Packet: 48-66 | THW6

(48) $y = \sec^{-1}\frac{1}{t},\ 0 < t < 1$

$$\frac{dy}{dt} = \frac{1 \cdot -\frac{1}{t^2}}{\frac{1}{t}\sqrt{(\frac{1}{t})^2 - 1}} = \frac{-\frac{1}{t}}{\sqrt{\frac{1-t^2}{t^2}}}$$

$$\frac{dy}{dt} = \frac{-\frac{1}{t} \cdot t}{\sqrt{1-t^2}} = \boxed{\frac{-1}{\sqrt{1-t^2}}}$$

(49) $y = x\sin^{-1}x + \sqrt{1-x^2}$

$$\frac{dy}{dx} = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} + \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}$$

$$= \boxed{\sin^{-1}x}$$

(50) $$\int_0^1 \frac{4ds}{\sqrt{4-s^2}} = \int_0^1 \frac{4ds}{2\sqrt{1-(\frac{s}{2})^2}}$$

$$= 2\int_0^1 \frac{ds}{\sqrt{1-(\frac{s}{2})^2}}$$

$u = \frac{s}{2}$
$du = \frac{1}{2}ds$
$[0,1] \to [0,\frac{1}{2}]$

$$= 2 \cdot 2\int_0^{1/2} \frac{du}{\sqrt{1-u^2}}$$

$$= 4\left[\sin^{-1}u\right]_0^{1/2} = 4(\sin^{-1}\tfrac{1}{2} - \sin^{-1}0)$$

$$= 4(\tfrac{\pi}{6} - 0) = \boxed{\frac{2\pi}{3}}$$

(51) $$\int \frac{dx}{2+(x-1)^2} = \int \frac{dx}{2(1+(\frac{x-1}{\sqrt{2}})^2)}$$

$u = \frac{x-1}{\sqrt{2}}$
$du = \frac{1}{\sqrt{2}}dx$

$$= \frac{1}{2} \cdot \sqrt{2}\int \frac{du}{1+u^2} = \frac{\sqrt{2}}{2}\tan^{-1}u + C$$

$$= \boxed{\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C}$$

(52) $$\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}}$$

$u = 2x-1$
$du = 2dx$

$$= \int \frac{dx}{\frac{2x-1}{2}\sqrt{(\frac{2x-1}{2})^2-1}} = \frac{1}{2}\int \frac{du}{\frac{u}{2}\sqrt{u^2-1}}$$

$$= \int \frac{du}{u\sqrt{u^2-1}} = \boxed{\sec^{-1}(2x-1) + C}$$

(53) $$\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\, d\theta}{1+(\sin\theta)^2}$$

even func.
$u = \sin\theta$
$du = \cos\theta\, d\theta$
$[0,\frac{\pi}{2}] \to [0,1]$

$$= 2 \cdot 2\int_0^{\pi/2} \frac{\cos\theta\, d\theta}{1+(\sin\theta)^2}$$

$$= 4\int_0^1 \frac{du}{1+u^2} = 4\tan^{-1}u\Big|_0^1$$

$$= 4(\tfrac{\pi}{4} - 0) = \boxed{\pi}$$

(54) $$\int \frac{y\,dy}{\sqrt{1-(y^2)^2}}$$

$u = y^2$
$du = 2y\,dy$

$$= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + C$$

$$= \boxed{\frac{1}{2}\sin^{-1}y^2 + C}$$

(55) $$\int_{-1}^0 \frac{6dt}{\sqrt{3-2t-t^2}}$$

$-(t^2+2t-3)$
$-(t^2+2t+1)+3+1$
$4-(t+1)^2$

$$= \int_{-1}^0 \frac{6dt}{\sqrt{4-(t+1)^2}} = \frac{6}{2}\int_{-1}^0 \frac{dt}{\sqrt{1-(\frac{t+1}{2})^2}}$$

$u = \frac{t+1}{2}$
$du = \frac{1}{2}dt$
$[-1,0] \to [0,\frac{1}{2}]$

$$= 3 \cdot 2\int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}}$$

$$= 6\left[\sin^{-1}u\right]_0^{1/2} = 6(\tfrac{\pi}{6} - 0)$$

$$= 6(\tfrac{\pi}{6}) = \boxed{\pi}$$

(56) $$\int_1^2 \frac{8dx}{x^2-2x+1+2-1} = \int_1^2 \frac{8dx}{(x-1)^2+1}$$

$u = x-1$
$du = dx$
$[1,2] \to [0,1]$

$$= 8\int_0^1 \frac{du}{u^2+1} = \left[8\tan^{-1}u\right]_0^1 = 8(\tfrac{\pi}{4} - 0) = \boxed{2\pi}$$

