^1 + \frac{2}{2!}(x-1)^2 - \frac{6}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4 - \frac{120}{5!}(x-1)^5 - \ldots \\ P_5(x) &= \boxed{1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5} \end{aligned}$$

4. To express $f(x) = \frac{1}{x^2}$ as a Taylor expansion about $c = 1$, we evaluate f and its derivatives at $c = 1$:

$$\begin{aligned} f(x) &= \frac{1}{x^2} & f(1) &= 1 \\ f'(x) &= \frac{-2}{x^3} & f'(1) &= -2 \\ f''(x) &= \frac{6}{x^4} & f''(1) &= 6 \\ f'''(x) &= \frac{-24}{x^5} & f'''(1) &= -24 \\ f^{(4)}(x) &= \frac{120}{x^6} & f^{(4)}(1) &= 120 \\ f^{(5)}(x) &= \frac{-720}{x^7} & f^{(5)}(1) &= -720 \end{aligned}$$

The Taylor Expansion of f centered at c is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^k(c)}{k!}(x-c)^k \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^n(c)}{n!}(x-c)^n + \ldots. \end{aligned}$$

Here,

$$\begin{aligned} f(x) &= 1 - 2(x-1) + \frac{6}{2!}(x-1)^2 - \frac{24}{3!}(x-1)^3 + \frac{120}{4!}(x-1)^4 - \frac{720}{5!}(x-1)^5 + \ldots \\ P_5(x) &= \boxed{1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 - 6(x-1)^5} \end{aligned}$$

5. To express $f(x) = \cos x$ as a Taylor expansion about $c = \frac{\pi}{2}$, we evaluate f and its derivatives at $c = \frac{\pi}{2}$:

$$\begin{aligned} f(x) &= \cos x & f\left(\frac{\pi}{2}\right) &= \cos\frac{\pi}{2} = 0 \\ f'(x) &= -\sin x & f'\left(\frac{\pi}{2}\right) &= -\sin\frac{\pi}{2} = -1 \\ f''(x) &= -\cos x & f''\left(\frac{\pi}{3}\right) &= -\cos\frac{\pi}{2} = 0 \\ f'''(x) &= \sin x & f'''\left(\frac{\pi}{2}\right) &= \sin\frac{\pi}{2} = 1 \\ f^{(4)}(x) &= \cos x & f^{(4)}\left(\frac{\pi}{2}\right) &= \cos\frac{\pi}{2} = 0 \\ f^{(5)}(x) &= -\sin x & f^{(5)}\left(\frac{\pi}{2}\right) &= -\sin\frac{\pi}{2} = -1 \end{aligned}$$

The Taylor expansion of f centered at c is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(c-x)^n + \ldots \end{aligned}$$

Here,

$$\begin{aligned} f(x) &= 0 - 1\cdot\left(x - \frac{\pi}{2}\right) + \frac{0}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 + \frac{0}{4!}\left(x-\frac{\pi}{2}\right)^4 - \frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5 + \ldots \\ P_5(x) &= \boxed{-\left(x-\frac{\pi}{2}\right) + \frac{1}{6}\left(x-\frac{\pi}{2}\right)^3 - \frac{1}{120}\left(x-\frac{\pi}{2}\right)^5} \end{aligned}$$

6. To express $f(x) = \sin x$ as a Taylor expansion about $c = \frac{\pi}{4}$, we evaluate f and its derivatives at $c = \frac{\pi}{4}$:

$$
\begin{aligned}
f(x) &= \sin x & \qquad f\left(\frac{\pi}{4}\right) &= \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \\
f'(x) &= \cos x & \qquad f'\left(\frac{\pi}{4}\right) &= \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \\
f''(x) &= -\sin x & \qquad f''\left(\frac{\pi}{4}\right) &= -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2} \\
f'''(x) &= -\cos x & \qquad f'''\left(\frac{\pi}{4}\right) &= -\cos x = -\frac{\sqrt{2}}{2} \\
f^{(4)}(x) &= \sin x & \qquad f^{(4)}\left(\frac{\pi}{4}\right) &= \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \\
f^{(5)}(x) &= \cos x & \qquad f^{(5)}\left(\frac{\pi}{4}\right) &= \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}
\end{aligned}
$$

The Taylor Expansion of f centered at c is

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^k(c)}{k!}(x-c)^k \\
&= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^n(c)}{n!}(x-c)^n + \ldots
\end{aligned}
$$

Here,

$$
\begin{aligned}
f(x) &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\frac{\sqrt{2}}{2}}{2!}\left(x-\frac{\pi}{4}\right)^2 - \frac{\frac{\sqrt{2}}{2}}{3!}\left(x-\frac{\pi}{4}\right)^3 + \frac{\frac{\sqrt{2}}{2}}{4!}\left(x-\frac{\pi}{4}\right)^4 + \frac{\frac{\sqrt{2}}{2}}{5!}\left(x-\frac{\pi}{4}\right)^5 + \ldots \\
P_5(x) &= \boxed{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(x-\frac{\pi}{4}\right)^4 + \frac{\sqrt{2}}{240}\left(x-\frac{\pi}{4}\right)^5}
\end{aligned}
$$

7. The Taylor expansion at $c = 0$ is the Maclaurin expansion.

The Maclaurin expansion for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^n}{n!} + \ldots$

To determine the Maclaurin expansion for e^{2x}, substitute $2x$ for x.

$$
\begin{aligned}
P_5(x) &= 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \frac{(2x)^5}{5!} \\
&= \boxed{1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5}
\end{aligned}
$$

8. The Taylor Expansion at $c = 0$ is the Maclaurin expansion.

The Maclaurin expansion for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \ldots + \frac{x^n}{n!} + \ldots$

To determine the Maclaurin expansion for e^{-x}, substitute $-x$ for x.

$$
\begin{aligned}
P_5(x) &= 1 - x + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} \\
&= \boxed{1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}}
\end{aligned}
$$

9. To express $f(x) = \frac{1}{1-2x}$ as a Taylor expansion about $c = 0$ is to determine the Maclaurin expansion for $f(x) = \frac{1}{1-2x}$.
We evaluate f and its derivatives at $c = 0$:

$$
\begin{aligned}
f(x) &= \frac{1}{1-2x} & f(0) &= 1 \\
f'(x) &= \frac{2}{(1-2x)^2} & f'(0) &= 2 \\
f''(x) &= \frac{8}{(1-2x)^3} & f''(0) &= 8 \\
f'''(x) &= \frac{48}{(1-2x)^4} & f'''(0) &= 48 \\
f^{(4)}(x) &= \frac{384}{(1-2x)^4} & f^{(4)}\left(\frac{\pi}{2}\right) &= 384 \\
f^{(5)}(x) &= \frac{3840}{(1-2x)^5} & f^{(5)}(0) &= 3840
\end{aligned}
$$

The Taylor expansion of f centered at c is

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\
&= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(c-x)^n + \ldots
\end{aligned}
$$

Here,

$$
\begin{aligned}
P_5(x) &= 1 + 2x + \frac{8}{2!}x^2 + \frac{48}{3!}x^3 + \frac{384}{4!}x^4 - \frac{3840}{5!}x^5 \\
&= \boxed{1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5}
\end{aligned}
$$

10. To express $f(x) = \frac{1}{1+x}$ as a Taylor expansion about $c = 0$ is to determine the Maclaurin expansion for $f(x) = \frac{1}{1+x}$.
We evaluate f and its derivatives at $c = 0$:

$$
\begin{aligned}
f(x) &= \frac{1}{1+x} & f(0) &= 1 \\
f'(x) &= \frac{-1}{(1+x)^2} & f'(0) &= -1 \\
f''(x) &= \frac{2}{(1+x)^3} & f''(0) &= 2 \\
f'''(x) &= \frac{-6}{(1+x)^4} & f'''(0) &= -6 \\
f^{(4)}(x) &= \frac{24}{(1+x)^5} & f^{(4)}(0) &= 24 \\
f^{(5)}(x) &= \frac{-120}{(1+x)^6} & f^{(5)}(0) &= -120
\end{aligned}
$$

The Taylor Expansion of f centered at c is

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^k(c)}{k!}(x-c)^k \\
&= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^n(c)}{n!}(x-c)^n + \ldots
\end{aligned}
$$

Here,

$$\begin{aligned} P_5(x) &= 1 - x + \frac{2x^2}{2!} - \frac{6x^3}{3!} + \frac{24x^4}{4!} - \frac{120x^5}{5!} \\ &= \boxed{1 - x + x^2 - x^3 + x^4 - x^5} \end{aligned}$$

11. To express $f(x) = x \ln x$ as a Taylor expansion about $c = 1$, we evaluate f and its derivatives at $c = 1$:

$$\begin{aligned} f(x) &= x \ln x & f(1) &= 0 \\ f'(x) &= \ln x + 1 & f'(1) &= 1 \\ f''(x) &= \frac{1}{x} & f''(1) &= 1 \\ f'''(x) &= -\frac{1}{x^2} & f'''(1) &= -1 \\ f^{(4)}(x) &= \frac{2}{x^3} & f^{(4)}(1) &= 2 \\ f^{(5)}(x) &= -\frac{6}{x^4} & f^{(5)}(1) &= -6 \end{aligned}$$

The Taylor expansion of f centered at c is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots \end{aligned}$$

Here,

$$\begin{aligned} f(x) &= 0 + 1 \cdot (x-1) + \frac{1}{2!}(x-1)^2 - \frac{1}{3!}(x-1)^3 + \frac{2}{4!}(x-1)^4 - \frac{6}{5!}(x-1)^5 + \ldots \\ P_5(x) &= \boxed{(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 - \frac{1}{20}(x-1)^5} \end{aligned}$$

12. To express $f(x) = xe^x$ as a Taylor expansion about $c = 1$, we first consider $f(x) = xe^x$ and its derivatives at $c = 1$:

$$\begin{aligned} f(x) &= xe^x & f(1) &= e \\ f'(x) &= (x+1)e^x & f'(1) &= 2e \\ f''(x) &= (x+2)e^x & f''(1) &= 3e \\ f'''(x) &= (x+3)e^x & f'''(1) &= 4e \\ f^{(4)}(x) &= (x+4)e^x & f^{(4)}(1) &= 5e \\ f^{(5)}(x) &= (x+5)e^x & f^{(5)}(1) &= 6e \end{aligned}$$

The Taylor Expansion of e^x centered at c is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^k(c)}{k!}(x-c)^k \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^n(c)}{n!}(x-c)^n + \ldots \end{aligned}$$

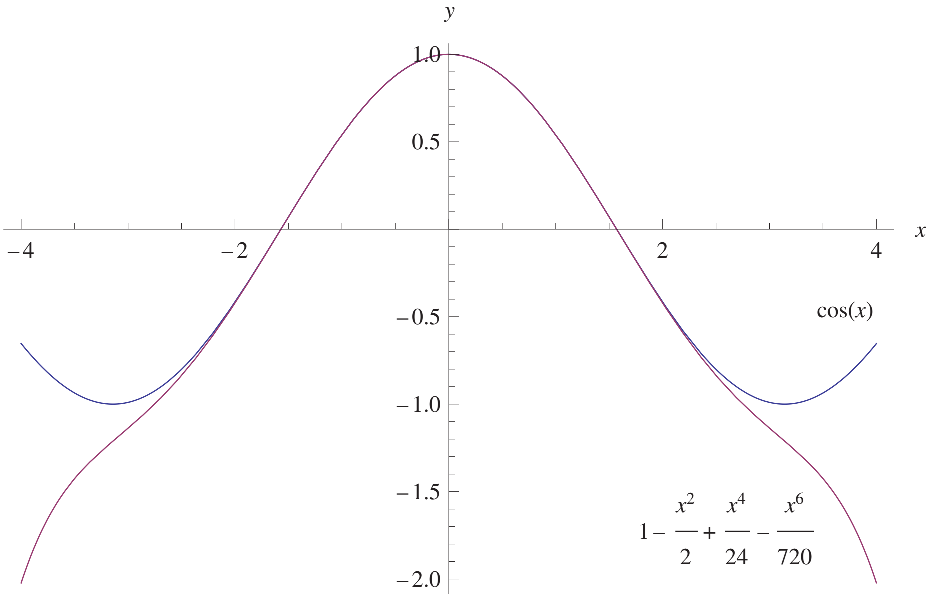
Here,

$$\begin{aligned} P_5(x) &= e + 2e(x-1) + \frac{3e}{2!}(x-1)^2 + \frac{4e}{3!}(x-1)^3 + \frac{5e}{4!}(x-1)^4 + \frac{6e}{5!}(x-1)^5 \\ &= \boxed{e + 2e(x-1) + \frac{3e}{2}(x-1)^2 + \frac{2e}{3}(x-1)^3 + \frac{5e}{24}(x-1)^4 + \frac{e}{20}(x-1)^5} \end{aligned}$$

13. (a) The Maclaurin series approximation to four nonzero terms of $y = \cos x$ is

$$\begin{aligned} \cos x &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\ &\approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \\ &= \boxed{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}.} \end{aligned}$$

(b) The figure below shows the graph of both the function and its Maclaurin approximation found in part (a).



(c) Evaluating the approximation at $x = \frac{\pi}{90}$, we get

$$\cos\left(\frac{\pi}{90}\right) \approx 1 - \frac{1}{2!}\left(\frac{\pi}{90}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{90}\right)^4 - \frac{1}{6!}\left(\frac{\pi}{90}\right)^6 \approx \boxed{0.9994.}$$

(d) Since the series is alternating, the error in the approximation is given by the next term's absolute value in the computation of part (c), which is

$$\frac{1}{8!}\left(\frac{\pi}{90}\right)^8 \approx \boxed{5.47 \times 10^{-17}.}$$

(e) Since the error is given by $\frac{1}{8!}x^8$, and we need this to be less than 0.0001 or 10^{-4}, we solve:

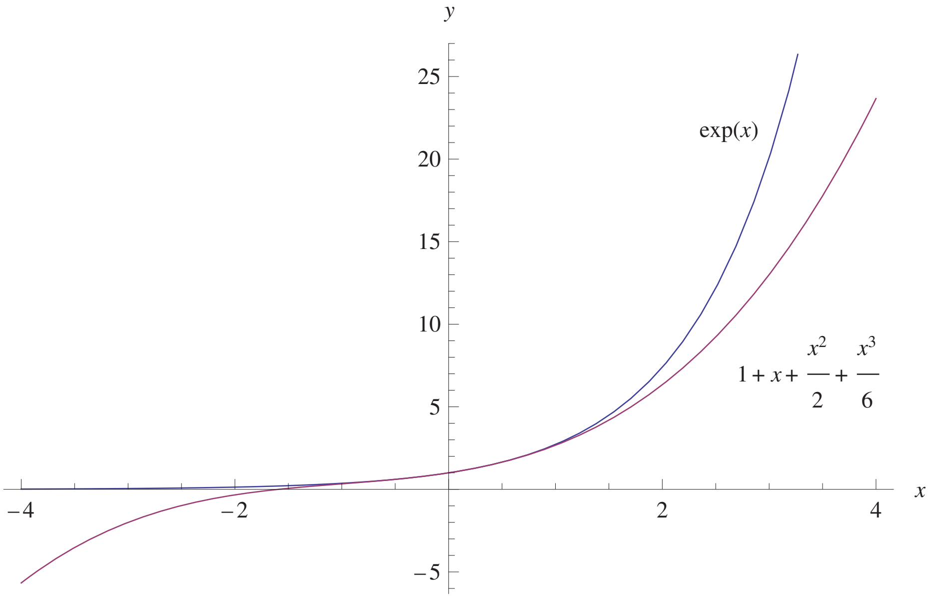
$$\begin{aligned} \frac{1}{8!}x^8 &< 10^{-4} \\ x^8 &< 8!\,10^{-4} \end{aligned}$$

Taking the eighth root of both sides of this last inequality, we obtain the approximate interval for x to be $\boxed{-1.1904 < x < 1.1904.}$

14. **(a)** The Maclaurin series approximation to four terms of $y = e^x$ is

$$\begin{aligned} e^x &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \\ &\approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \\ &= \boxed{1 + x + \frac{x^2}{2} + \frac{x^3}{6}.} \end{aligned}$$

(b) The figure below shows the graph of both the function and its Maclaurin approximation found in part (a).



(c) Evaluating the approximation at $x = \frac{1}{2}$, we get

$$e^{1/2} \approx 1 + \frac{1}{2} + \frac{1}{2!}\left(\frac{1}{2}\right)^2 + \frac{1}{3!}\left(\frac{1}{2}\right)^3 \approx \boxed{1.6458.}$$

(d) The actual value of $e^{1/2} \approx 1.6487$, so the error is $\boxed{0.0029.}$

(e) For the error to be less than 0.0001, we require

$$e^x - \left[1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right] < 0.0001.$$

Since the series is not alternating, we don't have an estimate for the error term. We proceed by trial and error to find $\boxed{|x| < 0.0838.}$

15. (a) The Maclaurin series of $f(x) = \sqrt[3]{1+x}$ is given by the Binomial Series:

$$f(x) = \sqrt[3]{1+x} = (1+x)^{1/3} = \sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} x^k$$

$$= \boxed{1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}x^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!} + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{4!}x^4 + \cdots + \binom{\frac{1}{2}}{n}x^n + \cdots .}$$

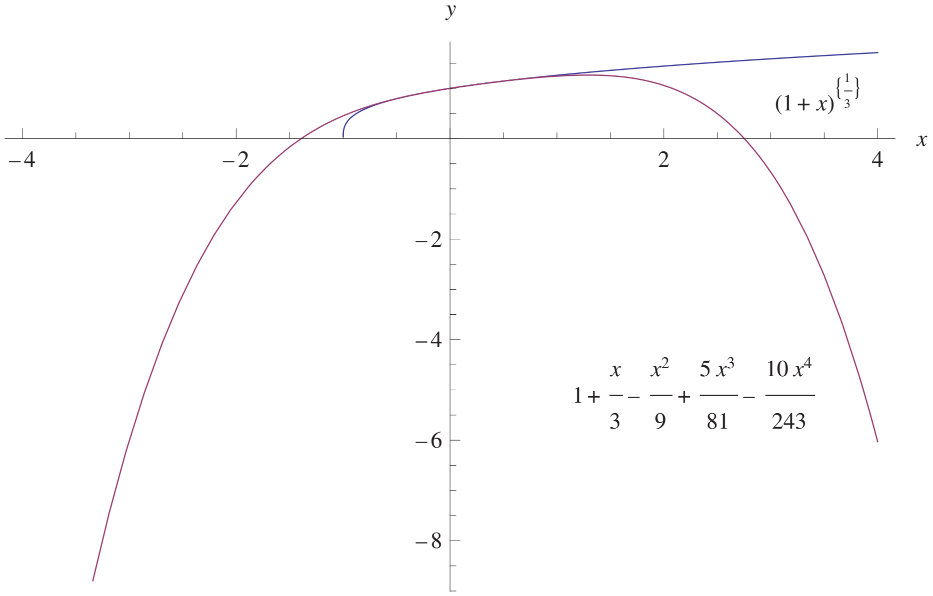
(b) By the theorem on p.721, since $m = \frac{1}{3} > 0$, but not an integer, the binomial series converges for $\boxed{[-1, 1].}$

(c) Using the first five terms of the Maclaurin series found in part (a), we get

$$f(x) \approx 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}x^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!} + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{4!}x^4$$

$$= \boxed{1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4.}$$

(d) The figure below shows the graph of both the function and its Maclaurin approximation found in part (c).



(e) From the graphs we see that the approximation is good for values of x in the range of convergence $[-1, 1]$, and not so good outside that range.

(f) To approximate $\sqrt[3]{0.9}$, we substitute $x = -0.1$ into the approximation found in part (c):

$$\sqrt[3]{0.9} = \sqrt[3]{1-0.1} \approx 1 + \frac{1}{3}(-0.1) - \frac{1}{9}(-0.1)^2 + \frac{5}{81}(-0.1)^3 - \frac{10}{243}(-0.1)^4 \approx \boxed{0.9655.}$$

Since the series is a convergent alternating series, the error in the approximation is bounded from above by the absolute value of the next term of the approximation, which is

$$E(x) \leq \left| \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)}{5!}x^5 \right|$$

$$= \left| \frac{22}{729}x^5 \right| .$$

So the error is less than or equal to

$$E(-0.1) = \left|\frac{22}{729}(-0.1)^5\right| = \boxed{3.018 \times 10^{-7}.}$$

16. (a) The Maclaurin series of $y = \frac{1}{\sqrt{4+x}}$ is given by the Binomial Series:

$$\frac{1}{\sqrt{4+x}} = (4+x)^{-1/2} = 4^{-1/2} \cdot \left(1+\frac{x}{4}\right)^{-1/2} = \frac{1}{2} \cdot \left(1+\frac{x}{4}\right)^{-1/2} = \boxed{\frac{1}{2}\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\left(\frac{x}{4}\right)^k}. \text{ More fully:}$$

$$= \frac{1}{2}\left[\begin{array}{l} 1+\left(-\frac{1}{2}\right)\frac{x}{4}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{4}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{4}\right)^3+ \\ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{4!}\left(\frac{x}{4}\right)^4+\cdots+\binom{-\frac{1}{2}}{n}\left(\frac{x}{4}\right)^n+\cdots \end{array}\right]$$

$$= \frac{1}{2}\left[1-\frac{1}{8}x+\frac{3}{128}x^2-\frac{5}{1024}x^3+\frac{35}{32,768}x^4+\cdots+\binom{-\frac{1}{2}}{n}\left(\frac{x}{4}\right)^n+\cdots\right]$$

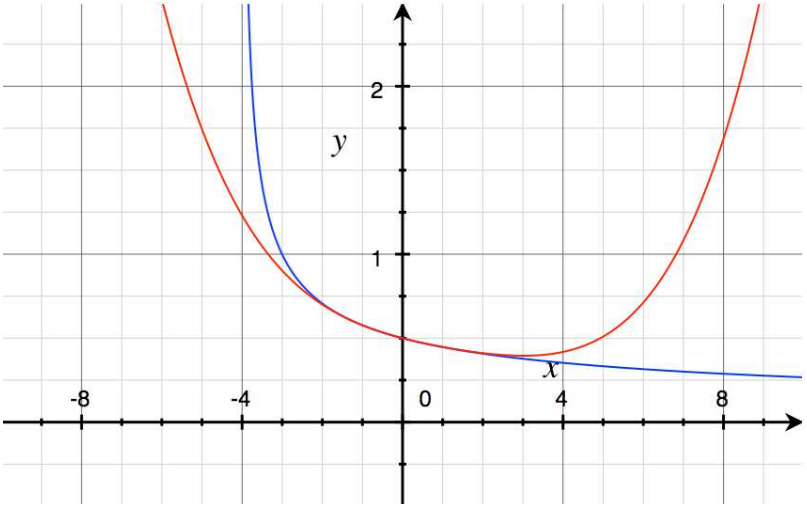
$$= \frac{1}{2}-\frac{1}{16}x+\frac{3}{256}x^2-\frac{5}{2048}x^3+\frac{35}{65,536}x^4+\cdots+\frac{1}{2}\binom{-\frac{1}{2}}{n}\left(\frac{x}{4}\right)^n+\cdots$$

(b) By the Theorem on p. 721, since m satisfies $-1 < m < 0$, $\frac{x}{4}$ must be in $(-1, 1]$, that is, $-1 < \frac{x}{4} \leq 1$, that is, $-4 < x \leq 4$. So the interval of convergence is $\boxed{(-4, 4]}$.

(c) Keeping the first five terms of the Maclaurin expansion found in (a), we have the desired approximation $P_4(x)$:

$$y = \frac{1}{\sqrt{4+x}} \approx P_4(x) = \boxed{\frac{1}{2}-\frac{x}{16}+\frac{3}{256}x^2-\frac{5}{2048}x^3+\frac{35}{65,536}x^4}$$

(d) The figure below shows the graph of both $y = \frac{1}{\sqrt{4+x}}$ (blue) and $y = P_4(x)$ (red).



(e) From the graphs we see that the approximation is reasonable for values of x in the range of convergence $(-4, 4]$ of the series (and excellent from about $x = -2$ to $x = 2$), and not good outside that range.

(f) To approximate $\frac{1}{\sqrt{4.2}}$ we substitute $x = 0.2$ into the approximation found in part (c):

$$\frac{1}{\sqrt{4.2}} \approx P_4(0.2) = \frac{1}{2} - \frac{0.2}{16} + \frac{3}{256}0.2^2 - \frac{5}{2048}0.2^3 + \frac{35}{65,536}0.2^4 \approx \boxed{0.4879501}$$

Since the series is a converging, alternating series, the absolute value of the error in the approximation is bounded from above by the absolute value of the next term in the approximation. Here, this is

$$\begin{aligned} |E_4(x)| &\leq \frac{1}{2}\left|\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right)}{5!}\left(\frac{x}{4}\right)^5\right| \\ &= \frac{1}{2}\left|-\frac{63}{262,144}x^5\right| \\ &= \frac{63}{524,288}\left|x^5\right| \end{aligned}$$

So the error is less than or equal to

$$\frac{63}{524,288}\left|0.2^5\right| \approx \boxed{3.845 \times 10^{-8}}$$

17. (a) From p.708, Example 9, the Maclaurin series of $y = \tan^{-1} x$ is given by Gregory's series:

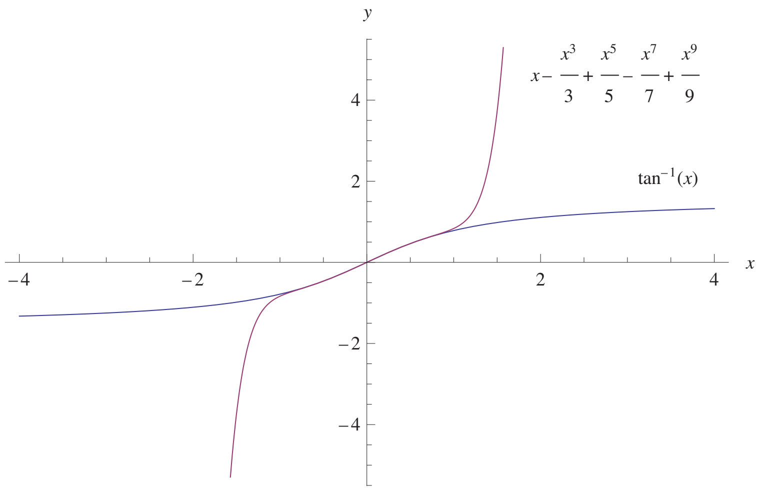
$$\boxed{y = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}.}$$

(b) From p.708, Example 9, it is seen that the series converges in the interval $[-1, 1]$.

(c) Keeping the first five nonzero terms of Gregory's series, we get the desired approximation

$$\boxed{y = \tan^{-1} x \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}.}$$

(d) The figure below shows the graph of both the function and its Maclaurin approximation found in part (c).



(e) We see that the approximation matches the function best in the interval of convergence $[-1, 1]$ of the Maclaurin series, and diverges rapidly from the function outside that range.

18. (a) Using the formula for the sum of geometric series, we see that the Maclaurin series is

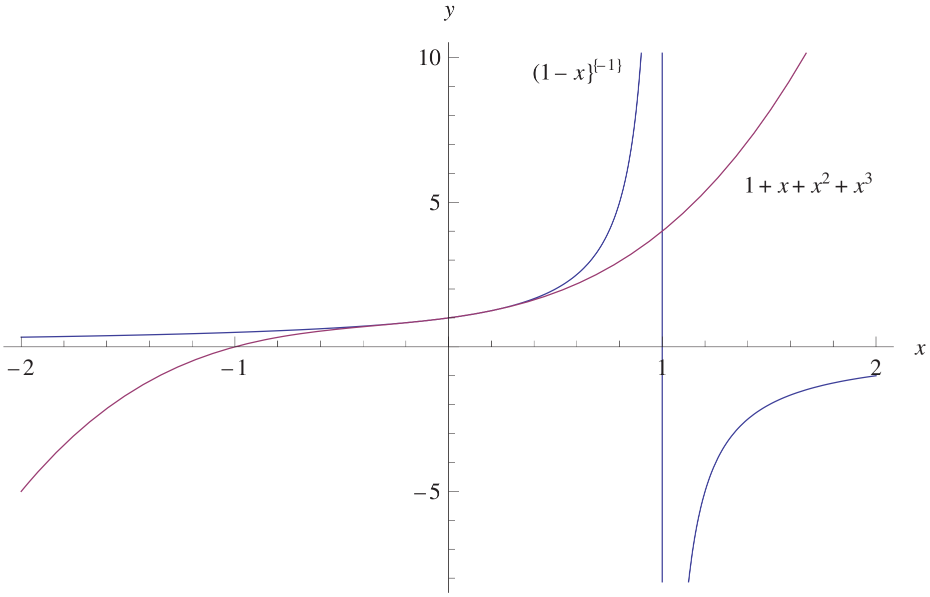
$$\boxed{y = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots .}$$

(b) Since the geometric series $\sum\limits_{k=0}^{\infty} r^k$ converges if $|r| < 1$, the series above converges for $|x| < 1$, or in the open interval $\boxed{(-1, 1).}$

(c) Using the first four nonzero terms of the Maclaurin series found in (a), we get

$$\boxed{y = \frac{1}{1-x} \approx 1 + x + x^2 + x^3.}$$

(d) The figure below shows the graph of both the function and its Maclaurin approximation found in part (c).



(e) We see from the graph that the approximation matches the function pretty well on the interval $(-0.5, 0.5)$. In the range $(-1, -0.5)$, which is within the range of convergence, the deviation is quite small, but in the range $(0.5, 1)$, which is also in the range of convergence, the deviation is large. Also, beyond $x = 1$, there is no agreement as to even the correct quadrant of the approximation and the function, which assumes another branch (as it is a hyperbola with asymptote $x = 1$). This is to be expected, because the series does not converge for values of $x > 1$.

19. To evaluate $\int_0^1 \sin x^2\, dx$, we first find the Maclaurin series for the integrand. From p.717, we write the Maclaurin series of $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots .$$

So the Maclaurin series of $\sin x^2$ is

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots + (-1)^n \frac{x^{4n+2}}{(2n+1)!} + \cdots .$$

Approximating to four nonzero terms, we get

$$\sin x^2 \approx x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}.$$

Integrating both sides we have

$$\begin{aligned}\int_0^1 \sin x^2\, dx &\approx \int_0^1 \left[x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}\right] dx \\ &= \left[\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!}\right]_0^1 \\ &= \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} \\ &= \frac{258,019}{831,600} \\ &\approx \boxed{0.310.}\end{aligned}$$

20. To evaluate $\int_0^1 \cos x^2\, dx$, we first find the Maclaurin series for the integrand. The Maclaurin series of $\cos x$ (see p.717) is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots .$$

So the Maclaurin series of $\cos x^2$ is

$$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots + (-1)^n \frac{x^{4n}}{(2n)!} + \cdots .$$

Approximating to four nonzero terms, we get

$$\cos x^2 \approx 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!}.$$

Integrating both sides, we have

$$\begin{aligned}\int_0^1 \cos x^2\, dx &\approx \int_0^1 \left[1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!}\right] dx \\ &= \left[x - \frac{x^5}{5\cdot 2!} + \frac{x^9}{9\cdot 4!} - \frac{x^{13}}{13\cdot 6!}\right]_0^1 \\ &= 1 - \frac{1}{10} + \frac{1}{216} - \frac{1}{9360} \\ &= \frac{25,399}{28,080} \\ &\approx \boxed{0.905.}\end{aligned}$$

21. Use the first four terms of the MacLaurin expansion for e^x to approximate $\int_0^{0.1} e^{x^2}\,dx$:
Replace x by x^2 in the Maclaurin expansion for e^x to obtain

$$\begin{aligned}\int_0^1 e^{x^2}dx &= \int_0^1 \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!}\right)dx\\ &= \left[x + \frac{x^3}{3} + \frac{x^5}{2!5} + \frac{x^7}{3!7}\right]_0^1 = \left[1 + \frac{(1)^3}{3} + \frac{(1)^5}{2!5} + \frac{(1)^7}{3!7}\right]\\ &\approx \boxed{1.457}\end{aligned}$$

22. Use a MacLaurin expansion to approximate $\int_0^{0.1} e^{-x^2}dx$:
Replace x by $-x^2$ in the Maclaurin expansion for e^x to obtain

$$\begin{aligned}\int_0^{0.1} e^{-x^2}dx &= \int_0^{0.1} \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!}\right)dx\\ &= \left[x - \frac{x^3}{3} + \frac{x^5}{2!5} - \frac{x^7}{3!7}\right]_0^{0.1} = \left[0.1 - \frac{(0.1)^3}{3} + \frac{(0.1)^5}{2!5} - \frac{(0.1)^7}{3!7}\right]\\ &\approx \boxed{0.100}\end{aligned}$$

23. The integrand in $\int_0^{0.2} \sqrt[3]{1+x^4}\,dx$ can be expanded as a Binomial Series:

$$\begin{aligned}\sqrt[3]{1+x^4} &= (1+x^4)^{1/3} = \sum_{k=0}^{\infty}\binom{\frac{1}{3}}{k}(x^4)^k\\ &= 1 + \frac{1}{3}x^4 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^8 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^{12} + \cdots + \binom{\frac{1}{3}}{n}x^{4n} + \cdots\\ &\approx 1 + \frac{1}{3}x^4 - \frac{1}{9}x^8 + \frac{5}{81}x^{12}\end{aligned}$$

which is the integrand approximated to four nonzero terms. Integrating both sides, we have

$$\begin{aligned}\int_0^{0.2} \sqrt[3]{1+x^4}\,dx &\approx \int_0^{0.2}\left[1 + \frac{1}{3}x^4 - \frac{1}{9}x^8 + \frac{5}{81}x^{12}\right]dx\\ &= \left[x + \frac{x^5}{5\cdot 3} - \frac{x^9}{9\cdot 9} + \frac{5x^{13}}{13\cdot 81}\right]_0^{0.2}\\ &= (0.2) + \frac{(0.2)^5}{15} - \frac{(0.2)^9}{81} + \frac{5(0.2)^{13}}{1053}\\ &\approx 0.2 + 2.133\times 10^{-5} - 6.321\times 10^{-9} + 3.889\times 10^{-12}\\ &\approx 0.2000213\\ &\approx \boxed{0.2000}\end{aligned}$$

which is correct to three decimal places.

24. The integrand in $\int_0^{1/2} \sqrt[3]{1+x}\,dx$ can be expanded as a Binomial Series:

$$\begin{aligned}\sqrt[3]{1+x} &= (1+x)^{1/3} = \sum_{k=0}^{\infty}\binom{\frac{1}{3}}{k}x^k\\ &= 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^3 + \cdots + \binom{\frac{1}{3}}{n}x^n + \cdots\\ &\approx 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3\end{aligned}$$

which is the integrand expanded to four nonzero terms. Integrating both sides, we have

$$\begin{aligned}\int_0^{1/2} \sqrt[3]{1+x}\,dx &\approx \int_0^{1/2} \left[1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3\right] dx\\ &= \left[x+\frac{x^2}{2\cdot 3}-\frac{x^3}{3\cdot 9}+\frac{5}{4\cdot 81}x^4\right]_0^{1/2}\\ &= \frac{1}{2}+\frac{1}{6}\left(\frac{1}{2}\right)^2-\frac{1}{27}\left(\frac{1}{2}\right)^3+\frac{5}{324}\left(\frac{1}{2}\right)^4\\ &= \frac{2789}{5184}\\ &\approx \boxed{0.538.}\end{aligned}$$

25. The integrand in $\int_0^{1/2} \frac{1}{\sqrt[3]{1+x^2}}\,dx$ can be expanded as a Binomial Series:

$$\begin{aligned}\frac{1}{\sqrt[3]{1+x^2}} &= (1+x^2)^{-1/3} = \sum_{k=0}^{\infty}\binom{-\frac{1}{3}}{k}(x^2)^k\\ &= 1-\frac{1}{3}x^2+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}x^4+\frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}x^6+\cdots+\binom{-\frac{1}{3}}{n}x^{2n}+\cdots\\ &\approx 1-\frac{1}{3}x^2+\frac{2}{9}x^4-\frac{14}{81}x^6\end{aligned}$$

which is the integrand expanded to four nonzero terms. Integrating both sides, we have

$$\begin{aligned}\int_0^{1/2}\frac{1}{\sqrt[3]{1+x^2}}\,dx &\approx \int_0^{1/2}\left[1-\frac{1}{3}x^2+\frac{2}{9}x^4-\frac{14}{81}x^6\right] dx\\ &= \left[x-\frac{x^3}{3\cdot 3}+\frac{2}{5\cdot 9}x^5-\frac{14}{7\cdot 81}x^7\right]_0^{1/2}\\ &= \left(\frac{1}{2}\right)-\frac{1}{9}\left(\frac{1}{2}\right)^3+\frac{2}{45}\left(\frac{1}{2}\right)^5-\frac{14}{567}\left(\frac{1}{2}\right)^7\\ &= \frac{12,631}{25,920}\\ &\approx \boxed{0.487.}\end{aligned}$$

26. The integrand in $\int_0^{0.2} \frac{1}{\sqrt{1+x^3}}\,dx$ can be expanded as a Binomial Series:

$$\begin{aligned}\frac{1}{\sqrt{1+x^3}} &= (1+x^3)^{-1/2} = \sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}(x^3)^k\\ &= 1-\frac{1}{2}x^3+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}x^6+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}x^9+\cdots+\binom{-\frac{1}{2}}{n}x^{3n}+\cdots\\ &\approx 1-\frac{1}{2}x^3+\frac{3}{8}x^6-\frac{5}{16}x^9\end{aligned}$$

which is the integrand expanded to four nonzero terms. Integrating both sides, we have

$$\begin{aligned}\int_0^{0.2} \frac{1}{\sqrt{1+x^3}}\,dx &\approx \int_0^{0.2}\left[1-\frac{1}{2}x^3+\frac{3}{8}x^6-\frac{5}{16}x^9\right]dx\\ &= \left[x-\frac{1}{2\cdot 4}x^4+\frac{3}{8\cdot 7}x^7-\frac{5}{16\cdot 10}x^{10}\right]_0^{0.2}\\ &= 0.2-\frac{(0.2)^4}{8}+\frac{3}{56}(0.2)^7-\frac{5}{160}(0.2)^{10}\\ &\approx \boxed{0.1998.}\end{aligned}$$

Applications and Extensions

27. We use the recursion relation

$$\ln(N+1)=\ln N+2\left[\frac{1}{2N+1}+\frac{1}{3}\left(\frac{1}{2N+1}\right)^3+\frac{1}{5}\left(\frac{1}{2N+1}\right)^5+\cdots\right].$$

Substituting $N=3$, and repeatedly using the recursion relation we get

$$\begin{aligned}\ln 4 &= \ln 3+2\left[\frac{1}{2\cdot 3+1}+\frac{1}{3}\left(\frac{1}{2\cdot 3+1}\right)^3+\frac{1}{5}\left(\frac{1}{2\cdot 3+1}\right)^5+\cdots\right]\\ &= \ln 2+2\left[\frac{1}{2\cdot 2+1}+\frac{1}{3}\left(\frac{1}{2\cdot 2+1}\right)^3+\frac{1}{5}\left(\frac{1}{2\cdot 2+1}\right)^5+\cdots\right]+2\left[\frac{1}{7}+\frac{1}{3}\left(\frac{1}{7}\right)^3+\frac{1}{5}\left(\frac{1}{7}\right)^5+\cdots\right]\\ &= \ln 1+2\left[\frac{1}{2\cdot 1+1}+\frac{1}{3}\left(\frac{1}{2\cdot 1+1}\right)^3+\frac{1}{5}\left(\frac{1}{2\cdot 1+1}\right)^5+\cdots\right]+2\left[\frac{1}{5}+\frac{1}{3}\left(\frac{1}{5}\right)^3+\frac{1}{5}\left(\frac{1}{5}\right)^5+\cdots\right]\\ &\quad +2\left[\frac{1}{7}+\frac{1}{3}\left(\frac{1}{7}\right)^3+\frac{1}{5}\left(\frac{1}{7}\right)^5+\cdots\right]\\ &= 0+2\left[\left\{\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right\}+\frac{1}{3}\left\{\left(\frac{1}{3}\right)^3+\left(\frac{1}{5}\right)^3+\left(\frac{1}{7}\right)^3\right\}+\frac{1}{5}\left\{\left(\frac{1}{3}\right)^5+\left(\frac{1}{5}\right)^5+\left(\frac{1}{7}\right)^5\right\}+\cdots\right]\\ &\approx \boxed{1.386147}\end{aligned}$$

28. (a) The Maclaurin series of $\sin x$ is (see p.716-717, Example 3)

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots.$$

So the Maclaurin series of $3\sin\left(\frac{x}{2}\right)$ is

$$\begin{aligned}3\sin\left(\frac{x}{2}\right) &= 3\left[\left(\frac{x}{2}\right)-\frac{1}{3!}\left(\frac{x}{2}\right)^3+\frac{1}{5!}\left(\frac{x}{2}\right)^5-\cdots+(-1)^n\frac{1}{(2n+1)!}\left(\frac{x}{2}\right)^{2n+1}+\cdots\right]\\ &= \boxed{\frac{3x}{2}-\frac{x^3}{16}+\frac{x^5}{1280}-\cdots+(-1)^n\frac{x^{2n+1}}{2^{2n+1}(2n+1)!}+\cdots.}\end{aligned}$$

(b) As shown in Example 3 on p.716-717, the interval of convergence of the Maclaurin series for $\sin x$ is $(-\infty, \infty)$. So the interval of convergence of the Maclaurin series for $3\sin\left(\frac{x}{2}\right)$ is also $\boxed{(-\infty, \infty).}$

(c) The error in using the first n nonzero terms of a convergent alternating series is bounded from above by the absolute value of the $n+1$st term. The error terms are all monomials which take their maximum values on the interval $(-2, 2)$ at the endpoints (as limits of $x \to -2^+$ or $x \to 2^-$) of the interval. We proceed by trial and error:

The error in keeping just the first term is $\left|-\frac{x^3}{16}\right|$, which as $x \to 2^-$ evaluates to $\left|\frac{2^3}{16}\right| = 0.5$.