(57) $$\int \frac{x^2+2x-1}{x^2+9}dx$$

$$\begin{array}{r} 1 \\ x^2+9 \overline{\big) x^2+2x-1} \\ -x^2 \quad\ \ominus 9 \\ \hline 2x-10 \end{array}$$

$$= \int dx + \int \frac{2x-10}{x^2+9}dx$$

$$= x + 2\int \frac{x}{x^2+9}dx - 10\int \frac{1}{x^2+9}dx$$

$u_1 = x^2+9,\ du_1 = 2x\,dx$

$u_2 = \frac{x}{3}$
$du_2 = \frac{1}{3}dx$

$$= x + \frac{2}{2}\int \frac{du_1}{u_1} - \frac{10}{9}\int \frac{1}{(\frac{x}{3})^2+1}dx$$

$$= x + \ln|u_1| - \frac{10}{9} \cdot 3\int \frac{1}{u_2^2+1}du_2 = \longrightarrow$$

(57) $x+\ln|x^2+9|-\frac{10}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$

$$\boxed{x+\ln(x^2+9)-\frac{10}{3}\tan^{-1}\left(\frac{x}{3}\right)+C}$$

(58) $\int \frac{e^{\sin^{-1}x}dx}{\sqrt{1-x^2}}$ $\qquad u=\sin^{-1}x$, $du=\frac{dx}{\sqrt{1-x^2}}$

$$=\int e^u du = e^u + C$$

$$=\boxed{e^{\sin^{-1}x}+C}$$

(59) $\int \frac{dy}{(\tan^{-1}y)(1+y^2)}$ $\qquad u=\tan^{-1}y$, $du=\frac{dy}{1+y^2}$

$$=\int \frac{du}{u} = \ln|u|+C$$

$$=\boxed{\ln|\tan^{-1}y|+C}$$

(60) $\int \frac{dx}{\sqrt{x}(x+1)((\tan^{-1}\sqrt{x})^2+9)}$ $\qquad u=\tan^{-1}\sqrt{x}$, $du=\frac{1/2\sqrt{x}}{x+1}dx$

$$=2\int\frac{du}{u^2+9}=\frac{2}{9}\int\frac{du}{\left(\frac{u}{3}\right)^2+1}$$

$$=\frac{2}{9}\cdot 3\tan^{-1}\left(\frac{u}{3}\right)+C$$

$$=\boxed{\frac{2}{3}\tan^{-1}\left(\frac{\tan^{-1}\sqrt{x}}{3}\right)+C}$$

(61) $\lim_{x\to\infty} x\tan^{-1}\frac{2}{x}=\lim_{x\to\infty}\frac{\tan^{-1}2/x}{1/x}$

$$\overset{LH}{=}\lim_{x\to\infty}\frac{\frac{1}{1+\left(\frac{2}{x}\right)^2}\cdot\frac{-2}{x^2}}{-\frac{1}{x^2}}$$

$$=\lim_{x\to\infty}\frac{2}{1+\left(\frac{2}{x}\right)^2}=\boxed{2}$$

(62) $\lim_{x\to0^+}\frac{(\tan^{-1}\sqrt{x})^2}{x\sqrt{x+1}}$ $\qquad \left(\frac{0}{0}\right)$

$$\overset{LH}{=}\lim_{x\to0^+}\frac{\cancel{2}\tan^{-1}\sqrt{x}\cdot\frac{1}{(1+x)\cancel{2}\sqrt{x}}}{\sqrt{x+1}+\frac{x}{2\sqrt{x+1}}}$$

$$=\lim_{x\to0^+}\frac{\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}(1+x)}}{\frac{2(x+1)+x}{2\sqrt{x+1}}}=\lim_{x\to0^+}\frac{\tan^{-1}\sqrt{x}}{\underset{\sqrt{1+x}}{\sqrt{x}(1+x)}}\cdot\frac{2\sqrt{x+1}}{3x+2}$$

$$=\lim_{x\to0^+}\frac{2\tan^{-1}\sqrt{x}}{(3x+2)\sqrt{x^2+x}}=\lim_{x\to0^+}\frac{\frac{\cancel{2}}{1+x}\cdot\frac{1}{\cancel{2}\sqrt{x}}}{3\sqrt{x^2+x}+\frac{(3x+2)(2x+1)}{2\sqrt{x^2+x}}}$$

$$=\lim_{x\to0^+}\frac{\frac{1}{\sqrt{x}(1+x)}}{\frac{6(x^2+x)+(3x+2)(2x+1)}{2\sqrt{x^2+x}}}$$

$$=\lim_{x\to0^+}\frac{2\cancel{\sqrt{x}}\sqrt{x+1}}{\cancel{\sqrt{x}}(1+x)}\cdot\frac{1}{6x^2+6x+6x^2+7x+2}$$

$$=\lim_{x\to0^+}\frac{2\sqrt{x+1}}{(1+x)(12x^2+13x+2)}=\frac{2}{2}=\boxed{1}$$

(63) verify by taking the derivative of the integral.