The error in keeping the first two terms is $\left|\frac{x^5}{1280}\right|$, which as $x \to 2^-$ evaluates to $\left|\frac{2^5}{1280}\right| = 0.025$.

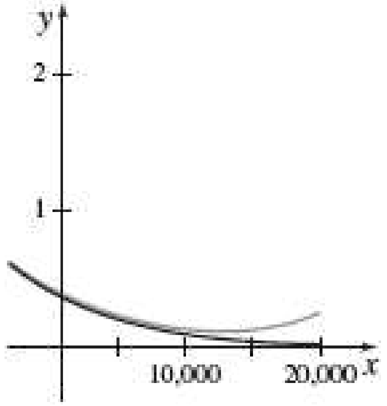
The error is already less than the required 0.1, so that only $\boxed{two}$ terms are needed in order to approximate f on the interval $(-2, 2)$ to an error less than or equal to 0.1. (As a check, $3\sin\left(\frac{2}{2}\right) = 3\sin 1 \approx 2.524$, while $\frac{3x}{2} - \frac{x^3}{16}$ evaluated at $x = 2$ gives 2.5, and $|2.524 - 2.5| = 0.024 < 0.1$.)

29. (a) To determine the Taylor polynomial, $P_4(x)$ centered at $x = 0$, begin with the Maclaurin expansion for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

Substitute $-\frac{\ln 2}{5600}x$ for x to obtain

$$\begin{aligned} e^{[(-\ln 2)/5600]x} &= 1 + \left(-\frac{\ln 2}{5600}x\right) + \frac{\left(-\frac{\ln 2}{5600}x\right)^2}{2!} + \frac{\left(-\frac{\ln 2}{5600}x\right)^3}{3!} + \frac{\left(-\frac{\ln 2}{5600}x\right)^4}{4!} + \ldots \\ &= 1 - \frac{\ln 2}{5600}x + \frac{1}{2}\left(\frac{\ln 2}{5600}\right)^2 x^2 - \frac{1}{6}\left(\frac{\ln 2}{5600}\right)^3 x^3 + \frac{1}{24}\left(\frac{\ln 2}{5600}\right)^4 x^4 + \ldots \\ P_4(x) &= 0.34\left[1 - \frac{\ln 2}{5600}x + \frac{1}{2}\left(\frac{\ln 2}{5600}\right)^2 x^2 - \frac{1}{6}\left(\frac{\ln 2}{5600}\right)^3 x^3 + \frac{1}{24}\left(\frac{\ln 2}{5600}\right)^4 x^4\right] \\ &= \boxed{0.34 - 0.34\left(\frac{\ln 2}{5600}\right)x + 0.17\left(\frac{\ln 2}{5600}\right)^2 x^2 - \frac{0.17}{3}\left(\frac{\ln 2}{5600}\right)^3 x^3 + \frac{0.17}{12}\left(\frac{\ln 2}{5600}\right)^4 x^4} \end{aligned}$$

(b)

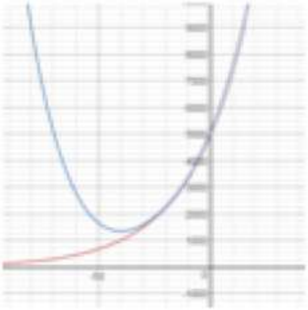


30. (a) To determine the Taylor polynomial, $P_4(x)$ centered at $x = 0$, begin with the Maclaurin expansion for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots.$

Substitute $0.04x$ for x to obtain

$$\begin{aligned} e^{0.04x} &= 1 + (0.04x) + \frac{(0.04x)^2}{2!} + \frac{(0.04x)^3}{3!} + \frac{(0.04x)^4}{4!} + \ldots \\ P_4(x) &= 5000\left[1 + (0.04x) + \frac{(0.04x)^2}{2!} + \frac{(0.04x)^3}{3!} + \frac{(0.04x)^4}{4!}\right] \\ &= \boxed{5000 + 200x + 4x^2 + \frac{4}{75}x^3 + \frac{1}{1875}x^4} \end{aligned}$$

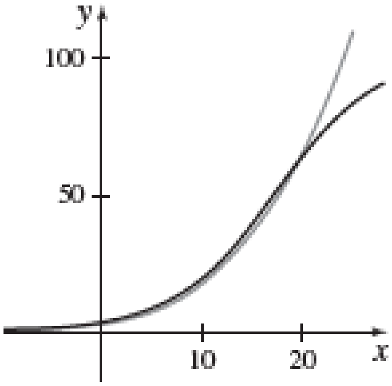
(b)



31. **(a)** To express $f(x) = \frac{100}{1+30.2e^{-0.2x}}$ as a Taylor expansion about $c = 0$, we use technology to obtain

$$P_3(x) = \boxed{3.205 + 0.6205x + 0.0581x^2 + 0.0034x^3}$$

(b)



32. **(a)** To express $f(x) = 100e^{-3e^{-0.2x}}$ as a Taylor expansion about $c = 0$, we first consider $f(x) = 100e^{-3e^{-0.2x}}$ and its derivatives at $c = 0$:

$$\begin{aligned}
f(x) &= 100e^{-3e^{-0.2x}} & f(0) &= 100e^{-3} \\
f'(x) &= 60e^{-3e^{-0.2x}-0.2x} & f'(0) &= 60e^{-3} \\
f''(x) &= 12\big(3 - e^{-0.2x}\big)e^{-3e^{-0.2x}-0.4x} & f''(0) &= 24e^{-3} \\
f'''(x) &= \frac{12}{5}\big(9 - 9e^{0.2x} + e^{0.4x}\big)e^{-3e^{-0.2x}-0.6x} & f'''(0) &= \frac{12e^{-3}}{5}
\end{aligned}$$

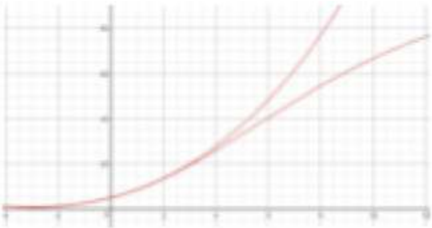
The Taylor Expansion of $f(x)$ centered at c is

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^k(c)}{k!}(x-c)^k \\
&= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots
\end{aligned}$$

Here,

$$\begin{aligned}
P_3(x) &= 100e^{-3} + 60e^{-3}(x) + \frac{24e^{-3}}{2!}(x)^2 + \frac{12e^{-3}}{5 \cdot 3!}(x)^3 \\
&= \frac{100}{e^3} + \frac{60x}{e^3} + \frac{12x^2}{e^3} + \frac{2x^3}{5e^3} \\
&= \boxed{4.979 + 2.988x + 0.597x^2 + 0.020x^3}
\end{aligned}$$

(b)



33. **(a)** From p.716, we write the Maclaurin series of $\sin x$ as

$$F(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots .$$

Since this series is a convergent alternating series, the error in using n terms of the series as an approximation is bounded from above by the absolute value of the $n+1$st term.
For $x = 4^\circ = 4^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{45}$, keeping just the first term in the Maclaurin expansion will result in an error that is less than absolute value of the second term, or $\left|-\frac{x^3}{3!}\right|$. At $x = 4^\circ = \frac{\pi}{45}$, the error bound will be

$$\left|-\frac{\left(\frac{\pi}{45}\right)^3}{3!}\right| \approx 0.00005 < 0.001.$$

So we just need the first term to give an answer correct to three decimal places:

$$\sin 4^\circ \approx \frac{\pi}{45} \approx \boxed{0.0698.}$$

(b) From p.717, we write the Maclaurin series of $\cos x$ as

$$F(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots .$$

Since the series is a convergent alternating series, the error in using n terms of the series as an approximation is bounded from above by the absolute value of the $n+1$st term.
For $x = 15^\circ = 15^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{12}$, keeping just the first two terms in the Maclaurin expansion will result in an error that is less than the absolute value of the third term, or $\left|\frac{x^4}{4!}\right|$. At $x = 15^\circ = \frac{\pi}{12}$, the error bound will be

$$\left|\frac{\left(\frac{\pi}{12}\right)^4}{4!}\right| \approx 0.0002 < 0.001.$$

So keeping the first two terms, we have an answer that is correct to three decimal places:

$$\cos 15^\circ \approx 1 - \frac{1}{2!}\left(\frac{\pi}{12}\right)^2 \approx \boxed{0.9659.}$$

(c) From p.708, using Gregory's series, we write the Maclaurin expansion of $\tan^{-1} x$ as

$$F(x) = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)} + \cdots .$$

Since the series is a convergent alternating series, the error in using n terms of the series as an approximation is bounded from above by the absolute value of the $n+1$st term.

For $x = 0.05$, the error in keeping just the first two terms in the Maclaurin expansion will result in an error that is less than the absolute value of the third term, or $\left|\frac{x^5}{5}\right|$. At $x = 0.05$, the error bound will be

$$\left|\frac{(0.05)^5}{5}\right| = 6.25 \times 10^{-8} < 0.001.$$

So keeping the first two terms, we get an answer correct to three decimal places:

$$\tan^{-1} 0.05 \approx 0.05 - \frac{(0.05)^3}{3} \approx \boxed{0.04496.}$$

34. **(a)** We have, using the standard result (or using the substitution $u = \tan x$, $du = \sec^2 x\, dx$)

$$\int_0^1 \frac{1}{1+x^2}\, dx = \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}.}$$

(b) Using the geometric series sum, provided that $|x| < 1$, we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty}(-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots + (-1)^n x^{2n} + \cdots .$$

Integrating term by term, we have, also using the result of part (a):

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2}\, dx = \frac{\pi}{4} &= \lim_{b\to 1^-} \int_0^b \left[1 - x^2 + x^4 - x^6 + x^8 - \cdots + (-1)^n x^{2n} + \cdots\right] dx \\ &= \lim_{b\to 1^-} \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots\right]_0^b \\ \text{or,} \quad & \boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^n}{2n+1} + \cdots}\end{aligned}$$

which is Leibniz's result.
(c) Keeping the first ten terms, we have

$$\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} \approx \boxed{0.7604}$$

giving $\pi \approx 4(0.7604) = 3.0416$. The actual value is $\pi \approx 3.1416$, rounded to four decimal places. So we see that the agreement is pretty poor, accurate to only about

$$\left|\frac{3.1416 - 3.0416}{3.1416}\right| \approx 3\%.$$

(d) To get an approximation of π accurate to the tenth decimal place, we need the error to be smaller than 10^{-10}. Since the series for $\frac{\pi}{4}$ is a convergent alternating series, the error in using n terms of the series as an approximation is bounded from above by the absolute value of the $n+1$st term. So we need, noting that to get the series for π, we need to multiply the series for $\frac{\pi}{4}$ by 4,

$$4 \cdot \left|\frac{(-1)^n}{2n+1}\right| \leq 10^{-10}$$

or $2n + 1 \geq 4 \cdot 10^{10}$, or $n \geq 2 \cdot 10^{10} - \frac{1}{2}$, or $\boxed{n \approx 2 \times 10^{10}}$ terms of the series.

35. Gregory's series is (see p. 708)

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots .$$

At $x = \frac{1}{2}$, we have, approximating with the first four terms of the series,

$$\tan^{-1}\left(\frac{1}{2}\right) \approx \frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 - \frac{1}{7}\left(\frac{1}{2}\right)^7 = \frac{6,229}{13,440}.$$

At $x = \frac{1}{3}$, we have, approximating with the first four terms of the series,

$$\tan^{-1}\left(\frac{1}{3}\right) \approx \frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 - \frac{1}{7}\left(\frac{1}{3}\right)^7 = \frac{24,628}{76,545}.$$

Since

$$\tan^{-1} 1 = \frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right),$$

we have

$$\begin{aligned} \pi &= 4\left[\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)\right] \\ &\approx 4\left[\frac{6,229}{13,440} + \frac{24,628}{76,545}\right] \\ &= \frac{1,538,665}{489,888} \\ &\approx \boxed{3.14085} \end{aligned}$$

an answer that is correct to three decimal places.

36. The general kinetic energy formula may be expanded using the Binomial Series as follows:

$$\begin{aligned} K_{\text{gen.}}(v) &= mc^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right) \\ &= mc^2\left(\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right) \\ &= mc^2\left(\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\left(-\frac{v^2}{c^2}\right)^k - 1\right) \\ &= mc^2\left[1 - \frac{1}{2}\left(-\frac{v^2}{c^2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(-\frac{v^2}{c^2}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\left(-\frac{v^2}{c^2}\right)^3 + \cdots - 1\right] \\ &= mc^2\left[\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right] \\ &= \frac{1}{2}mv^2 + \frac{3}{8}mv^2\left(\frac{v^2}{c^2}\right) + \frac{5}{16}mv^2\left(\frac{v^4}{c^4}\right) + \cdots \\ &= \frac{1}{2}mv^2\left[1 + \frac{3}{4}\left(\frac{v}{c}\right)^2 + \frac{5}{8}\left(\frac{v}{c}\right)^4 + \cdots\right] \\ &\approx \frac{1}{2}mv^2 \end{aligned}$$

if $\frac{v}{c} \to 0$. So we see that the general kinetic energy formula reduces to the classical one $K = \frac{1}{2}mv^2$ at speeds $v \to 0$, that is, for low speeds compared to the speed of light.

Challenge Problems

37. To express $f(x) = \sin x - \lambda x$ as a Taylor expansion about $c = \pi$, we first consider $f(x) = \sin x - \lambda x$ and its derivatives at $c = \pi$:

$$\begin{aligned} f(x) &= \sin x - \lambda x & f(\pi) &= -\lambda\pi \\ f'(x) &= \cos x - \lambda & f'(\pi) &= -1 - \lambda \\ f''(x) &= -\sin x & f''(\pi) &= 0 \end{aligned}$$

The Taylor expansion of $f(x)$ centered at c is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots \end{aligned}$$

Here,

$$\begin{aligned} P_2(x) &= -\lambda\pi + (-1-\lambda)(x-\pi) + \frac{0}{2!}(x-\pi)^2 \\ &= -\lambda\pi - x + \pi - \lambda x + \lambda\pi \\ &= \boxed{-(1+\lambda)x + \pi} \end{aligned}$$

$$\begin{aligned} \cot x &= \lambda x \\ \cot x - \lambda x &= 0 \\ f(x) &= 0 \\ P_2(x) &= 0 \\ -(1+\lambda)x + \pi &= 0 \\ (-1-\lambda)x &= -\pi \\ x &= \boxed{\frac{\pi}{1+\lambda}} \end{aligned}$$

38. To express $f(x) = \cot x - \lambda x$ as a Taylor expansion about $c = \frac{\pi}{2}$, we first consider $f(x) = \cot x - \lambda x$ and its derivatives at $c = \frac{\pi}{2}$:

$$\begin{aligned} f(x) &= \cot x - \lambda x & f\left(\frac{\pi}{2}\right) &= \frac{-\pi\lambda}{2} \\ f'(x) &= -\csc^2 x - \lambda & f'\left(\frac{\pi}{2}\right) &= -1 - \lambda \\ f''(x) &= -2\csc x(-\csc x \cot x) & & \\ &= \frac{2\cos x}{\sin^3 x} & f''\left(\frac{\pi}{2}\right) &= 0 \end{aligned}$$

The Taylor Expansion of $f(x)$ centered at c is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{k}(c)}{k!}(x-c)^k \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{n}(c)}{n!}(x-c)^n + \ldots \end{aligned}$$

Here,

$$\begin{aligned} P_2(x) &= \frac{-\pi\lambda}{2} + (-1-\lambda)\left(x - \frac{\pi}{2}\right) + \frac{0}{2!}\left(x - \frac{\pi}{2}\right)^2 \\ &= \frac{-\pi\lambda}{2} - x + \frac{\pi}{2} - \lambda x + \frac{\pi\lambda}{2} \\ &= \boxed{-x + \tfrac{\pi}{2} - \lambda x} \end{aligned}$$

$$\begin{aligned} \cot x &= \lambda x \\ \cot x - \lambda x &= 0 \\ f(x) &= 0 \\ P_2(x) &= 0 \\ -x + \frac{\pi}{2} - \lambda x &= 0 \\ (-1-\lambda)x &= -\frac{\pi}{2} \\ x &= \frac{-\frac{\pi}{2}}{-1-\lambda} \\ &= \boxed{\frac{\pi}{2(1+\lambda)}} \end{aligned}$$

39. (a) $a_k = (-1)^{k+1}\int_0^{\pi/k} \sin(kx)\,dx$. Set $kx = u$. Then $x = \frac{u}{k}$, and $dx = \frac{1}{k}\,du$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{k}$, $u = \pi$. So the integral becomes

$$\begin{aligned} a_k &= (-1)^{k+1}\frac{1}{k}\int_0^{\pi} \sin u\,du \\ &= \frac{(-1)^{k+1}}{k}\left[-\cos u\right]\Big|_0^{\pi} \\ &= \frac{(-1)^{k+1}}{k}[-\cos\pi - (-\cos 0)] \\ &= \boxed{\frac{2}{k}(-1)^{k+1}.} \end{aligned}$$

(b) The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{2}{k}(-1)^{k+1} = 2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ is a constant multiple of the alternating harmonic series, which converges, so it converges as well.

(c) On p.707-708, in Example 8 of Section 8.8, it is demonstrated that

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = \ln 2 \approx 0.693.$$

So we have

$$\sum_{k=1}^{\infty} a_k = 2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} \approx 2(0.693) = 1.386.$$

So we conclude that

$$\boxed{1 \le \sum_{k=1}^{\infty} a_k \le \frac{3}{2}.}$$

40. The Maclaurin expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

So the Maclaurin expansion of e^{x^3} is

$$e^{x^3} = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \cdots + \frac{x^{3n}}{n!} + \cdots .$$

The Maclaurin series of xe^{x^3} is given by

$$xe^{x^3} = x + x^4 + \frac{x^7}{2!} + \frac{x^{10}}{3!} + \cdots + \frac{x^{3n+1}}{n!} + \cdots = \boxed{\sum_{k=0}^{\infty} \frac{x^{3k+1}}{k!}.}$$

41. (a) Some terms of the Maclaurin expansion of $\sin^{-1} x$ were obtained in Example 11 of Section 8.9, p.721. To the ninth order in x, it is

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots .$$

So we have for $x = 1$,

$$\sin^{-1} 1 = \boxed{\frac{\pi}{2} = 1 + \frac{1}{6} + \frac{3}{40} + \frac{5}{112} + \frac{35}{1152} + \cdots}$$

and for $x = \frac{1}{2}$, we have

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} = \frac{1}{2} + \frac{1}{6}\left(\frac{1}{2}\right)^3 + \frac{3}{40}\left(\frac{1}{2}\right)^5 + \frac{5}{112}\left(\frac{1}{2}\right)^7 + \frac{35}{1152}\left(\frac{1}{2}\right)^9 + \cdots$$

$$\text{or,} \quad \boxed{\frac{\pi}{6} = \frac{1}{2} + \frac{1}{48} + \frac{3}{1280} + \frac{5}{14,336} + \frac{35}{589,824} + \cdots .}$$

(b) Evaluating the terms shown in the first expression, we get $\boxed{\frac{\pi}{2} \approx 1.571}$ correct to three decimal places.

(c) Evaluating the terms shown in the second expression, we get $\boxed{\frac{\pi}{6} \approx 0.524}$ correct to three decimal places.