$$\frac{d}{dx}\left[x(\sin^{-1}x)^2-2x+2\sqrt{1-x^2}\sin^{-1}x+C\right]$$

$$=(\sin^{-1}x)^2+\frac{2x\sin^{-1}x}{\sqrt{1-x^2}}-2+\frac{\cancel{2}(-2x)}{\cancel{2}\sqrt{1-x^2}}\sin^{-1}x+\frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}}$$

$$=(\sin^{-1}x)^2$$

(64) $\int dy=\int\frac{dx}{x\sqrt{x^2-1}}$; $y(2)=\pi$

$$y=\sec^{-1}x+C$$

$$\pi=\sec^{-1}2+C$$

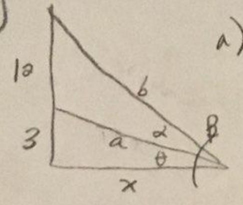
$$\pi=\frac{\pi}{3}+C$$

$$\frac{2\pi}{3}=C$$

$$\boxed{y=\sec^{-1}x+\frac{2\pi}{3}\ ,\ x>1}$$

THW 6 cont

(65)



a) $\cot\theta = \frac{x}{3}$

$\theta = \cot^{-1}\left(\frac{x}{3}\right)$

$\cot\beta = \frac{x}{15}$

$\beta = \cot^{-1}\left(\frac{x}{15}\right)$

$$\boxed{\alpha = \cot^{-1}\left(\frac{x}{15}\right) - \cot^{-1}\left(\frac{x}{3}\right)}$$

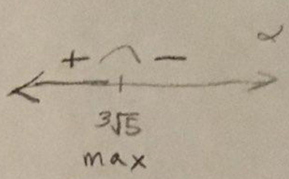
b) $$\frac{d\alpha}{dx} = -\frac{1}{1+\left(\frac{x}{15}\right)^2}\cdot\frac{1}{15} + \frac{1}{1+\left(\frac{x}{3}\right)^2}\cdot\frac{1}{3}$$

$$\frac{d\alpha}{dx} = \frac{9\cdot\frac{1}{3}}{9+x^2} - \frac{225\cdot\frac{1}{15}}{225+x^2} = \frac{675+3x^2-135-15x^2}{(9+x^2)(225+x^2)} = \frac{540-12x^2}{(9+x^2)(225+x^2)}$$

$$\frac{d\alpha}{dx} = \frac{540-12x^2}{(x^2+225)(x^2+9)} \; ; \; \frac{d\alpha}{dx} = 0 \rightarrow 540-12x^2=0$$

$$x^2 = 45$$

$$x = 3\sqrt{5}$$



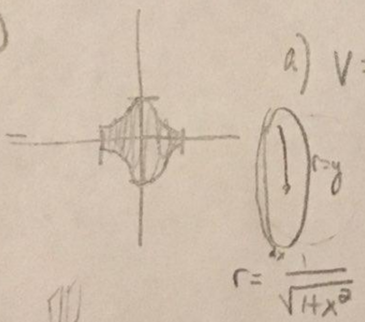
Since $\frac{d\alpha}{dx} > 0$ for $x \in (0, 3\sqrt{5})$ and $\frac{d\alpha}{dx} < 0$ for $x > 3\sqrt{5}$, there is a local max at $x = 3\sqrt{5}$.

$x = 0 \rightarrow \alpha = \frac{\pi}{4} - \frac{\pi}{4} = 0$

$x = 3\sqrt{5} \rightarrow \alpha = \cot^{-1}\left(\frac{3\sqrt{5}}{15}\right) - \cot^{-1}\left(\frac{3\sqrt{5}}{3}\right) \doteq 0.730 \approx 41.810°$

∴ The maximum viewing angle is approximately 41.810°.

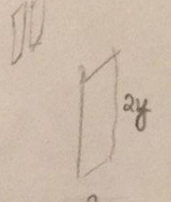
(66)



a) $$V = \int_{-1}^{1} \pi\left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx = 2\pi\int_0^1 \frac{dx}{1+x^2} = 2\pi\tan^{-1}x$$

$$= 2\pi\left(\tan^{-1}1 - \tan^{-1}0\right) =$$

$$= 2\pi\left(\frac{\pi}{4}\right) = \boxed{\frac{\pi^2}{2}}$$



$2y = \frac{2}{\sqrt{1+x^2}}$

b) $$V = \int_{-1}^{1}\left(\frac{2}{\sqrt{1+x^2}}\right)^2 dx = 2\int_0^1 \frac{4}{1+x^2}\,dx$$

$$= 8\tan^{-1}x\Big|_0^1$$

$$= 8\left(\tan^{-1}1 - \tan^{-1}0\right)$$

$$= 8\left(\frac{\pi}{4}\right) = \boxed{2\pi}$$