AP® Practice Problems

1. For $y = \cos x$

$$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$P_4(0.2) = 1 - \frac{(0.2)^2}{2!} + \frac{(0.2)^4}{4!} \approx \boxed{0.980}$$

CHOICE B

2. $P_3(x) = 4 + 2(x-2) + 3(x-2)^2 + \frac{1}{2}(x-2)^3$

$$f'(x) = 2 + 6(x-2) + \frac{3}{2}(x-2)^2$$
$$f''(x) = 6 + 3(x-2)^2$$
$$f''(2) = 6 + 3(2-2)^2 = \boxed{6}$$

CHOICE D

3. $P_n(x) = \frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^n(c)}{n!}(x-c)^n.$

Determine $P_3(x)$ centered at -1:

$$\begin{aligned}P_3(x) &= \frac{f(-1)}{0!}(x+1)^0 + \frac{f'(-1)}{1!}(x+1)^1 + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3\\ &= 4 - 3(x+1) + \frac{3}{2!}(x+1)^2 + \frac{2}{3!}(x+1)^3\\ &= \boxed{4 - 3(x+1) + \frac{3}{2}(x+1)^2 + \frac{1}{3}(x+1)^3}\end{aligned}$$

CHOICE D

4. (a) Utilize the MacLaurin expansion for

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\begin{aligned}f(x) = e^{-x} &= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \ldots + \frac{(-x)^n}{n!} + \ldots\\ &= \boxed{1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots + \frac{(-x)^n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}}\end{aligned}$$

(b) $$\begin{aligned}e^{-0.1} &\approx 1 - (0.1) + \frac{(0.1)^2}{2!}\\ &= 1 - 0.1 + 0.005\\ &= \boxed{0.905}\end{aligned}$$

(c) Since the series is a convergent series of alternating terms, the absolute value of the error after the nth term is less than or equal to the absolute value of the $n+1$st term. By trial-and-error:

4th term ($n = 3$): $\left|\frac{(-1)^3(0.1)^3}{3!}\right| = 0.0001\bar{6} > 0.00001$

5th term ($n = 4$): $\left|\frac{(-1)^4(0.1)^4}{4!}\right| = 0.0000041\bar{6} < 0.00001$

Since the absolute value of the 5th term is less than the desired error, $\boxed{\text{4 terms}}$ are sufficient.

5. (a) $f(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$

Using the first four nonzero terms of the Maclaurin expansion for

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

we find

$$e^{-x^2/2} = 1 + \frac{-x^2}{2} + \frac{\left(\frac{-x^2}{2}\right)^2}{2!} + \frac{\left(\frac{-x^2}{2}\right)^3}{3!} = 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6$$

$$\boxed{\frac{1}{\sqrt{2\pi}}e^{-x^2/2} = \frac{1}{\sqrt{2\pi}}\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right)}$$

(b)
$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx &= \int_0^1 \frac{1}{\sqrt{2\pi}}\left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}\right)dx \\ &= \frac{1}{\sqrt{2\pi}}\int_0^1 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}\right)dx \\ &= \frac{1}{\sqrt{2\pi}}\left[x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336}\right]_0^1 \\ &= \frac{1}{\sqrt{2\pi}}\left[1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} - 0\right] \\ &= \boxed{\frac{1}{\sqrt{2\pi}}\cdot\frac{479}{560} \approx 0.341}\end{aligned}$$

(c) A bound on the error in the approximation of a convergent Alternating Series by using terms a_0 through a_n is given by $|E_n| \le a_{n+1}$.
Here,

$$\begin{aligned}|E_4| \le a_5 &= \int_0^1 \frac{1}{\sqrt{2\pi}}\cdot\frac{\left(\frac{x^2}{2}\right)^4}{4!}\,dx = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{2^4}\cdot\frac{1}{4!}\int_0^1 x^8\,dx \\ &= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{384}\left[\frac{x^9}{9}\right]_0^1 = \boxed{\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{3456} \approx 1.154\times 10^{-4}}\end{aligned}$$

(d) According to the Error Estimate for a Convergent Alternating Series theorem (p. 682), the error estimate given in Part (c) above is valid if the conditions of the Alternating Series Test (p. 680) are met, namely, (i) the series is an alternating series of the form $\sum\limits_{k=1}^{\infty}(-1)^{k+1}a_k$ or $\sum\limits_{k=1}^{\infty}(-1)^k a_k$; (ii) $\lim\limits_{n\to\infty} a_n = 0$; and (iii) the a_k are non-decreasing.

(i) The series is $\displaystyle\int_0^1\left[\sum_{k=0}^{\infty}\frac{1}{\sqrt{2\pi}}\cdot\frac{\left(-\frac{x^2}{2}\right)^k}{k!}\right]dx = \int_0^1\left[\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{\sqrt{2\pi}\cdot 2^k\cdot k!}\right]dx =$

$$\sum_{k=0}^{\infty}\left[\int_0^1 (-1)^k\frac{x^{2k}}{\sqrt{2\pi}\cdot 2^k\cdot k!}dx\right] = \sum_{k=0}^{\infty}\left[(-1)^k\frac{1}{2^k\cdot k!}\cdot\int_0^1 x^{2k}\,dx\right] =$$

$$\sum_{k=0}^{\infty}\left\{(-1)^k\frac{1}{\sqrt{2\pi}\cdot 2^k\cdot k!}\cdot\left[\frac{x^{2k+1}}{2k+1}\right]_0^1\right\} = \sum_{k=0}^{\infty}(-1)^k\frac{1}{\sqrt{2\pi}\cdot 2^k\cdot k!(2k+1)},$$

which can be rewritten $= \sum\limits_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt{2\pi}\cdot 2^{k-1}\cdot(k-1)![2(k-1)+1]}$ to show that it is the first type.

(ii) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{2\pi}\cdot 2^n \cdot n!(2n+1)} = 0.$

(iii) $$\frac{a_{k+1}}{a_k} = \frac{\frac{1}{\sqrt{2\pi}\cdot 2^{k+1}\cdot(k+1)![2(k+1)+1]}}{\frac{1}{\sqrt{2\pi}\cdot 2^k\cdot k!(2k+1)}} = \frac{\sqrt{2\pi}\cdot 2^k\cdot k!(2k+1)}{\sqrt{2\pi}\cdot 2^{k+1}\cdot(k+1)!(2k+2)}$$
$$= \frac{2k+1}{2(k+1)(2k+2)} = \frac{2k+1}{4k^2+8k+4}.$$

The terms are non-increasing if

$$\frac{a_{k+1}}{a_k} \le 1$$
$$\frac{2k+1}{4k^2+8k+4} \le 1$$
$$2k+1 \le 4k^2+8k+4$$
$$0 \le 4k^2+6k+3$$
$$0 \le 4\left(k^2+\frac{3}{2}k+\frac{9}{16}\right)+3-4\cdot\frac{9}{16}$$
$$0 \le 4\left(k+\frac{3}{2}\right)^2+\frac{3}{4}$$

which is true for all k.

The conditions are met, so the use of the error bound is valid.

(e) The error, E_n, of a convergent Alternating Series using n terms is numerically less than or equal to the $(n+1)st$ term of the series. (i.e., $|E_n| \le a_{n+1}$).

Here, $|E_n| < a_{n+1} \le a_5 = \frac{1}{\sqrt{2\pi}}\int_0^1 \frac{x^8}{384}\,dx$

If the numbers a_k, where $a_k > 0$ of an alternating series satisfy two conditions: $\lim\limits_{n\to\infty}(a_n) = 0$ and the a_k are nonincreasing, then the alternating series converges.

The series $\int_0^1 \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx = \int_0^1 \frac{1}{\sqrt{2\pi}}\left(1-\frac{x^2}{2}+\frac{x^4}{8}-\frac{x^6}{48}\right)dx =$ $\frac{1}{\sqrt{2\pi}}\left[x-\frac{x^3}{6}+\frac{x^5}{40}-\frac{x^7}{336}\right]_0^1$ is an alternating series, where

$$a_n = \frac{x^{2n+1}}{(2n+1)(2^n)(n!)}$$

For $\frac{1}{\sqrt{2\pi}}\left[1-\frac{1}{6}+\frac{1}{40}-\frac{1}{336}-0\right]$

$$a_n = \frac{1}{(2n+1)(2^n)(n!)}$$

We begin by confirming that $\lim\limits_{n\to\infty}(a_n) = \lim\limits_{n\to\infty}\frac{1}{(2n+1)(2^n)(n!)} = 0$. Next, using the Algebraic Ratio test, we verify that the terms $a_k = \frac{1}{(2n+1)(2^n)(n!)}$ are nonincreasing. Since

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(2(n+1)+1)(2^{n+1})(n+1)!}}{\frac{1}{(2n+1)(2^n)(n!)}} = \frac{(2n+1)(2^n)(n!)}{(2(n+1)+1)(2^{n+1})(n+1)!}$$
$$= \frac{(2n+1)(2^n)(n!)}{(2n+3)(2^n)(2)(n+1)(n!)} = \frac{(2n+1)}{(2n+3)(2)(n+1)} = \frac{1}{2}\left(\frac{2n+1}{2n+3}\right)\left(\frac{1}{n+1}\right)$$
$$= \frac{1}{2}\left(\frac{2+\frac{1}{n}}{2+\frac{3}{n}}\right)\left(\frac{1}{n+1}\right) < 1 \text{ for all } n \ge 1 \text{ the terms } a_k$$

are nonincreasing. By the Alternating Series Test, the series converges.

Chapter 8 Review Exercises

1. The nth term of the sequence is $s_n = \frac{(-1)^{n+1}}{n^4}$. Setting $n = 1, 2, 3, 4, 5$, we get the first five terms of the sequence to be

$$\begin{aligned}\{s_n\} &= \left\{\frac{(-1)^{1+1}}{1^4}, \frac{(-1)^{2+1}}{2^4}, \frac{(-1)^{3+1}}{3^4}, \frac{(-1)^{4+1}}{4^4}, \frac{(-1)^{5+1}}{5^4}, \cdots\right\}\\ &= \left\{1, -\frac{1}{2^4}, \frac{1}{3^4}, -\frac{1}{4^4}, \frac{1}{5^4}, \cdots\right\}.\end{aligned}$$

So the first five terms of the sequence are

$$\boxed{1, -\frac{1}{16}, \frac{1}{81}, -\frac{1}{256}, \frac{1}{625}.}$$

2. The nth term of the sequence is $s_n = \frac{2^n}{3^n}$. Setting $n = 1, 2, 3, 4, 5$, we get the first five terms of the sequence to be

$$\begin{aligned}\{s_n\} &= \left\{\frac{2^1}{3^1}, \frac{2^2}{3^2}, \frac{2^3}{3^3}, \frac{2^4}{3^4}, \frac{2^5}{3^5}, \cdots\right\}\\ &= \left\{\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \frac{32}{243}, \cdots\right\}.\end{aligned}$$

So the first five terms of the sequence are

$$\boxed{\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \frac{32}{243}.}$$

3. The terms of the sequence

$$2, -\frac{3}{2}, \frac{9}{8}, -\frac{27}{32}, \frac{81}{128}, \cdots$$

can be written as

$$(-1)^{1+1}\frac{3^0}{2^{0-1}}, (-1)^{2+1}\frac{3^1}{2^{2-1}}, (-1)^{3+1}\frac{3^2}{2^{4-1}}, (-1)^{4+1}\frac{3^3}{2^{6-1}}, (-1)^{5+1}\frac{3^4}{2^{8-1}}, \cdots .$$

So the nth term of the sequence is

$$\begin{aligned}s_n &= (-1)^{n+1}\frac{3^{n-1}}{2^{2n-3}}\\ &= (-1)^{n-1} \cdot (-1)^2 \cdot 2 \cdot \frac{3^{n-1}}{2^{2n-2}}\\ &= (-1)^{n-1}2 \cdot \frac{3^{n-1}}{4^{n-1}}\\ &= \boxed{2\left(-\frac{3}{4}\right)^{n-1}.}\end{aligned}$$

4. The sequence is $\{s_n\} = \left\{1 + \frac{n}{n^2+1}\right\}$. Using the sum and difference property of sequences, we have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 + \frac{n}{n^2+1}\right) = \lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{n}{n^2+1} = 1 + 0 = \boxed{1.}$$

5. The sequence is $\{s_n\} = \left\{\ln \frac{n+2}{n}\right\} = \{\ln b_n\}$, where $b_n = \frac{n+2}{n}$. The sequence $\{b_n\}$ converges to

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n+2}{n} = \lim_{n\to\infty}\left(1+\frac{2}{n}\right) = 1+0=1.$$

So by the theorem on p.636, setting $f(x) = \ln x$, which is continuous at $x = 1$ (the value of the limit of b_n), we conclude that the limit of the original sequence is

$$\ln(1) = \boxed{0.}$$

6. The sequence is $\{s_n\} = \{\tan^{-1} n\}$. Let $f(x) = \tan^{-1} x$ be a related function of the sequence. Since $f'(x) = \frac{1}{1+x^2} > 0$ for all $x \geq 1$, the function is monotonically increasing, which means the sequence is also monotonically increasing for $n \geq 1$. Also,

$$\tan^{-1} n \leq \frac{\pi}{2}$$

for all positive nonzero integers n, so the sequence $\{s_n\}$ is bounded from above. Since the sequence $\{s_n\}$ is an increasing sequence that is bounded from above, it $\boxed{\textit{converges}}$, and the limit of the sequence is

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \tan^{-1} n = \boxed{\frac{\pi}{2}.}$$

7. The sequence is $\{s_n\} = \left\{\frac{(-1)^n}{(n+1)^2}\right\}$. Since

$$-\frac{1}{(n+1)^2} \leq \frac{(-1)^n}{(n+1)^2} \leq \frac{1}{(n+1)^2},$$

and the sequences $\left\{-\frac{1}{(n+1)^2}\right\}$ and $\left\{\frac{1}{(n+1)^2}\right\}$ are both convergent, and converge to the limit of 0, since $\lim\limits_{n\to\infty} \frac{1}{(n+1)^2} = 0$, we have by the Squeeze Theorem of sequences that

$$\lim_{n\to\infty} s_n = \boxed{0.}$$

8. The sequence is $\{s_n\} = \left\{\frac{e^n}{(n+2)^2}\right\}$. To determine if the sequence is increasing or decreasing, apply the Algebraic Ratio Test:

$$\frac{s_{n+1}}{s_n} = \frac{\frac{e^{n+1}}{(n+3)^2}}{\frac{e^n}{(n+2)^2}} = e\left(\frac{n+2}{n+3}\right)^2.$$

We show that

$$2\left(\frac{n+2}{n+3}\right)^2 > 1.$$

Then this will mean that $e\left(\frac{n+2}{n+3}\right)^2 > 1$ as well, since $e > 2$. Expanding the desired inequality, we get

$$\begin{aligned}
2\left(\frac{n+2}{n+3}\right)^2 &> 1\\
2(n+2)^2 &> (n+3)^2\\
2(n^2+4n+4) &> n^2+6n+9\\
2n^2+8n+8 &> n^2+6n+9\\
n^2+2n-1 &> 0\\
n^2+2n+1-2 &> 0\\
(n+1)^2-2 &> 0\\
(n+1)^2 &> 2,
\end{aligned}$$

which is true for all $n \geq 1$. So this shows that the sequence is $\boxed{increasing}$ since we have shown that $\frac{s_{n+1}}{s_n} > 1$. Also the sequence is $\boxed{bounded\ from\ below}$ by its first terrm

$$s_1 = \frac{e}{(1+2)^2} = \frac{e}{9}.$$

It is $\boxed{not\ bounded\ from\ above}$ since

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{e^n}{(n+2)^2} = \lim_{x\to\infty} \frac{e^x}{(x+2)^2} = \lim_{x\to\infty} \frac{e^x}{2(x+2)} = \lim_{x\to\infty} \frac{e^x}{2} = \infty,$$

where we used a related function of the nth term of the sequence and applied L'Hôpital's rule to it. Since the sequence is increasing, and it is not bounded from above, it $\boxed{diverges.}$

9. The sequence $\{s_n\} = \{n!\}$ $\boxed{diverges}$ since $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} n! = \infty$.

10. The sequence $\{s_n\} = \left\{\left(\frac{5}{8}\right)^n\right\}$ is of the form $\{s_n\} = \{r^n\}$ where $-1 < r = \frac{5}{8} < 1$. Since $\lim\limits_{n\to\infty} r^n = 0$ for $-1 < r < 1$, we have $\lim\limits_{n\to\infty} s_n = 0$, that is, the sequence $\boxed{converges}$ to the limit of $\boxed{0.}$

11. The sequence $\{s_n\} = \left\{\left(-\frac{1}{2}\right)^n\right\}$ is of the form $\{s_n\} = \{r^n\}$ where $-1 < r = -\frac{1}{2} < 1$. Since $\lim\limits_{n\to\infty} r^n = 0$ for $-1 < r < 1$, we have $\lim\limits_{n\to\infty} s_n = 0$, that is, the sequence $\boxed{converges}$ to the limit of $\boxed{0.}$

12. The sequence is $\{s_n\} = \{(-1)^n + e^{-n}\}$. As $n \to \infty$, $\lim\limits_{n\to\infty} e^{-n} = 0$ but $\lim\limits_{n\to\infty} (-1)^n$ does not exist. Since $\lim\limits_{n\to\infty} s_n$ does not exist, the sequence $\boxed{diverges.}$

13. The sequence $\{s_n\} = \left\{1 + \frac{2}{n}\right\}$ is decreasing since using the Algebraic Difference Test, we have

$$s_{n+1} - s_n = 1 + \frac{2}{n+1} - 1 - \frac{2}{n} = 2\left(\frac{1}{n+1} - \frac{1}{n}\right) = -\frac{2}{n(n+1)} < 0$$

for $n \geq 1$. It's also bounded from below by 1, since $s_n = 1 + \frac{2}{n} > 1$ for all $n \geq 1$. Since the sequence is decreasing and bounded from below, it $\boxed{converges.}$

14. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(-1)^k}{4^{k-1}}$. The fifth partial sum of the series is

$$\begin{aligned} S_5 &= \sum_{k=1}^{5} \frac{(-1)^k}{4^{k-1}} \\ &= \frac{(-1)^1}{4^{1-1}} + \frac{(-1)^2}{4^{2-1}} + \frac{(-1)^3}{4^{3-1}} + \frac{(-1)^4}{4^{4-1}} + \frac{(-1)^5}{4^{5-1}} \\ &= -\frac{1}{4^0} + \frac{1}{4} - \frac{1}{4^2} + \frac{1}{4^3} - \frac{1}{4^4} \\ &= -1 + \frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \frac{1}{256} \\ &= \boxed{-\frac{205}{256}.} \end{aligned}$$

15. The telescoping series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \left(\frac{4}{k+4} - \frac{4}{k+5}\right)$. The nth partial sum of the series is

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \left(\frac{4}{k+4} - \frac{4}{k+5}\right) \\ &= \left(\frac{4}{1+4} - \frac{4}{1+5}\right) + \left(\frac{4}{2+4} - \frac{4}{2+5}\right) + \cdots + \left(\frac{4}{n-1+4} - \frac{4}{n-1+5}\right) + \left(\frac{4}{n+4} - \frac{4}{n+5}\right) \\ &= \left(\frac{4}{5} - \frac{4}{6}\right) + \left(\frac{4}{6} - \frac{4}{7}\right) + \cdots + \left(\frac{4}{n+3} - \frac{4}{n+4}\right) + \left(\frac{4}{n+4} - \frac{4}{n+5}\right) \\ &= \frac{4}{5} - \frac{4}{n+5}. \end{aligned}$$

The limit of the sequence of partial sums is

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{4}{5} - \frac{4}{n+5}\right) = \lim_{n\to\infty} \frac{4}{5} - \lim_{n\to\infty} \frac{4}{n+5} = \frac{4}{5} - 0 = \boxed{\frac{4}{5}}$$

is the sum of the telescoping series.

16. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\cos^2(k\pi)}{k}$. Since $\cos^2(n\pi) = 1$ for any value of n, we may write the series as $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k}$. But this is the harmonic series which $\boxed{\textit{diverges.}}$

17. The series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} -(\ln 2)^k = \sum\limits_{k=1}^{\infty} (-\ln 2)(\ln 2)^{k-1}$ is a geometric series of the form $\sum\limits_{k=1}^{\infty} ar^{k-1}$ where $a = (-\ln 2)$ and $r = \ln 2 \approx 0.693 < 1$. Since $|r| < 1$, the geometric series converges to a sum of

$$\frac{a}{1-r} = \frac{-\ln 2}{1-\ln 2} = \boxed{\frac{\ln 2}{\ln 2 - 1}.}$$

18. The series can be written $\sum\limits_{k=0}^{\infty} a_k = \sum\limits_{k=0}^{\infty} \frac{e}{3^k} = e + \sum\limits_{k=1}^{\infty} \left(\frac{e}{3}\right)\left(\frac{1}{3}\right)^{k-1}$. The second term is a geometric series of the form $\sum\limits_{k=1}^{\infty} ar^{k-1}$ where $a = \frac{e}{3}$ and $r = \frac{1}{3}$. Since $|r| < 1$, the geometric series converges to a sum of

$$\frac{a}{1-r} = \frac{\frac{e}{3}}{1-\frac{1}{3}} = \frac{e/3}{2/3} = \frac{e}{2}.$$

The original series then converges to a sum of $e + \frac{e}{2} = \boxed{\frac{3e}{2}.}$

19. The series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(4^{1/3}\right)^k = \sum_{k=1}^{\infty} 4^{1/3} \cdot \left(4^{1/3}\right)^{k-1}$ is of the form of a geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ where $a = 4^{1/3}$ and $r = 4^{1/3} > 1$. Since $|r| > 1$, the series $\boxed{\textit{diverges.}}$

20. The number $r = 0.123123123\cdots$ can be written and then summed into a rational number form, using the properties of convergent geometric series as shown below:

$$\begin{aligned} r &= 0.123123123123\cdots \\ &= 0.123 + 0.123 \times 10^{-3} + 0.123 \times 10^{-6} + 0.123 \times 10^{-9} + \cdots \\ &= 0.123 \left[1 + 10^{-3} + \left(10^{-3}\right)^2 + \left(10^{-3}\right)^3 + \cdots\right] \\ &= 0.123 \cdot \sum_{k=1}^{\infty} \left(10^{-3}\right)^{k-1} \\ &= 0.123 \cdot \frac{1}{1-10^{-3}} \\ &= 0.123 \cdot \frac{10^3}{10^3-1} = 0.123 \left(\frac{1000}{999}\right) = \frac{123}{999} = \boxed{\frac{41}{333}.} \end{aligned}$$

21. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{3k-2}{k}$. Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n-2}{n} - \lim_{n\to\infty} \left(3 - \frac{2}{n}\right) = \lim_{n\to\infty} 3 - \lim_{n\to\infty} \frac{2}{n} = 3 - 0 = 3 \neq 0,$$

the series $\boxed{\textit{diverges}}$ by the Divergence Test.

22. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\ln k}{k^2}$. A related function of the nth term of the series is $f(x) = \frac{\ln x}{x^2}$, which is continuous and decreasing on $[1, \infty)$, and for which $f(k) = a_k$ for all positive nonzero integers k. To see it is decreasing, compute $f'(x)$:

$$f'(x) = \frac{x^2 \cdot \frac{1}{x} - \ln x \cdot 2x}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3} < 0$$

for $x > \sqrt{e}$. So the series decreases for $n > 1$. By the Integral Test,

$$I = \int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^b f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx.$$

Let $u = \ln x$, $dv = \frac{dx}{x^2}$. Then $du = \frac{dx}{x}$, $v = -\frac{1}{x}$. So

$$\begin{aligned}
I &= \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \left[\left[\ln x \cdot -\frac{1}{x} \right]_1^b + \int_1^b \frac{dx}{x^2} \right] \\
&= \lim_{b\to\infty} \left[\left(-\frac{\ln b}{b} + \frac{\ln 1}{1} \right) + \left[-\frac{1}{x} \right]_1^b \right] \\
&= -\lim_{b\to\infty} \frac{\ln b}{b} + 0 + \lim_{b\to\infty} \left(-\frac{1}{b} + 1 \right) \\
&= -\lim_{b\to\infty} \frac{\frac{1}{b}}{1} + (-0 + 1) = -0 + (-0 + 1) = 1,
\end{aligned}$$

where L'Hôpital's rule was used on the first term in the final line. Since the limit is a positive real number, by the Integral Test, the series $\boxed{\textit{converges.}}$

23. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{4k^2+9}$. A related function of the nth term of the series is $f(x) = \frac{1}{4x^2+9}$, which is continuous and decreasing on $[1, \infty)$, and for which $f(k) = a_k$ for all positive nonzero integers k. To apply the integral test, we have to evaluate

$$I = \int_1^\infty f(x)\,dx = \int_1^\infty \frac{dx}{4x^2+9}.$$

Let $x = \frac{3}{2}\tan\theta$. Then $dx = \frac{3}{2}\sec^2\theta\,d\theta$. When $x = 1$, $\tan\theta = \frac{2}{3}$, or $\theta = \tan^{-1}\frac{2}{3}$. As $n \to \infty$, $\theta \to \frac{\pi}{2}$. So we have

$$\begin{aligned}
I &= \int_{\tan^{-1}\frac{2}{3}}^{\frac{\pi}{2}} \frac{\frac{3}{2}\sec^2\theta\,d\theta}{4\left(\frac{9}{4}\tan^2\theta\right)+9} = \frac{3}{2\cdot 9} \int_{\tan^{-1}\frac{2}{3}}^{\frac{\pi}{2}} \frac{\sec^2\theta\,d\theta}{1+\tan^2\theta} \\
&= \frac{1}{6} \int_{\tan^{-1}\frac{2}{3}}^{\frac{\pi}{2}} \frac{\sec^2\theta\,d\theta}{\sec^2\theta} = \frac{1}{6}[\theta]\Big|_{\tan^{-1}\frac{2}{3}}^{\frac{\pi}{2}} \\
&= \frac{1}{6}\left[\frac{\pi}{2} - \tan^{-1}\frac{2}{3} \right].
\end{aligned}$$

Since $\tan^{-1} x < \frac{\pi}{2}$ for any x, the integral is a positive real number, so by the Integral Test, the series $\boxed{\textit{converges.}}$

24. The p-series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^{5/2}}$ $\boxed{\textit{converges}}$ since $p = \frac{5}{2} > 1$. The bounds for the sum for a general convergent p-series are

$$\frac{1}{p-1} < \sum_{k=1}^{\infty} \frac{1}{k^p} < 1 + \frac{1}{p-1},$$

so for this particular p-series they will be

$$\left(\frac{1}{\frac{5}{2}-1}, 1 + \frac{1}{\frac{5}{2}-1} \right) = \left(\frac{2}{3}, \frac{5}{3} \right).$$

So the series has a $\boxed{\textit{lower bound } \frac{2}{3}}$ and $\boxed{\textit{upper bound } \frac{5}{3}.}$

25. The series is $\sum\limits_{k=5}^{\infty} a_k = \sum\limits_{k=5}^{\infty} \left[\frac{1}{k^5} \cdot \frac{1}{2^k}\right]$. We have

$$a_n = \frac{1}{n^5} \cdot \frac{1}{2^n} < \frac{1}{n^5} = b_n$$

for all $n \geq 5$. Since the series $\sum\limits_{k=5}^{\infty} b_k = \sum\limits_{k=5}^{\infty} \frac{1}{k^5}$ is a convergent p-series (since $p = 5 > 1$) from the fifth term forward, and since the omission of a finite number of terms does not affect convergence or divergence properties of a p-series, the original series $\sum\limits_{k=5}^{\infty} a_k$ also $\boxed{\textit{converges}}$ by the Comparison Test.

26. The series $\sum\limits_{k=1}^{\infty} \left[\frac{3}{5^k} - \left(\frac{2}{3}\right)^{k-1}\right] = \sum\limits_{k=1}^{\infty} \left[\frac{3}{5} \cdot \left(\frac{1}{5}\right)^{k-1} - \left(\frac{2}{3}\right)^{k-1}\right]$ can be written as a difference of two geometric series, $\sum\limits_{k=1}^{\infty} \frac{3}{5} \cdot \left(\frac{1}{5}\right)^{k-1} - \sum\limits_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k-1}$each of which converges absolutely (since $|r| = \frac{1}{5} < 1$ and $|r| = \frac{2}{3} < 1$), so the original series also $\boxed{\textit{converges.}}$

27. The series $\sum\limits_{k=1}^{\infty} \frac{3}{k^5}$ is a constant multiple of a convergent p-series (since $p = 5 > 1$), so it also $\boxed{\textit{converges.}}$

28. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k+1}}$. The nth term of the series satisfies

$$a_n = \frac{1}{\sqrt{n+1}} > \frac{1}{n+1} = b_n$$

for $n \geq 1$ since $n + 1 > \sqrt{n+1}$ for $n \geq 1$. But the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{3} \cdots + \frac{1}{n} + \cdots$ is a harmonic series from the second term forward, and since the convergence or divergence properties of a series is not affected by the omission of a finite number of terms, the series $\sum\limits_{k=1}^{\infty} b_k$ diverges, and so we see that the original series $\boxed{\textit{diverges}}$ as well by the Comparison Test.

29. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k+1}{k^{k+1}}$. The nth term of the series behaves like

$$a_n = \frac{n+1}{n^{n+1}} = \frac{n+1}{n} \cdot \frac{1}{n^n} = \left(1 + \frac{1}{n}\right) \cdot \frac{1}{n^n} \approx \frac{1}{n^n} = b_n$$

for large values of n. We evaluate

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+1}{n^{n+1}}}{\frac{1}{n^n}} = \lim_{n\to\infty} (n+1) \cdot \left(\frac{n^n}{n^{n+1}}\right) = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$$

Since the limit is a positive real number and the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^k}$ is the convergent "k-to-the-k" series, the original series $\sum\limits_{k=1}^{\infty} a_k$ is also $\boxed{\textit{convergent}}$ by the Limit Comparison Test.

30. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{4}{k\,3^k}$. The nth term of the series satisfies

$$a_n = \frac{4}{n\,3^n} \leq \frac{4}{3^n} = b_n$$

for $n \geq 1$. Since the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{4}{3^n} = \sum\limits_{k=1}^{\infty} \frac{4}{3}\left(\frac{1}{3}\right)^{k-1}$ is a convergent geometric series (since $|r| = \frac{1}{3} < 1$), the original series $\boxed{\textit{converges}}$ by the Comparison Test.

31. The series is $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k = \sum\limits_{k=1}^{\infty} (-1)^{k+1} \frac{k+2}{k(k+1)}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+2}{n(n+1)} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{n\left(1+\frac{1}{n}\right)} = 0.$$

By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+3}{(n+1)(n+2)}}{\frac{n+2}{n(n+1)}} = \frac{n(n+3)}{(n+2)^2} = \frac{n^2+3n}{n^2+4n+4} < 1.$$

So by the Alternating Series Test, the series $\boxed{\textit{converges.}}$
To find the sum of the series to an accuracy of 0.001, we first estimate how many terms we must keep in the summation of the series. The error in keeping the first $n-1$ terms of a convergent alternating series is bounded from above by the absolute value of the nth term; so we require $\frac{n+2}{n(n+1)} = 0.001$. This is equivalent to

$$\begin{aligned} \frac{n+2}{n(n+1)} &= \frac{1}{1000} \\ 1000n + 2000 &= n^2 + n \\ n^2 - 999n - 2000 &= 0. \end{aligned}$$

Solving the quadratic equation, we find that the smallest number greater than the positive solution is $n = 1001$. Carrying out the sum of the first 1000 terms of the series (since $n - 1 = 1001 - 1 = 1000$), we get

$$\sum_{k=1}^{1000} (-1)^{k+1} \frac{k+2}{k(k+1)} \approx \boxed{1.079}$$

which is correct to three decimal places.

32. The series is $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k = \sum\limits_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{e^k}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{e^n} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0,$$

using L'Hôpital's rule repeatedly on a related function of the nth term. By the Algebraic Ratio Test,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} = \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{e} < 1,$$

if $n + 1 < \sqrt{e}n$ or $(\sqrt{e} - 1)n > 1$, or $n > \frac{1}{\sqrt{e}-1} \approx 1.54$. So for $n \geq 2$, the series decreases. Since the convergence property does not depend on the behavior of a finite number of terms, we conclude that by the Algebraic Ratio Test, the series converges, even if its sum begins at $n = 1$.

To find the sum of the series to an accuracy of 0.001, we first estimate how many terms we must keep in the summation of the series. The error in keeping the first $n-1$ terms of a convergent alternating series is bounded from above by the absolute value of the nth term; so we require $\frac{n^2}{e^n} = 0.001$, or $e^n = 1000n^2$. By a process of trial and error, we find that $n \approx 12$. Carrying out the sum of the first 11 terms of the series (since $n-1 = 12-1 = 11$) we get

$$\sum_{k=1}^{11} (-1)^{k+1} \frac{k^2}{e^k} \approx \boxed{0.091}$$

which is correct to three decimal places.

33. The series is $\sum\limits_{k=1}^{\infty} (-1)^k a_k = \sum\limits_{k=1}^{\infty} (-1)^k \frac{3}{\sqrt[3]{k}}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin \frac{3}{\sqrt[3]{n}} = 0.$$

By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{3}{\sqrt[3]{n+1}}}{\frac{3}{\sqrt[3]{n}}} = \sqrt[3]{\frac{n}{n+1}} < 1,$$

for $n \geq 1$. So, by the Alternating Series Test, the series converges. This series cannot be summed by exact methods, so we have to approximate the sum by using the error estimate. The error in keeping the first $n-1$ terms of a convergent alternating series is bounded from above by the absolute value of the nth term; so we require

$$a_n = \frac{3}{\sqrt[3]{n}} = 10^{-3} \text{ or } \sqrt[3]{n} = 3 \times 10^3 \text{ or } n = 27 \times 10^9.$$

Using a CAS, we do the computation to these many terms, and obtain the sum of approximately $\boxed{-1.715}$, an answer that is correct to three decimal places. (The exact answer is $3(2^{2/3} - 1)\zeta\left(\frac{1}{3}\right)$, where $\zeta(s)$ is the Riemann zeta function introduced in Problem 85, Section 8.3, p.670.)

34. The series is $\sum\limits_{k=1}^{\infty} \sin\left(\frac{\pi}{2}k\right)$. The sequence of partial sums has the form $\{S_n\} = \{1, 0, -1, 0, 1, \cdots\}$. It does not converge, so the series $\boxed{\textit{diverges}}$.

35. The series is $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

By the Algebraic Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \frac{\sqrt{n}}{\sqrt{n+1}} < 1$$

for all $n \geq 1$. So, by the Alternating Series Test, the series converges. The series of absolute values $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is a p-series with $0 < p = \frac{1}{2} < 1$, so it is divergent. Since the original series converges, but the series of absolute values diverges, the original series $\boxed{\textit{converges conditionally.}}$

36. The series is $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\cos k}{k^3}$. The series of absolute values is $\sum\limits_{k=1}^{\infty} |a_k| = \sum\limits_{k=1}^{\infty} \frac{|\cos k|}{k^3}$. The nth term satisfies

$$|a_n| = \frac{|\cos n|}{n^3} \leq \frac{1}{n^3} = b_n$$

for all $n \geq 1$. Since the series $\sum\limits_{k=1}^{\infty} b_k = \sum\limits_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series (since $p = 3 > 1$), the series of absolute values converges by the Comparison Test, and so the original series is $\boxed{\textit{absolutely convergent.}}$

37. The nth term of the series

$$\frac{1}{2} - \frac{4}{2^3+1} + \frac{9}{3^3+1} - \frac{16}{4^3+1} + \cdots$$

is

$$a_n = (-1)^{n+1} \frac{n^2}{n^3+1}.$$

It satisfies

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^2}{n^3+1} = 0.$$

By the Algebraic Ratio Test,

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{\frac{(n+1)^2}{(n+1)^3+1}}{\frac{n^2}{n^3+1}} = \frac{(n+1)^2}{n^2} \cdot \frac{(n^3+1)}{[(n+1)^3+1]} \\ &= \frac{(n^2+2n+1)(n^3+1)}{n^2[(n^3+3n^2+3n+1)+1]} = \frac{n^5+2n^4+n^3+n^2+2n+1}{n^5+3n^4+3n^3+2n^2} < 1, \end{aligned}$$

provided we have

$$\begin{aligned} n^5+2n^4+n^3+n^2+2n+1 &< n^5+3n^4+3n^3+2n^2 \\ \text{or,} \qquad 2n+1 &< n^4+2n^3+n^2 \\ \text{or,} \qquad 2n+1 &< n^2(n^2+2n+1) \end{aligned}$$

which is true for all $n \geq 1$. So the series of absolute values decreases as well. By the Alternating Series Test, the original series converges. The nth term of the series of absolute values $\sum\limits_{k=1}^{\infty} |a_k| = \sum\limits_{k=1}^{\infty} \frac{k^2}{k^3+1}$ behaves for large values of n like

$$|a_n| = \frac{n^2}{n^3+1} = \frac{n^2}{n^3\left(1+\frac{1}{n^3}\right)} \approx \frac{1}{n} = |b_n|.$$

Comparing with the divergent harmonic series $\sum\limits_{k=1}^{\infty} |b_k| = \sum\limits_{k=1}^{\infty} \frac{1}{k}$, we get

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \frac{\frac{n^2}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1.$$

Since the limit is a positive real number and the series $\sum_{k=1}^{\infty} |b_k|$ is divergent, it means that the series of absolute values $\sum_{k=1}^{\infty} |a_k|$ is also divergent by the Limit Comparison Test. Since the original series converges, but the series of absolute values diverges, the original series is $\boxed{\textit{conditionally convergent.}}$

38. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{2^k}{k!}$ is a series of nonzero terms. By the Ratio Test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)n!}\right| = 2 \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

Since the limit is less than 1, the series $\boxed{\textit{converges}}$ by the Ratio Test.

39. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k!}{e^{k^2}}$ is a series of nonzero terms. By the Ratio Test,

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{(n+1)!}{e^{(n+1)^2}}}{\frac{n!}{e^{n^2}}}\right| = \lim_{n\to\infty} \left|\frac{e^{n^2}}{e^{(n^2+2n+1)}} \cdot \frac{(n+1)n!}{n!}\right| \\ &= \lim_{n\to\infty} \frac{n+1}{e^{2n+1}} = \lim_{x\to\infty} \frac{x+1}{e^{2x+1}} = \lim_{x\to\infty} \frac{1}{2e^{2x+1}} \\ &= 0,\end{aligned}$$

applying L'Hôpital's rule to a related function of the ratio. Since the limit is less than 1, by the Ratio Test, we conclude that the series $\boxed{\textit{converges.}}$

40. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{2^k}{(k+3)^{k+1}}$ is a series of nonzero terms. By the Root Test,

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\left|\frac{2^n}{(n+3)^{n+1}}\right|} = 2 \lim_{n\to\infty} \frac{1}{(n+3)^{(n+1)/n}} = 2 \lim_{n\to\infty} \frac{1}{(n+3)(n+3)^{1/n}} \\ &= 2 \lim_{n\to\infty} \frac{1}{n+3} \cdot \lim_{n\to\infty} \frac{1}{n^{1/n}} \cdot \lim_{n\to\infty} \frac{1}{\left(1+\frac{3}{n}\right)^{1/n}} \\ &= 2 \cdot 0 \cdot 1 \cdot \frac{1}{(1+0)^0} \\ &= 0,\end{aligned}$$

where we have used the limit result $\lim_{n\to\infty} n^{1/n} = 1$ (see solutions to Exercise 43 of Section 8.6 for a proof). Since the limit is less than 1, by the Root Test, the series $\boxed{\textit{converges.}}$

41. We use the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left|\left[\ln\left(e^4 + \frac{1}{n^2}\right)\right]^n\right|} = \lim_{n\to\infty} \ln\left(e^4 + \frac{1}{n^2}\right) = \ln e^4 = 4 > 1$$

Therefore the series $\boxed{\textit{diverges}}$.

42. The series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2^{k+1}}{3^k} = \sum_{k=1}^{\infty}\frac{(-2)^{k+1}}{3^k} = \sum_{k=1}^{\infty}\frac{(-2)^2}{3}\cdot\left(-\frac{2}{3}\right)^{k-1} = \sum_{k=1}^{\infty}\frac{4}{3}\left(-\frac{2}{3}\right)^{k-1}$ is a geometric series of the form $\sum_{k=1}^{\infty} ar^{k-1}$ with $a = \frac{4}{3}$ and $r = -\frac{2}{3}$. Since $0 < |r| < 1$, the series $\boxed{\textit{converges.}}$

43. The series can be written $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k}\right) = \sum_{k=1}^{\infty}\ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{\infty}(\ln(k+1)-\ln k)$. The nth partial sum is

$$\begin{aligned}S_n &= \sum_{k=1}^{n}(\ln(k+1)-\ln k)\\ &= (\ln 2-\ln 1)+(\ln 3-\ln 2)+\cdots+(\ln n-\ln(n-1))+(\ln(n+1)-\ln n)\\ &= \ln(n+1)-\ln 1 = \ln(n+1)-0\\ &= \ln(n+1).\end{aligned}$$

Since

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\ln(n+1) = \infty,$$

the sequence of partial sums $\{S_n\}$ diverges, which means that the series $\boxed{\textit{diverges}}$ as well.

44. The series is $\sum_{k=5}^{\infty} a_k = \sum_{k=5}^{\infty}\frac{3}{k\sqrt{k-4}}$. The nth term satisfies

$$a_n = \frac{3}{n\sqrt{n-4}} < \frac{3}{(n-4)\sqrt{n-4}} = \frac{3}{(n-4)^{3/2}} = b_n,$$

for all $n \geq 5$. Consider the series $\sum_{k=5}^{\infty} b_k = \sum_{k=5}^{\infty}\frac{3}{(k-4)^{3/2}}$. Setting $k-4 = k'$, we get $\sum_{k'=1}^{\infty}\frac{3}{k'^{3/2}}$. This is a constant multiple of a convergent p-series (since $p = \frac{3}{2} > 1$), so the series $\sum_{k=5}^{\infty} b_k$ converges. By the Comparison Test, this means that the original series $\sum_{k=5}^{\infty} a_k$ $\boxed{\textit{converges}}$ as well.

45. $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\frac{1}{\left(1+\frac{k^2+1}{k^2}\right)^k}$ is a series of nonzero terms. Using the Root Test,

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{1}{\left(1+\frac{n^2+1}{n^2}\right)^n}\right|} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{n^2+1}{n^2}\right)} = \lim_{n\to\infty}\frac{1}{1+\left(1+\frac{1}{n^2}\right)} = \frac{1}{1+(1+0)} = \frac{1}{2} < 1.$$

Since the limit is less than 1, the series $\boxed{\textit{converges}}$ by the Root Test.

46. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}\frac{2\cdot 4\cdot 6\cdots(2k)}{1\cdot 3\cdot 5\cdots(2k-1)}$. We can write the nth term as

$$a_n = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)} = \frac{(2\cdot 4\cdot 6\cdots(2n))^2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-1)(2n)} = \frac{[2^n\cdot n!]^2}{(2n)!} = \frac{2^{2n}(n!)^2}{(2n)!}.$$

The Ratio Test can be used since $a_n \neq 0$ for all $n \geq 1$. We have

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{2^{2(n+1)}((n+1)!)^2}{(2(n+1))!}}{\frac{2^{2n}(n!)^2}{(2n)!}}\right| \\
&= \lim_{n\to\infty}\left|\frac{2^{2n+2}}{2^{2n}}\cdot\left(\frac{(n+1)n!}{n!}\right)^2\cdot\frac{(2n)!}{(2n+2)(2n+1)(2n)!}\right| \\
&= 2^2\lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right| \\
&= 4\lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n}\right)^2}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}\right| \\
&= 4\cdot\left|\frac{1+0}{(2+0)(2+0)}\right| = 4\cdot\frac{1}{4} = 1.
\end{aligned}$$

Since the limit is equal to 1, the Ratio Test is inconclusive. However, observe that using

$$\left|\frac{a_{n+1}}{a_n}\right| = 4\left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right|$$

we can show that $\left|\frac{a_{n+1}}{a_n}\right| > 1$. This happens when

$$\begin{aligned}
4\left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right| &> 1 \\
4(n+1)^2 &> (2n+2)(2n+1) \\
4(n^2+2n+1) &> 4n^2+4n+2n+2 \\
4n^2+8n+4 &> 4n^2+6n+2 \\
2n+4 &> 2 \\
\text{or,}\qquad 2n &> -2,
\end{aligned}$$

which is true for all $n \geq 1$. So the terms of the series are increasing. Since the first term, $a_1 = \frac{2^2(1!)}{2!} = 2$, and the terms of the series are increasing, it would mean that

$$\lim_{n\to\infty} a_n > a_1 = 2 \neq 0.$$

So the series $\boxed{\textit{diverges}}$ by the Divergence Test.

47. The series is $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^2}{(1+k^3)\ln(\sqrt[3]{1+k^3})}$. Consider a related function of the nth term of the series,

$$f(x) = \frac{x^2}{(1+x^3)\ln(\sqrt[3]{1+x^3})}.$$

This is continuous and defined on $[1,\infty)$, decreasing, and such that $f(k) = a_k$ for all $k \geq 1$. (To see that it is decreasing for $[\sqrt[3]{2},\infty)$ we can use the following argument:

$$f'(x) = -\frac{[3x^4+(x^3-2)x\ln(1+x^3)]}{(1+x^3)^2(\ln(1+x^3))^2}.$$

Since denominator is always positive being the product of two squared terms, the sign of $f'(x)$ is negative if the numerator is positive. (Note the minus sign outside.) The numerator is positive at least for $x^3 - 2 \geq 0$ or $x \geq \sqrt[3]{2} \approx 1.26$.) the Using the Integral Test, we compute

$$I = \int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{x^2\,dx}{(1+x^3)\ln(\sqrt[3]{1+x^3})}.$$

Let $u = 1 + x^3$. Then $du = 3x^2\,dx$. When $x = 1$, $u = 2$. When $x = b$, $u = 1 + b^3$. The integral becomes

$$I = \lim_{b\to\infty} \int_2^{1+b^3} \frac{\frac{1}{3}\,du}{\frac{1}{3}u\ln u} = \lim_{b\to\infty} \int_2^{1+b^3} \frac{du}{u\ln u}.$$

Let $v = \ln u$. Then $dv = \frac{du}{u}$. When $u = 2$, $v = \ln 2$. When $u = 1 + b^3$, $v = \ln(1 + b^3)$. So we have

$$I = \lim_{b\to\infty} \int_{\ln 2}^{\ln(1+b^3)} \frac{dv}{v} = \lim_{b\to\infty} [\ln|v|]\Big|_{\ln 2}^{\ln(1+b^3)} = \lim_{b\to\infty} [\ln(\ln(1+b^3)) - \ln(\ln 2))] = \infty.$$

Since the limit of the improper integral is not finite, by the Integral Test, the series $\boxed{\textit{diverges.}}$

48. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k^{10}}{2^k}$ is a series of nonzero terms. By the Ratio Test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{n^{10+1}}{2^{n+1}}}{\frac{n^{10}}{2^n}}\right| = \lim_{n\to\infty} \left|\frac{2^n}{2\cdot 2^n} \cdot \frac{n^{10}\cdot n}{n^{10}}\right| = \frac{1}{2}\lim_{n\to\infty} n = \infty.$$

Since the limit is ∞, by the Ratio Test, the series $\boxed{\textit{diverges.}}$

49. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{\left(1+\frac{1}{k^2}\right)^{k^2}}{2^k}$ is a series of nonzero terms. Using the Root Test,

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\left|\frac{\left(1+\frac{1}{n^2}\right)^{n^2}}{2^n}\right|} = \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2/n} \\
&= \frac{1}{2}\left[\lim_{n^2\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2}\right]^{\lim\limits_{n\to\infty} 1/n} \\
&= \frac{1}{2}\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right]^{0} \\
&= \frac{1}{2}[e]^0 = \frac{1}{2} < 1,
\end{aligned}$$

where we used the substitution $m = n^2$ and the standard result that $\lim\limits_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e$. Since the limit is less than 1, by the Ratio Test, the series $\boxed{\textit{converges.}}$

50. $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \left(\frac{k^2+1}{k}\right)^k$ is a series of nonzero terms. Using the Root Test,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n^2+1}{n}\right)^n\right|} = \lim_{n\to\infty}\left|\frac{n^2+1}{n}\right| = \infty.$$

Since the limit is ∞, by the Root Test, the series $\boxed{\textit{diverges.}}$

51. The series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{k!}{3\,k^k}$ is a series of nonzero terms. Using the Ratio Test,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{3\,(n+1)^{n+1}}}{\frac{n!}{3\,n^n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)n!}{n!}\cdot\frac{n^n}{(n+1)(n+1)^n}\right| = \lim_{n\to\infty}\left(\frac{n}{1+n}\right)^n \\
&= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n} \\
&= \frac{1}{e} < 1,
\end{aligned}$$

since $e > 1$, where we have used the standard result that $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$. Since the limit is less than 1, by the Ratio Test, the series $\boxed{\textit{converges.}}$

52. The series is $\sum\limits_{k=1}^{\infty}(-1)^{k+1}a_k = \sum\limits_{k=1}^{\infty}(-1)^{k+1}\frac{k+2}{3k-2}$. The absolute value of the nth term satisfies

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n+2}{3n-2} = \lim_{n\to\infty}\frac{1+\frac{2}{n}}{3-\frac{2}{n}} = \frac{1}{3} \neq 0.$$

Since the limit is nonzero, the series $\boxed{\textit{diverges}}$ by the Divergence Test.

53. $\sum\limits_{k=2}^{\infty}\frac{6}{k(\ln k)^3}$ is a series whose terms are positive.

To find bounds on the sum we must determine whether $f(x) = \frac{6}{x(\ln x)^3}$ is a continuous, positive, decreasing function.

First, $f(x) = \frac{6}{x(\ln x)^3}$ is a positive and continuous function for all $x > 1$.

Next, find $f'(x)$:

$$f'(x) = -\frac{6}{x^2(\ln x)^3} - \frac{18}{x^2(\ln x)^4} < 0 \quad \text{for all } x > 1,$$

So $f(x) = \frac{6}{x(\ln x)^3}$ is also a decreasing function on that interval.

Next, find $\int_2^{\infty}\frac{6}{x(\ln x)^3}\,dx$:

$$\begin{aligned}
\int_2^{\infty}\frac{6}{x(\ln x)^3}\,dx &= \lim_{b\to\infty}\int_2^b\frac{6}{x(\ln x)^3}\,dx = \lim_{b\to\infty}\left[\frac{-3}{(\ln x)^2}\right]_2^b \\
&= \lim_{b\to\infty}\left[\frac{-3}{(\ln b)^2}+\frac{3}{(\ln 2)^2}\right] = \frac{3}{(\ln 2)^2}
\end{aligned}$$

Since $\int_2^{\infty}\frac{6}{x(\ln x)^3}\,dx$ converges and $a_1 = \frac{3}{(\ln 2)^3}$, then

$$\int_2^{\infty}\frac{6}{x(\ln x)^3}\,dx < \sum_{k=2}^{\infty}\frac{6}{k(\ln k)^3} < a_1 + \int_2^{\infty}\frac{6}{x(\ln x)^3}\,dx$$

$$\frac{3}{(\ln 2)^2} < \sum_{k=2}^{\infty}\frac{6}{k(\ln k)^3} < \frac{3}{(\ln 2)^3} + \frac{3}{(\ln 2)^2}$$

Therefore

$$\boxed{6.244 < \sum_{k=2}^{\infty}\frac{6}{k(\ln k)^3} < 15.252}$$

54. $\sum\limits_{k=1}^{\infty} \frac{1}{k^{4/3}}$ is a convergent p-series for which the bounds are

$$\frac{1}{\frac{4}{3}-1} < \sum_{k=1}^{\infty} \frac{1}{k^{4/3}} < 1 + \frac{1}{\frac{4}{3}-1}$$
$$3 < \sum_{k=1}^{\infty} \frac{1}{k^{4/3}} < 4$$

Therefore

$$\boxed{6 < \sum_{k=1}^{\infty} \frac{2}{k^{4/3}} < 8}$$

55. (a) The power series $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} \frac{(x-3)^{3k-1}}{k^2}$ is a series of nonzero terms if $x \neq 3$. (If $x = 3$, the series sums to 0.) Applying the Ratio Test,

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left| \frac{\frac{(x-3)^{3(n+1)-1}}{(n+1)^2}}{\frac{(x-3)^{3n-1}}{n^2}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(x-3)^{3n-1+3}}{(x-3)^{3n-1}} \cdot \frac{n^2}{(n+1)^2} \right| = |x-3|^3 \left(\lim_{n\to\infty} \frac{n}{n+1} \right)^2 \\
&= |x-3|^3 \left(\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \right)^2 = |x-3|^3 \cdot \left(\frac{1}{1+0} \right)^2 \\
&= |x-3|^3 < 1
\end{aligned}$$

for convergence of the series by the Ratio Test. So the power series converges in the interval $-1 < x - 3 < 1$, or $2 < x < 4$. The radius of convergence is $\boxed{R = 1}$.

(b) At $x = 2$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(2-3)^{3k-1}}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^{3k-1}}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^{3k+3}(-1)^{-4}}{k^2} = \sum_{k=1}^{\infty} \frac{[(-1)^3]^{k+1}}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}.$$

The series of absolute values $\sum\limits_{k=1}^{\infty} |a_k| = \sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series (since $p = 2 > 1$), so the original series is absolutely convergent, hence converges. So the power series converges at $x = 2$.
At $x = 4$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(4-3)^{3k-1}}{k^2} = \sum_{k=1}^{\infty} \frac{(1)^{3k-1}}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

which is a convergent p-series (since $p = 2 > 1$). So the power series converges at $x = 4$.
Concluding, the interval of convergence of the power series is $\boxed{2 \leq x \leq 4.}$

56. **(a)** The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{x^k}{\sqrt[3]{k}}$ is a series of nonzero terms if $x \neq 0$. (If $x = 0$, the series sums to 0.) By the Ratio Test,

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{\sqrt[3]{n+1}}}{\frac{x^n}{\sqrt[3]{n}}}\right| \\ &= |x| \lim_{n\to\infty} \sqrt[3]{\frac{n}{n+1}} = |x| \sqrt[3]{\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)} \\ &= |x| \sqrt[3]{\frac{1}{1+0}} = |x| < 1\end{aligned}$$

for convergence of the series by the Ratio Test. So the power series converges in the interval $-1 < x < 1$, and the radius of convergence is $\boxed{R = 1}$.

(b) At $x = -1$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k}}.$$

The absolute value of the nth term satisfies

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0.$$

By the Algebraic Ratio Test, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{\sqrt[3]{n+1}}}{\frac{1}{\sqrt[3]{n}}}\right| = \sqrt[3]{\frac{n}{n+1}} < 1$$

for all $n \geq 1$. So the absolute value of the terms of the series are decreasing. By the Alterbating Series Test, the series converges. So the power series converges at $x = -1$. At $x = 1$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(1)^k}{\sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/3}}.$$

This is a p-series with $0 < p = \frac{1}{3} < 1$, so it diverges. So the power series diverges at $x = 1$. Concluding, the interval of convergence of the power series is $\boxed{-1 \leq x < 1.}$

57. **(a)** The power series $\sum_{k=0}^{\infty} (-1)^k a_k = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\,(k+1)} \left(\frac{x}{2}\right)^{2k+1}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) Using the Ratio Test,

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!\,(n+2)}\left(\frac{x}{2}\right)^{2(n+1)+1}}{\frac{1}{n!\,(n+1)}\left(\frac{x}{2}\right)^{2n+1}}\right| \\ &= \left|\frac{x}{2}\right|^2 \lim_{n\to\infty}\left|\frac{n+1}{n+2}\cdot\frac{n!}{(n+1)n!}\right| = \left|\frac{x}{2}\right|^2 \cdot \lim_{n\to\infty}\frac{1}{n+2} \\ &= 0\end{aligned}$$

for any value of x. So the radius of convergence of the power series is $\boxed{R = \infty}$.

(b) Since the series converges for any value of x, the interval of convergence of the series is $\boxed{-\infty < x < \infty.}$

58. (a) The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k^k}{(k!)^2} x^k$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 0.) Using the Ratio Test, we have

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{(n+1)^{n+1}x^{n+1}}{((n+1)!)^2}}{\frac{n^n x^n}{(n!)^2}} \right| \\ &= |x| \lim_{n\to\infty} \left| \frac{(n+1)(n+1)^n}{n^n} \cdot \left(\frac{n!}{(n+1)n!} \right)^2 \right| \\ &= |x| \lim_{n\to\infty} \left| \left(\frac{n+1}{n} \right)^n \cdot \frac{1}{n+1} \right| \\ &= |x| \cdot \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n \cdot \lim_{n\to\infty} \frac{1}{n+1} \\ &= |x| \cdot e \cdot 0 = 0, \end{aligned}$$

where the standard limit result $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$ has been used. Since the limit is less than 1 for any value of x, the radius of convergence of the power series is $\boxed{R = \infty.}$

(b) Since the series converges for any value of x, the interval of convergence of the series is $\boxed{-\infty < x < \infty.}$

59. (a) The power series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(x-1)^k}{k}$ is a series of nonzero terms for $x \neq 1$. (At $x = 1$, the series converges to 0.) Using the Ratio Test, we get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-1)^{n+1}}{n+1}}{\frac{(x-1)^n}{n}} \right| = |x-1| \lim_{n\to\infty} \left(\frac{n}{n+1} \right) = |x-1| \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}} \right) = |x-1| \cdot \frac{1}{1+0} = |x-1| < 1$$

for convergence of the series by the Ratio Test. So the power series converges in the interval $-1 < x - 1 < 1$, or $0 < x < 2$, and the radius of convergence is $\boxed{R = 1.}$

(b) At $x = 0$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(0-1)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$$

which, being a constant multiple of the alternating harmonic series (namely the alternating harmonic series multiplied by -1), converges. So the power series converges for $x = 0$.
At $x = 2$, the series becomes

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{(2-1)^k}{k} = \sum_{k=1}^{\infty} \frac{(1)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

which is the divergent harmonic series. So the power series diverges for $x = 2$.
Concluding, the interval of convergence of the power series is $\boxed{0 \leq x < 2.}$

60. (a) The power series $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{3^k x^k}{5^k}$ is a series of nonzero terms for $x \neq 0$. (At $x = 0$, the series converges to 1.) By the Ratio Test, we get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{3^{n+1}x^{n+1}}{5^{n+1}}}{\frac{3^n x^n}{5^n}} \right| = \frac{3}{5}|x| < 1$$

for convergence of the series by the Ratio Test. So $|x| < \frac{5}{3}$, which means the power series converges in the interval $-\frac{5}{3} < x < \frac{5}{3}$, and its radius of convergence is $\boxed{R = \dfrac{5}{3}.}$

(b) At $x = -\frac{5}{3}$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{3^k}{5^k} \cdot \left(-\frac{5}{3}\right)^k = \sum_{k=0}^{\infty} (-1)^k.$$

The sequence of partial sums of this series is $\{S_n\} = \{1, 0, 1, 0, \cdots\}$ which does not converge, and this means the series does not converge either. So the power series diverges for $x = -\frac{5}{3}$. At $x = \frac{5}{3}$, the series becomes

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \frac{3^k}{5^k} \cdot \left(\frac{5}{3}\right)^k = \sum_{k=0}^{\infty} 1$$

which diverges. So the power series diverges for $x = \frac{5}{3}$.

Concluding, the interval of convergence of the power series is $\boxed{-\dfrac{5}{3} < x < \dfrac{5}{3}.}$

61. To express the function $f(x) = \frac{2}{x+3}$ as a power series centered at 0, we use the following function respresented as a geometric series: $g(x) = \frac{1}{1-x} = \sum\limits_{k=0}^{\infty} x^k$ for $-1 < x < 1$. So,

$$\begin{aligned} f(x) = \frac{2}{3+x} &= \frac{2}{3}\left(\frac{1}{1+\frac{x}{3}}\right) = \frac{2}{3}\left[\frac{1}{1-(-\frac{x}{3})}\right] \\ &= \frac{2}{3}\sum_{k=0}^{\infty}\left(-\frac{x}{3}\right)^k = \boxed{\frac{2}{3}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{3}\right)^k}, \end{aligned}$$

which converges for $\left|\frac{x}{3}\right| < 1$ or $-1 < \frac{x}{3} < 1$, or $-3 < x < 3$.

62. To express the function $f(x) = \frac{1}{1-3x}$ as a power series centered at 0, we use the following function represented as a geometric series: $g(x) = \frac{1}{1-x} = \sum\limits_{k=0}^{\infty} x^k$ for $-1 < x < 1$. So we have

$$f(x) = \frac{1}{1-3x} = g(3x) = \sum_{k=1}^{\infty}(3x)^k = \boxed{\sum_{k=0}^{\infty} 3^k x^k},$$

which converges for $|3x| < 1$ or $-1 < 3x < 1$ or $-\frac{1}{3} < x < \frac{1}{3}$.

63. **(a)** From Problem 60 above, the series centered about 0 for

$$\frac{1}{1-3x^2} = f(x^2) = \sum_{k=0}^{\infty} 3^k (x^2)^k = 1 + 3x^2 + 9x^4 + 27x^6 + \cdots + 3^n x^{2n} + \cdots.$$

Integrating both sides we get

$$\begin{aligned} \int_0^x \frac{1}{1-3t^2}\,dt &= \int_0^x \left[1 + 3t^2 + 9t^4 + 27t^6 + \cdots + 3^n t^{2n} + \cdots\right] dt \\ &= \boxed{x + x^3 + \frac{9}{5}x^5 + \frac{27}{7}x^7 + \cdots + \frac{3^n}{2n+1}x^{2n+1} + \cdots.} \end{aligned}$$

(b) To ascertain the correct number of terms to keep, we calculate the n for which the contribution to the sum becomes less than 0.001, for $x = \frac{1}{2}$. So, setting $x = \frac{1}{2}$ in the nth term, we get

$$\frac{3^n}{2n+1}\left(\frac{1}{2}\right)^{2n+1} \leq 0.001.$$

Proceeding by trial and error, we find that $n = 11$. So we have

$$\int_0^{1/2} \frac{1}{1-3x^2}\,dx \approx \sum_{k=0}^{11} \frac{3^k}{2k+1}\left(\frac{1}{2}\right)^{2k+1} \approx \boxed{0.758}$$

an answer that is correct to three decimal places.

64. To find the Taylor expansion of $f(x) = \frac{1}{1-2x}$ about $c = 1$, we begin by evaluating the function and its derivatives at $x = 1$.

$$\begin{aligned} f(x) &= \frac{1}{1-2x} & f(1) &= \frac{1}{1-2\cdot 1} = -1 \\ f'(x) &= \frac{2\cdot 1}{(1-2x)^2} & f'(1) &= \frac{2\cdot 1}{(1-2\cdot 1)^2} = 2\cdot 1! \\ f''(x) &= \frac{2^2\cdot 2\cdot 1}{(1-2x)^3} & f''(1) &= \frac{2^2\cdot 2!}{(1-2\cdot 1)^3} = -2^2\cdot 2! \\ f'''(x) &= \frac{2^3\cdot 3\cdot 2\cdot 1}{(1-2x)^4} & f'''(1) &= \frac{2^3\cdot 3!}{(1-2\cdot 1)^4} = 2^3\cdot 3! \\ &\vdots & &\vdots \end{aligned}$$

The Taylor expansion of the function about $c = 1$ is

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\ &= \frac{-1}{0!}(x-1)^0 + \frac{2\cdot 1!}{1!}(x-1)^1 - \frac{2^2\cdot 2!}{2!}(x-1)^2 + \frac{2^3\cdot 3!}{3!}(x-1)^3 - \cdots + (-1)^{n+1}\frac{2^n\cdot n!}{n!}(x-1)^n + \cdots \\ &= -1 + 2(x-1) - 2^2(x-1)^2 + 2^3(x-1)^3 - \cdots + (-1)^{n+1}2^n(x-1)^n + \cdots \\ &= \boxed{\sum_{k=0}^{\infty}(-1)^{k+1}2^k(x-1)^k.} \end{aligned}$$

65. To find the Taylor expansion of $f(x) = e^{x/2}$ about $c = 1$, we begin by evaluating the function and its derivatives at $x = 1$.

$$\begin{aligned} f(x) &= e^{x/2} & f(1) &= e^{1/2} \\ f'(x) &= \frac{1}{2}e^{x/2} & f'(1) &= \frac{1}{2}e^{1/2} \\ f''(x) &= \frac{1}{2^2}e^{x/2} & f''(1) &= \frac{1}{2^2}e^{1/2} \\ f'''(x) &= \frac{1}{2^3}e^{x/2} & f'''(1) &= \frac{1}{2^3}e^{1/2} \\ &\vdots & &\vdots \end{aligned}$$

The Taylor expansion of the function about $c = 1$ is

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\
&= \frac{1}{0!}e^{1/2}(x-1)^0 + \frac{1}{1!\,2}e^{1/2}(x-1)^1 + \frac{1}{2!\,2^2}e^{1/2}(x-1)^2 + \frac{1}{3!\,2^3}e^{1/2}(x-1)^3 + \cdots \\
&\qquad + \frac{1}{n!\,2^n}e^{1/2}(x-1)^n + \cdots \\
&= \boxed{\sum_{k=0}^{\infty} \frac{1}{k!\,2^k}e^{1/2}(x-1)^k.}
\end{aligned}$$

66. To find the Maclaurin expansion of $f(x) = 2x^3 - 3x^2 + x + 5$, we begin by evaluating the function and its derivatives at $x = 0$.

$$\begin{aligned}
f(x) &= 2x^3 - 3x^2 + x + 5 & f(0) &= 2(0)^3 - 3(0)^2 + 0 + 5 = 5 \\
f'(x) &= 6x^2 - 6x + 1 & f'(0) &= 6(0)^2 - 6(0) + 1 = 1 \\
f''(x) &= 12x - 6 & f''(0) &= -6 \\
f'''(x) &= 12 & f'''(0) &= 12 \\
f^{(4)}(x) &= 0 & f^{(4)}(0) &= 0.
\end{aligned}$$

The higher derivatives and their evaluations at $x = 0$ are all 0 as well. So the Maclaurin series is

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k \\
&= \frac{5}{0!}x^0 + \frac{1}{1!}x^1 + \frac{-6}{2!}x^2 + \frac{12}{3!}x^3 \\
&= 5 + x - 3x^2 + 2x^3 \\
&= \boxed{2x^3 - 3x^2 + x - 5.}
\end{aligned}$$

We see that the Maclaurin expansion of the function is equivalent to the function itself.

67. To find the Taylor series for $f(x) = \tan x$ about $c = \frac{\pi}{4}$, we begin by evaluating the function and its derivatives at $x = \frac{\pi}{4}$. Let $y = f(x)$, $y_1 = f\left(\frac{\pi}{4}\right)$, $y' = f'(x)$, $y_1' = f'\left(\frac{\pi}{4}\right)$, etc.

$$\begin{aligned}
y &= \tan x & y_1 &= \tan\left(\frac{\pi}{4}\right) = 1 \\
y' &= \sec^2 x = 1 + \tan^2 x = 1 + y^2 & y_1' &= 1 + y_1^2 = 1 + (1)^2 = 2 \\
y'' &= 2yy' & y_1'' &= 2y_1y_1' = 2(1)(2) = 4 \\
y''' &= 2y'^2 + 2yy'' & y_1''' &= 2y_1'^2 + 2y_1y_1'' \\
& & &= 2(2)^2 + 2(1)(4) = 16 \\
y^{(4)} &= 4y'y'' + 2y'y'' + 2yy''' & & \\
&= 6y'y'' + 2yy''' & y_1^{(4)} &= 6y_1'y_1'' + 2y_1y_1''' \\
& & &= 6(2)(4) + 2(1)(16) = 80 \\
y^{(5)} &= 6y''^2 + 6y'y''' + 2y'y''' + 2yy^{(4)} & & \\
&= 6y''^2 + 8y'y''' + 2yy^{(4)} & y_1^{(5)} &= 6y_1''^2 + 8y_1'y_1''' + 2y_1y_1^{(4)} \\
& & &= 6(4)^2 + 8(2)(16) + 2(1)(80) = 512 \\
&\vdots & &\vdots
\end{aligned}$$

The Taylor expansion of the function about $c = \frac{\pi}{4}$ is

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\
&= 1 + \frac{2}{1!}\left(x-\frac{\pi}{4}\right)^1 + \frac{4}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x-\frac{\pi}{4}\right)^3 + \frac{80}{4!}\left(x-\frac{\pi}{4}\right)^4 + \frac{512}{5!}\left(x-\frac{\pi}{4}\right)^5 + \cdots \\
&= \boxed{1 + 2\left(x-\frac{\pi}{4}\right) + 2\left(x-\frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x-\frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x-\frac{\pi}{4}\right)^4 + \frac{64}{15}\left(x-\frac{\pi}{4}\right)^5 + \cdots}
\end{aligned}
$$

68. The Maclaurin expansion of $f(x) = e^{-x}\sin x$ is obtained by multiplying the Maclaurin expansions of e^{-x} and $\sin x$ together. From Example 2, p.716 of Section 8.9, we get the Maclaurin expansion

$$g(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots .$$

So we have

$$g(-x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^n}{n!} + \cdots .$$

From p.717 of Section 8.9 the book, we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots .$$

So the first five terms of the Maclaurin series of $f(x)$ are

$$
\begin{aligned}
f(x) &= e^{-x}\sin x \\
&= \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots + (-1)^n\frac{x^n}{n!} + \cdots\right)\cdot\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots\right) \\
&= \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots\right)\cdot\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\
&= 1\cdot\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) - x\cdot\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \frac{x^2}{2}\cdot\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\
&\quad - \frac{x^3}{6}\cdot\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \frac{x^4}{24}\cdot\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) - \frac{x^5}{120}\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \cdots \\
&= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) + \left(-x^2 + \frac{x^4}{6} - \frac{x^6}{120} + \cdots\right) + \left(\frac{x^3}{2} - \frac{x^5}{12} + \frac{x^7}{240} - \cdots\right) \\
&\quad + \left(-\frac{x^4}{6} + \frac{x^6}{36} - \frac{x^8}{720} + \cdots\right) + \left(\frac{x^5}{24} - \frac{x^7}{144} + \frac{x^9}{2880} - \cdots\right) + \left(-\frac{x^6}{120} + \cdots\right) + \cdots \\
&= x - x^2 + x^3\left(\frac{1}{2} - \frac{1}{6}\right) + x^4\left(\frac{1}{6} - \frac{1}{6}\right) + x^5\left(\frac{1}{120} + \frac{1}{24} - \frac{1}{12}\right) + x^6\left(-\frac{1}{120} + \frac{1}{36} - \frac{1}{120}\right) - \cdots \\
&= \boxed{x - x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \frac{1}{90}x^6 - \cdots .}
\end{aligned}
$$

69. Since $(1+x)^m = \sum_{k=0}^{\infty}\binom{m}{k}x^k$, the binomial series of $f(x) = \frac{1}{(x+1)^4} = (1+x)^{-4}$ is given by

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty}\binom{-4}{k}x^k \\
&= \binom{-4}{0}x^0 + \binom{-4}{1}x^1 + \binom{-4}{2}x^2 + \binom{-4}{3}x^3 + \cdots + \binom{-4}{n}x^n + \cdots \\
&= 1 - 4x + \frac{(-4)(-4-1)}{2!}x^2 + \frac{(-4)(-4-1)(-4-2)}{3!}x^3 + \cdots \\
&= \boxed{1 - 4x + 10x^2 - 20x^3 + \cdots .}
\end{aligned}$$

Since $m = -4$ satisfies $m \leq -1$, by the conditions of the theorem on p.620, the binomial series of $\frac{1}{(x+1)^4}$ converges on the open interval $\boxed{(-1,1).}$

70. Since $(1+x)^m = \sum_{k=0}^{\infty}\binom{m}{k}x^k$, the binomial series of
$f(x) = \sqrt[3]{x^2-1} = (-1)^{1/3}(1-x^2)^{1/3} = -1(1-x^2)^{1/3}$ is given by

$$\begin{aligned}
f(x) &= -\sum_{k=0}^{\infty}\binom{\frac{1}{3}}{k}(-x^2)^k \\
&= -\left[\binom{\frac{1}{3}}{0}(-x^2)^0 + \binom{\frac{1}{3}}{1}(-x^2)^1 + \binom{\frac{1}{3}}{2}(-x^2)^2 + \binom{\frac{1}{3}}{3}(-x^2)^3 + \cdots + \binom{\frac{1}{3}}{n}(-x^2)^n + \cdots\right] \\
&= -\left[1 - \frac{1}{3}x^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^4 - \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^6 + \cdots\right] \\
&= \boxed{-1 + \frac{1}{3}x^2 + \frac{1}{9}x^4 + \frac{5}{81}x^6 + \cdots .}
\end{aligned}$$

Since $m = \frac{1}{3}$ satisfies $m > 0$ but m is not an integer, by the conditions of the theorem on p.620, the series converges on the closed interval $\boxed{[-1,1].}$

71. Since $(1+x)^m = \sum_{k=0}^{\infty}\binom{m}{k}x^k$, the binomial expansion of $f(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{-1/2}$ is given by

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}(-1)^k x^k \\
&= \binom{-\frac{1}{2}}{0}(-1)^0x^0 + \binom{-\frac{1}{2}}{1}(-1)^1x^1 + \binom{-\frac{1}{2}}{2}(-1)^2x^2 + \binom{-\frac{1}{2}}{3}(-1)^3x^3 + \cdots \\
&= 1 - \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 - \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3 + \cdots \\
&= \boxed{1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots .}
\end{aligned}$$

Since $m = -\frac{1}{2}$ satisfies $-1 < m < 0$, by the conditions of the theorem on p.620, the series converges on the half open interval $\boxed{(-1,1].}$

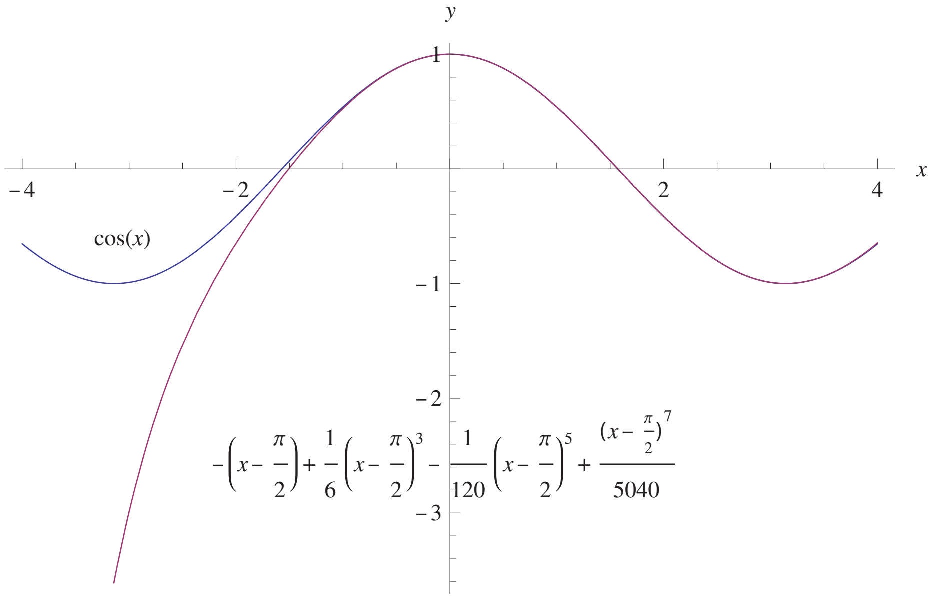
72. (a) To find the Taylor expansion to four nonzero terms of $y = \cos x$ about $x = \frac{\pi}{2}$, we compute the function and its derivatives at $x = \frac{\pi}{2}$ until four nonzero values result.

$$\begin{aligned}
y &= \cos x & \qquad y\left(\frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{2}\right) = 0 \\
y'(x) &= -\sin x & y'\left(\frac{\pi}{2}\right) &= -\sin\left(\frac{\pi}{2}\right) = \boxed{-1} \\
y''(x) &= -\cos x & y''\left(\frac{\pi}{2}\right) &= -\cos\left(\frac{\pi}{2}\right) = 0 \\
y'''(x) &= \sin x & y'''\left(\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2}\right) = \boxed{1} \\
y^{(4)}(x) &= \cos x & y^{(4)}\left(\frac{\pi}{2}\right) &= \cos\left(\frac{\pi}{2}\right) = 0 \\
y^{(5)} &= -\sin x & y^{(5)}\left(\frac{\pi}{2}\right) &= -\sin\left(\frac{\pi}{2}\right) = \boxed{-1} \\
y^{(6)}(x) &= -\cos x & y^{(6)}\left(\frac{\pi}{2}\right) &= -\cos\left(\frac{\pi}{2}\right) = 0 \\
y^{(7)}(x) &= \sin x & y^{(7)}\left(\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2}\right) = \boxed{1} \\
&\vdots & &\vdots
\end{aligned}$$

So the Taylor expansion to four nonzero terms of $y = \cos x$ about $c = \frac{\pi}{2}$ is given by

$$\begin{aligned}
y = f(x) = \cos x &= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \\
&= \frac{1}{1!}\left(x-\frac{\pi}{2}\right)^1 + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5 + \frac{1}{7!}\left(x-\frac{\pi}{2}\right)^7 - \cdots \\
&= \boxed{-\left(x-\frac{\pi}{2}\right) + \frac{1}{6}\left(x-\frac{\pi}{2}\right)^3 - \frac{1}{120}\left(x-\frac{\pi}{2}\right)^5 + \frac{1}{5040}\left(x-\frac{\pi}{2}\right)^7 - \cdots}
\end{aligned}$$

(b) The figure below shows the graph of both the function and its Maclaurin approximation found above in part (a).



(c) Set $x = 88° = \frac{22}{45}\pi$. Then $\left(x - \frac{\pi}{2}\right) = \left(\frac{22}{45}\pi - \frac{\pi}{2}\right) = -\frac{\pi}{90}$. So

$$\begin{aligned}\cos(88°) &\approx -\left(-\frac{\pi}{90}\right) + \frac{1}{6}\left(-\frac{\pi}{90}\right)^3 - \frac{1}{120}\left(-\frac{\pi}{90}\right)^5 + \frac{1}{5040}\left(-\frac{\pi}{90}\right)^7 \\ &\approx \boxed{0.0349.}\end{aligned}$$

(d) Since the series is a converging alternating one, the error in using this approximation is bounded from above by the absolute value of the next nonzero term, which is

$$\left|\frac{1}{9!}\left(x - \frac{\pi}{2}\right)^9\right|.$$

Evaluating this error bound for $x = 88°$, we have

$$\text{error} \le \left|\frac{1}{9!}\left(-\frac{\pi}{90}\right)^9\right| \approx \boxed{2.12 \times 10^{-19}.}$$

(e) We need to determine the x values for which the error term would be less than or equal to 0.0001, that is, we need

$$\begin{aligned}\left|\frac{1}{9!}\left(x - \frac{\pi}{2}\right)^9\right| &\le 0.0001 \\ -36.288 \le \left(x - \frac{\pi}{2}\right)^9 &\le 36.288 \\ -1.490414 \lessapprox x - \frac{\pi}{2} &\lessapprox 1.490414 \\ -1.490414 + \frac{\pi}{2} \lessapprox x &\lessapprox 1.490414 + \frac{\pi}{2} \\ 0.08038 \lessapprox x &\lessapprox 3.061210 \\ 4.6° \lessapprox x &\lessapprox 175.4°.\end{aligned}$$

So the x values for which the error in the approximation would be less than or equal to 0.0001 are approximately $\boxed{4.6° \le x \le 175.4°.}$

73. The Maclaurin series for $f(x) = e^x$ is given by (see p.714) by

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

To ensure the accuracy of the approximation of $e^{0.3}$ to three decimal places, let us find the number of terms we need to keep:

$$\frac{x^n}{n!} = 0.001$$

gives, by trial and error, $n = 4$. Keeping the first five nonzero terms of the expansion, we have

$$\begin{aligned}e^{0.3} &\approx 1 + 0.3 + \frac{(0.3)^2}{2!} + \frac{(0.3)^3}{3!} + \frac{(0.3)^4}{4!} \\ &\approx 1 + 0.3 + 0.045 + 0.0045 + 0.00033 \\ &\approx \boxed{1.34983}\end{aligned}$$

The exact value is $e^{0.3} \approx 1.34986$ so the value found using the first five terms is accurate to at least three decimal places.

74. The integrand in $\int_0^{1/2} \frac{1}{\sqrt{1-x^3}}\,dx$ can be expressed as a binomial series:

$$\begin{aligned}\frac{1}{\sqrt{1-x^3}} &= (1-x^3)^{-1/2} = \sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}(-x^3)^k\\ &= 1-\frac{1}{2}(-x^3)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-x^3)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}(-x^3)^3+\cdots\\ &\quad+\binom{-\frac{1}{2}}{n}(-x^3)^n+\cdots\\ &\approx 1+\frac{1}{2}x^3+\frac{3}{8}x^6+\frac{5}{16}x^9\end{aligned}$$

which is the integrand expanded to four nonzero terms. Integrating both sides between the limits, we have

$$\begin{aligned}\int_0^{1/2}\frac{1}{\sqrt{1-x^3}}\,dx &\approx \int_0^{1/2}\left[1+\frac{1}{2}x^3+\frac{3}{8}x^6+\frac{5}{16}x^9\right]dx\\ &= \left[x+\frac{1}{2\cdot 4}x^4+\frac{3}{8\cdot 7}x^7+\frac{5}{16\cdot 10}x^{10}\right]_0^{1/2}\\ &= (0.5)+\frac{1}{8}(0.5)^4+\frac{3}{56}(0.5)^7+\frac{1}{32}(0.5)^{10}\\ &\approx \boxed{0.5083.}\end{aligned}$$

75. The Maclaurin expansion of e^x is (see p.714)

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots.$$

So the Maclaurin expansion of e^{x^2} is

$$e^{x^2} = 1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\cdots+\frac{x^{2n}}{n!}+\cdots.$$

The integrand is expanded to four nonzero terms for the approximation:

$$\begin{aligned}\int_0^{1/2} e^{x^2}\,dx &\approx \int_0^{1/2}\left[1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}\right]dx\\ &= \left[x+\frac{x^3}{3}+\frac{x^5}{2!\,5}+\frac{x^7}{3!\,7}\right]_0^{1/2}\\ &= 0.5+\frac{(0.5)^3}{3}+\frac{(0.5)^5}{10}+\frac{(0.5)^7}{42}\\ &\approx \boxed{0.545.}\end{aligned}$$

AP® Review Problems

1. $P_n(x) = \frac{f(c)(x-c)^0}{0!}+\frac{f'(c)(x-c)^1}{1!}+\frac{f''(c)(x-c)^2}{2!}+\frac{f'''(c)(x-c)^3}{3!}\cdots+\frac{f^n(c)(x-c)^n}{n!}+\cdots$

Determine $P_4(x)$ centered at 2:

$$\begin{aligned} P_4(x) &= \frac{f(2)(x-2)^0}{0!} + \frac{f'(2)(x-2)^1}{1!} + \frac{f''(2)(x-2)^2}{2!} + \frac{f'''(2)(x-2)^3}{3!} + \frac{f^{iv}(2)(x-2)^4}{4!} \\ &= 3 + \frac{0(x-2)^1}{1!} + \frac{5(x-2)^2}{2!} + \frac{-4(x-2)^3}{3!} + \frac{-2(x-2)^4}{4!} \\ &= \boxed{3 + \frac{5}{2}(x-2)^2 - \frac{2}{3}(x-2)^3 - \frac{1}{12}(x-2)^4} \end{aligned}$$

CHOICE B

2. Since $0 < a_k \le b_k$ and $\sum\limits_{k=1}^{\infty} a_k$ diverges, $\sum\limits_{k=1}^{\infty} b_k$ must also diverge, and therefore $\boxed{\sum\limits_{k=1}^{\infty} b_k \neq 1}$.

CHOICE C

3. I. $\sum\limits_{K=1}^{\infty} k^{-3/2}$ is a p-series with $p = \frac{3}{2} > 1$ and therefore it $\boxed{\text{converges}}$.

II. $\sum\limits_{k=1}^{\infty} k^{-1}$ is a harmonic series and therefore it $\boxed{\text{diverges}}$.

III. $$\sum_{k=1}^{\infty} 2^{1-k} = \sum_{k=1}^{\infty} 2^1(2)^{-k} = \sum_{k=1}^{\infty} \frac{2}{2^k} = 2\sum_{k=1}^{\infty} \frac{1}{2^k}$$

is a Geometric Series with $0 < R = \frac{1}{2} < 1$, so therefore it $\boxed{\text{converges}}$.

CHOICE C

4. $$\begin{aligned} \sum_{k=1}^{\infty} \left(-\frac{5}{6}\right)^{k-1} &= \sum_{k=1}^{\infty} \left[\left(-\frac{5}{6}\right)^k \left(-\frac{5}{6}\right)^{-1}\right] = \sum_{k-1}^{\infty} \left[\left(-\frac{5}{6}\right)^k \left(-\frac{6}{5}\right)\right] \\ &= -\frac{6}{5}\sum_{k=1}^{\infty} \left(-\frac{5}{6}\right)^k, \end{aligned}$$

which is an Alternating Geometric Series for which $r = -\frac{5}{6}$, and since $\left|-\frac{5}{6}\right| < 1$, the series $\boxed{\text{converges}}$.

$$-\frac{6}{5}\sum_{k=1}^{\infty} \left(-\frac{5}{6}\right)^k = -\frac{6}{5} \cdot \frac{-\frac{5}{6}}{1 - \left(-\frac{5}{6}\right)} = -\frac{6}{5} \cdot \frac{-\frac{5}{6}}{\frac{11}{5}} = \frac{6}{5} \cdot \frac{5}{6} \cdot \frac{5}{11} = \boxed{\frac{6}{11}}$$

CHOICE B

5. $\sum\limits_{k=0}^{\infty} \frac{(x-2)^k}{k^2}$ is a power series centered at 2. We use the Ratio Test with $a_n = \frac{(x-2)^n}{n^2}$ and $a_{n+1} = \frac{(x-2)^{n+1}}{(n+1)^2}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(x-2)^{n+1}}{(n+1)^2}}{\frac{(x-2)^n}{n^2}}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}n^2}{(x-2)^n(n+1)^2}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^n(x-2)n^2}{(x-2)^n(n+1)^2}\right|$$

$$\lim_{n\to\infty} \left|\frac{(x-2)n^2}{(n+1)^2}\right| = |x-2| \lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^2\right| = |x-2| \lim_{n\to\infty} \left|\left(\frac{1}{1+\frac{1}{n}}\right)^2\right| = |x-2|$$

The series converges absolutely if $|x-2| < 1$, or equivalently if $1 < x < 3$.

The radius of convergence is $r = 1$. To find the interval of convergence, test the endpoints.

If $x = 1$,

$$\sum_{k=0}^{\infty} \frac{(x-2)^k}{k^2} = \sum_{k=0}^{\infty} \frac{(1-2)^k}{k^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k^2}$$

which is a convergent alternating series. The endpoint $x = 1$ is included in the interval of convergence.

If $x = 3$,

$$\sum_{k=0}^{\infty} \frac{(x-2)^k}{k^2} = \sum_{k=0}^{\infty} \frac{(3-2)^k}{k^2} = \sum_{k=0}^{\infty} \frac{(1)^k}{k^2}$$

which is a p-series which converges since $p = 2 > 1$. $x = 3$ is included in the interval of convergence.

The interval of convergence is $\boxed{1 \le x \le 3}$.

CHOICE D

6. $\sum\limits_{k=1}^{\infty} k^{-n/3} = \sum\limits_{k=1}^{\infty} \frac{1}{k^{n/3}}$, which is a p-series which converges if $\frac{n}{3} > 1$, that is, if $n > 3$.

$\sum\limits_{k=1}^{\infty} \frac{(-1)^{nk}}{k}$ will converge only if it is alternating. Therefore n must be odd.

$\sum\limits_{k=1}^{\infty} \left(\frac{n}{6}\right)^k$ is a Geometric Series that will converge provided that $\frac{n}{6} < 1$, that is, provided that $n < 6$.

The conditions required by the three series are summarized as requiring that n be an odd integer for which $3 < n < 6$.

The only choice which satisfies these requirements is $\boxed{n = 5}$.

CHOICE D

7. The language as stated in the problem is that of the Integral Test, for which the conclusion is that $\boxed{\sum\limits_{k=1}^{\infty} a_k \text{ converges if } \int_1^{\infty} f(x)\,dx \text{ converges.}}$

CHOICE D

8. $$\sum_{k=1}^{\infty} \frac{3^{k+1} + 4k!}{3^k \cdot k!} = \sum_{k=1}^{\infty} \left(\frac{3^{k+1}}{3^k \cdot k!} + \frac{4k!}{3^k \cdot k!} \right) = \sum_{k=1}^{\infty} \left(\frac{3}{k!} + \frac{4}{3^k} \right)$$

$$= 3\sum_{k=1}^{\infty} \frac{1}{k!} + 4\sum_{k=1}^{\infty} \frac{1}{3^k} = 3e + 4\left(\frac{\frac{1}{3}}{1 - \frac{1}{3}} \right) = 3e + 4\left(\frac{1}{2} \right) = \boxed{3e + 2}$$

CHOICE D

9. $\displaystyle\sum_{k=1}^{\infty}\frac{3}{(2k)^4}=\sum_{k=1}^{\infty}\frac{3}{16k^4}=\frac{3}{16}\sum_{k=1}^{\infty}\frac{1}{k^4}$

$\sum_{k=1}^{\infty}\frac{1}{k^4}$ is a convergent p-series for which the bounds are

$$\begin{aligned}\frac{1}{4-1}&<\sum_{k=1}^{\infty}\frac{1}{k^4}<1+\frac{1}{4-1}\\ \frac{1}{3}&<\sum_{k=1}^{\infty}\frac{1}{k^4}<\frac{4}{3},\\ \frac{3}{16}\cdot\frac{1}{3}&<\frac{3}{16}\sum_{k=1}^{\infty}\frac{1}{k^4}<\frac{3}{16}\cdot\frac{4}{3}\\ \frac{1}{16}&<\frac{3}{16}\sum_{k=1}^{\infty}\frac{1}{k^4}<\frac{1}{4}\\ &\boxed{\frac{1}{16}<\sum_{k=1}^{\infty}\frac{3}{(2k)^4}<\frac{1}{4}}\end{aligned}$$

10. Evaluate each series separately.

I. $$2-1+\frac{1}{2}-\frac{1}{4}+\frac{1}{8}+\ldots=\left(\frac{1}{2}\right)^{-1}-\left(\frac{1}{2}\right)^{0}+\left(\frac{1}{2}\right)^{1}-\left(\frac{1}{2}\right)^{2}+\left(\frac{1}{2}\right)^{3}$$
$$=\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{1}{2}\right)^{k-2}=4\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{1}{2}\right)^{k}.$$

$4\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{1}{2}\right)^k$ is an alternating Geometric series which $\boxed{\text{converges absolutely}}$, since $r=\frac{1}{2}<1$.

II. $$2+\frac{4}{2^{3/2}}+\frac{6}{3^{3/2}}+\frac{8}{4^{3/2}}+\ldots=\sum_{k=1}^{\infty}\frac{2k}{k^{3/2}}=2\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}$$

which is a $\boxed{\text{divergent}}$ p-series, since $p=\frac{3}{2}>1$.

III. $$6-4-\frac{8}{3}-\frac{16}{9}-\frac{32}{27}-\cdots=6-4-\frac{2^3}{3}-\frac{2^4}{3^2}-\frac{2^5}{3^3}-\cdots$$
$$=6-\frac{2^2}{3^0}-\frac{2^3}{3^1}-\frac{2^4}{3^2}-\frac{2^5}{3^3}-\cdots=6-\sum_{k=2}^{\infty}\frac{2^k}{3^{k-2}}$$
$$=6-\sum_{k=2}^{\infty}\frac{2^k}{3^k\cdot 3^{-2}}=6-9\sum_{k=2}^{\infty}\left(\frac{2}{3}\right)^k,$$

and $\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k$ is a convergent Geometric Series, since $r=\frac{2}{3}<1$, so the original series $\boxed{\text{converges}}$.

CHOICE B

11. $\sum\limits_{k=0}^{\infty}\left(\frac{x-1}{4}\right)^k$ is a power series centered at 1. We use the Ratio Test with $a_n = \left(\frac{x-1}{4}\right)^n$ and $a_{n+1} = \left(\frac{x-1}{4}\right)^{n+1}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{x-1}{4}\right)^{n+1}}{\left(\frac{x-1}{4}\right)^{n}}\right| = \lim_{n\to\infty}\left|\left(\frac{x-1}{4}\right)\right| = \left|\frac{x-1}{4}\right|$$

The series converges absolutely if $\left|\frac{x-1}{4}\right| < 1$, or equivalently if $-3 < x < 5$.

The radius of convergence is $R = 1$. To find the interval of convergence, test the endpoints.

If $x = -3$,

$$\sum_{k=0}^{\infty}\left(\frac{x-1}{4}\right)^k = \sum_{k=0}^{\infty}\left(\frac{-3-1}{4}\right)^k = \sum_{k=0}^{\infty}(-1)^k,$$

which is a divergent alternating series. The endpoint $x = 1$ is not included in the interval of convergence.

If $x = 5$,

$$\sum_{k=0}^{\infty}\left(\frac{x-1}{4}\right)^k = \sum_{k=0}^{\infty}\left(\frac{5-1}{4}\right)^k = \sum_{k=0}^{\infty}(1)^k,$$

which diverges, so $x = 5$ is not included in the interval of convergence.

The interval of convergence is $\boxed{-3 < x < 5}$.

CHOICE D

12.
$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\\ x^2\sin x &= x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)\\ &= \boxed{x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \cdots}\end{aligned}$$

CHOICE C

13. $$\sum_{k=1}^{\infty}\frac{4^{k+2}}{5^k} = \sum_{k=1}^{\infty}\frac{4^k\cdot 4^2}{5^k} = 16\sum_{k=1}^{\infty}\left(\frac{4}{5}\right)^k,$$

which is a $\boxed{\text{convergent}}$ Geometric Series since $r = \frac{4}{5} < 1$.

$$16\sum_{k=1}^{\infty}\left(\frac{4}{5}\right)^k = 16\cdot\frac{\frac{4}{5}}{1-\frac{4}{5}} = 16\cdot 4 = \boxed{64}$$

CHOICE B

14. $\sum\limits_{k=1}^{\infty} \frac{6}{(k+1)(k+2)}$ can be rewritten using Partial Fraction Decomposition as follows:

$$\begin{aligned}\frac{6}{(k+1)(k+2)} &= \frac{A}{k+1} + \frac{B}{k+2} \\ 6 &= A(k+2) + B(k+1)\end{aligned}$$

$$\begin{array}{ll} \text{Let } k=-2 & \text{Let } k=-1 \\ \quad 6=-B & \quad A=6 \\ \quad B=-6 & \end{array}$$

Rewrite $\frac{6}{(k+1)(k+2)} = \frac{6}{k+1} + \frac{-6}{k+2}$, so

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{6}{(k+1)(k+2)} &= \sum_{k=1}^{\infty}\left(\frac{6}{k+1} - \frac{6}{k+2}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{6}{k+1} - \frac{6}{k+2}\right) \\ &= \lim_{n\to\infty}\left[\left(\frac{6}{2}-\frac{6}{3}\right) + \left(\frac{6}{3}-\frac{6}{4}\right) + \left(\frac{6}{4}-\frac{6}{5}\right) + \ldots + \left(\frac{6}{n}-\frac{6}{n+1}\right) + \left(\frac{6}{n+1}-\frac{6}{n+2}\right)\right] \\ &= \lim_{n\to\infty}\left[\frac{6}{2} + \left(-\frac{6}{3}+\frac{6}{3}\right) + \left(-\frac{6}{4}+\frac{6}{3}\right) + \cdots + \left(-\frac{6}{n+1}+\frac{6}{n+1}\right) - \frac{6}{n+2}\right] \\ &= \lim_{n\to\infty}\left(\frac{6}{2} - \frac{6}{n-2}\right) = \frac{6}{2} = \boxed{3}.\end{aligned}$$

(This is a telescoping series, which $\boxed{\text{converges}}$.)

CHOICE B

15. The convergence or divergence of $\sum\limits_{k=1}^{\infty} \frac{(\ln k)^2}{k}$ can be evaluated using the Integral Test as follows:

Let $f(x) = \frac{(\ln x)^2}{x}$.

The Integral Test is applicable provided that $f(x) = \frac{(\ln x)^2}{x}$ is continuous, positive, and nonincreasing on the interval $[1, \infty)$.

Observe that on the interval $[1, \infty)$, $f(x)$ is always positive.

Determine $f'(x)$ to evaluate if $f(x)$ is continuous and nonincreasing.

$$f'(x) = \frac{2\ln x \cdot \frac{1}{x} \cdot x - 1 \cdot (\ln x)^2}{x^2} = \frac{2\ln x - (\ln x)^2}{x^2}$$

$f'(x) < 0$ for values of $x > 7.389$, so for large values of x, $f(x)$ is nonincreasing.

$f'(x)$ is defined on $[1, \infty)$ which implies that $f(x)$ is continuous on $[1, \infty)$.

With the preconditions to the Integral Test satisfied we proceed to evaluate the convergence or divergence of $\int_1^{\infty} \frac{(\ln x)^2}{x}\,dx$.

$$\int_1^{\infty} \frac{(\ln x)^2}{x}\,dx = \lim_{x\to b}\int_1^{b} \frac{(\ln x)^2}{x}\,dx = \lim_{x\to b}\left[\frac{\ln^3 x}{3}\right]_1^b = \frac{1}{3}\lim_{x\to b}\left[\ln^3 b - \ln^3 1\right] = \infty.$$

$\int_1^{\infty} \frac{(\ln x)^2}{x}\,dx$ diverges, so by the Integral Test $\sum\limits_{k=1}^{\infty} \frac{(\ln k)^2}{k}$ also $\boxed{\text{diverges}}$.

16. We begin by testing the series for absolute convergence.

The series of absolute values is

$$\sum_{k=1}^{\infty}\left|(-1)^{k+1}\left(\frac{\sqrt{k}}{k+2}\right)\right| = \sum_{k=1}^{\infty}\frac{\sqrt{k}}{k+2}.$$

Since for large n, $\frac{\sqrt{n}}{n+2} \approx \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ we compare the series $\sum\limits_{k=1}^{\infty}\frac{\sqrt{k}}{k+2}$ to the p-series $\sum\limits_{k=1}^{\infty}\frac{1}{k^{1/2}}$, which we know diverges.

With $a_n = \frac{\sqrt{n}}{n+2}$ and $b_n = \frac{1}{n^{1/2}}$,

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n+2}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty}\frac{n}{n+2} = \lim_{n\to\infty}\frac{1}{1+\frac{2}{n}} = 1.$$

Since the limit is a positive number and the p-series diverges, so by the Limit Comparison Test the series $\sum\limits_{k=1}^{\infty}\frac{\sqrt{k}}{k+2}$ diverges and $\sum\limits_{k=1}^{\infty}(-1)^{k+1}\left(\frac{\sqrt{k}}{k+2}\right)$ is not absolutely convergent.

We must proceed to determine if $\sum\limits_{k=1}^{\infty}(-1)^{k+1}\left(\frac{\sqrt{k}}{k+2}\right)$ is conditionally convergent.

We begin by confirming that $\lim\limits_{n\to\infty}(a_n) = \lim\limits_{n\to\infty}\frac{\sqrt{n}}{n+2} = 0$.

Next, using the Algebraic Ratio test, we verify that the terms $a_k = \frac{\sqrt{k}}{k+2}$ are nonincreasing.

Since

$$\frac{a_{n+1}}{a_n} = \frac{\frac{\sqrt{n+1}}{n+1+2}}{\frac{\sqrt{n}}{n+2}} = \frac{(n+2)\sqrt{n+1}}{(n+3)\sqrt{n}} = \frac{n+2}{n+3}\cdot\frac{n+1}{n} = \left(\frac{1+\frac{2}{n}}{1+\frac{3}{n}}\right)\left(1+\frac{1}{n}\right) = 1$$

for all $n \geq 1$, the terms a_k are nonincreasing.

By the Alternating Series Test, the series $\sum\limits_{k=1}^{\infty}(-1)^{k+1}\left(\frac{\sqrt{k}}{k+2}\right)$ converges.

Therefore $\sum\limits_{k=1}^{\infty}(-1)^{k+1}\left(\frac{\sqrt{k}}{k+2}\right)$ is $\boxed{\text{conditionally convergent}}$.

17. (a) $f(x) = \tan^{-1} x$.

To express a function f as a MacLaurin series, we begin by evaluating f and its derivatives at 0.

$$\begin{array}{ll}
f(x) = \tan^{-1} x & f(0) = 0 \\
f'(x) = \dfrac{1}{1+x^2} & f'(0) = 1 \\
f''(x) = \dfrac{-2x}{(1+x^2)^2} & f''(0) = 0 \\
f'''(x) = \dfrac{2(3x^2-1)}{(1+x^2)^3} & f'''(0) = -2 \\
f^{(4)}(x) = \dfrac{24x - 24x^3}{(1+x^2)^4} & f^{(4)}(0) = 0 \\
f^{(5)}(x) = \dfrac{120x^4 - 240x^2 + 24}{(1+x^2)^5} & f^{(5)}(0) = 24 \\
f^{(6)}(x) = \dfrac{-240x(3x^4 - 10x^2 + 3)}{(1+x^2)^6} & f^{(6)}(0) = 0
\end{array}$$

Then use the definition of a Maclaurin series,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0 + 1 \cdot x + \frac{0x^2}{2!} - \frac{2x^3}{3!} + \frac{0x^4}{4!} + \frac{24x^5}{5!} + \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots$$

The first five terms are $P_9(x) = \boxed{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}}$.

(b) $$g(x) = \int_0^x \tan^{-1} t \, dt = \int_0^x \left(\sum_{t=0}^{9} \frac{f^{(k)}(0)}{k!} t^k \right) dt$$

$$= \int_0^x \left(t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots \right) dt = \left[\frac{t^2}{2} - \frac{t^4}{3 \cdot 4} + \frac{t^6}{5 \cdot 6} - \frac{t^8}{7 \cdot 8} + \cdots \right]_0^x$$

$$= \boxed{\frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \frac{x^8}{56} + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k}}{2k(2k-1)}}$$

(c) The radius of convergence is $R = 1$, centered at 0, so the series converges absolutely if $-1 < x < 1$.

To find the interval of convergence, we test the endpoints, $x = -1$ and $x = 1$.

When $x = -1$ or $x = 1$, $\sum\limits_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k}}{2k(2k-1)}$ reduces to

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{(\pm 1)^{2k}}{2k(2k-1)} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2k(2k-1)},$$

which is convergent by the Alternating Series test.

So $x = -1$ and $x = 1$ are both included in the interval of convergence.

Consquently the interval of convergence is $\boxed{-1 \le x \le 1}$.

(d) Let $P_n(x)$ be the approximation of $\int_0^x \tan^{-1} x\,dx$ using the sum of terms a_1 through a_n of the series expansion found above.

Then $\int_0^{1/4} \tan^{-1} x\,dx \approx P_n\left(\frac{1}{4}\right)$.

The bound on the error in using this approximation is

$$|E_n| \le a_{n+1} = \frac{\left(\frac{1}{4}\right)^{2(n+1)}}{2(n+1)[2(n+1)-1]} = \frac{\left(\frac{1}{4}\right)^{2n+2}}{(2n+2)(2n-1)} \text{ for } P_n\left(\frac{1}{4}\right).$$

By trial and error,

$n = 1$: $a_{n+1} = a_2 = 0.000977 > 0.0001$
$n = 2$: $a_{n+1} = a_3 = 0.0000136 < 0.0001$

So we want

$$\begin{aligned} P_2\left(\frac{1}{4}\right) &= (-1)^{1-1}a_1 + (-1)^{2-1}a_2 \\ &= \frac{\left(\frac{1}{4}\right)^2}{2} - \frac{\left(\frac{1}{4}\right)^3}{12} \\ &= \frac{95}{3072} \\ &\approx \boxed{0.031} \end{aligned}$$