Transcendental Wksht: 67 - 82 | THW8

(67) a) $\int \text{sech}\, x\, dx = \tan^{-1}(\sinh x) + C$ $\quad \cosh^2 x - \sinh^2 x = 1$

$\frac{d}{dx}\left[\tan^{-1}(\sinh x) + C\right] = \frac{\cosh x}{1+\sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \frac{1}{\cosh x} = \text{sech}\, x$

$\text{sech}\, x = \frac{d}{dx}\left[\tan^{-1}(\sinh x) + C\right] \Rightarrow \int \text{sech}\, x\, dx = \frac{d}{dx}\int \left(\tan^{-1}(\sinh x) + C\right) dx$

$\Rightarrow \int \text{sech}\, x\, dx = \tan^{-1}(\sinh x) + C$

b) $\int \text{sech}\, x\, dx = \sin^{-1}(\tanh x) + C$

$\frac{d}{dx}\left[\sin^{-1}(\tanh x) + C\right] = \frac{\text{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\text{sech}^2 x}{\text{sech}\, x} = \text{sech}\, x$

(68) $\int x \coth^{-1} x\, dx = \frac{x^2-1}{2}\coth^{-1} x + \frac{x}{2} + C$

$\frac{d}{dx}\left[\frac{x^2-1}{2}\coth^{-1} x + \frac{x}{2} + C\right] = \frac{2x}{2}\coth^{-1} x + \frac{x^2-1}{2}\cdot\frac{1}{1-x^2} + \frac{1}{2}, \ |x|>1$

$= x\coth^{-1} x - \frac{1}{2} + \frac{1}{2}$

$= x \coth^{-1} x$

(69) $\int \sinh 2x\, dx = \boxed{\frac{\cosh 2x}{2} + C}$

(79) $\int \tanh \frac{x}{7}\, dx = \frac{\text{sech}^2\left(\frac{x}{7}\right)}{\frac{1}{7}} + C = \boxed{7\,\text{sech}^2\left(\frac{x}{7}\right) + C}$ or $7\ln\left|e^{x/7} + e^{-x/7}\right| + C$

(71) $\int \frac{\text{sech}\sqrt{t}\, \tanh\sqrt{t}}{\sqrt{t}}\, dt$ $\quad u = \sqrt{t}$, $\quad du = \frac{1}{2\sqrt{t}}\, dt$

$= 2\int \text{sech}\, u \tanh u\, du = -2\,\text{sech}\, u + C = \boxed{-2\,\text{sech}(\sqrt{t}) + C}$

) $\int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta\, d\theta = \int_{-\ln 4}^{-\ln 2} 2e^{\theta}\left(\frac{e^{\theta}+e^{-\theta}}{2}\right) d\theta = \int_{-\ln 4}^{-\ln 2} \left(e^{2\theta} + 1\right) d\theta$

$= \frac{1}{2}e^{2\theta}\Big|_{-\ln 4}^{-\ln 2} + \theta\Big|_{-\ln 4}^{-\ln 2} = \frac{1}{2}\left(e^{-2\ln 2} - e^{-2\ln 4}\right) + (-\ln 2 + \ln 4)$

$= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{16}\right) + \ln 2 = \frac{1}{2}\cdot\frac{\cancel{12}\ 3}{\cancel{64}\ 32} + \ln 2 = \boxed{\frac{3}{32} + \ln 2}$

73 $\int_1^2 \frac{\cosh(\ln t)}{t}\,dt$ $\quad u = \ln t$, $du = \frac{1}{t}dt$ $\quad [1,2] \to [0, \ln 2]$

$$= \int_0^{\ln 2} \cosh u\,du = \sinh u\Big|_0^{\ln 2} = \sinh(\ln 2) - \sinh(0)$$

$$= \frac{e^{\ln 2} - e^{-\ln 2}}{2} - \frac{e^0 - e^{-0}}{2} = \frac{2 - \frac{1}{2}}{2} - 0 = \boxed{\frac{3}{4}}$$

74 $\sinh^{-1}\left(-\frac{5}{12}\right) = x \to \sinh x = -\frac{5}{12} \to \frac{e^x - e^{-x}}{2} = -\frac{5}{12} \to e^x - e^{-x} = -\frac{5}{6}$

$6e^x(e^x - e^{-x}) = 6e^x\left(\frac{-5}{6}\right) \to 6e^{2x} - 6 = -5e^x$ $\quad$ Let $u = e^x$

$6u^2 + 5u - 6 = 0 \to (2u+3)(3u-2) = 0 \to u = -\frac{3}{2}$ or $\frac{2}{3}$

$\cancel{e^x = -\frac{3}{2}}$ or $e^x = \frac{2}{3} \to x = \boxed{\ln \frac{2}{3} \text{ or } \ln 2 - \ln 3}$

75 $\tanh^{-1}\left(-\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right) = \boxed{\frac{1}{2}\ln\frac{1}{3} \text{ or } \frac{\ln 3}{-2}}$ $\quad$ using the formula

76 $\operatorname{sech}^{-1}\left(\frac{3}{5}\right) = x \to \operatorname{sech} x = \frac{3}{5} \to \frac{2}{e^x + e^{-x}} = \frac{3}{5} \to 3e^x + 3e^{-x} = 10$

$3e^{2x} + 3 = 10e^x \to 3e^{2x} - 10e^x + 3 = 0 \to (3e^x - 1)(e^x - 3) = 0$

$e^x = \frac{1}{3}$ or $e^x = 3 \to x = \ln\frac{1}{3}$ or $\boxed{x = \ln 3}$

$x = -\ln 3 \to$ ~~but $x > 0$~~

77 $\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}} = \left[\sinh^{-1}\left(\frac{u}{a}\right)\right]_0^{2\sqrt{3}}$ $\quad$ where $u = x$ and $a = 2$

$$= \left[\sinh^{-1}\left(\frac{x}{2}\right)\right]_0^{2\sqrt{3}} = \sinh^{-1}(\sqrt{3}) - 0 = \boxed{\sinh^{-1}\sqrt{3}}$$

or $\sinh^{-1}\sqrt{3} = \ln(\sqrt{3} + 2)$

78 $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right)\Big|_{4/5}^{12/13}$ $\quad$ where $u = 4x$, $a = 1$ and $\left[\frac{1}{5}, \frac{3}{13}\right] \to \left[\frac{4}{5}, \frac{12}{13}\right]$

$$= -1\operatorname{sech}^{-1}\left(\frac{u}{1}\right)\Big|_{4/5}^{12/13} = -\left(\operatorname{sech}^{-1}\frac{12}{13} - \operatorname{sech}^{-1}\frac{4}{5}\right)$$

$$= \operatorname{sech}^{-1}\frac{4}{5} - \operatorname{sech}^{-1}\frac{12}{13}$$

$$= \ln\left(\frac{1+\sqrt{1-\left(\frac{4}{5}\right)^2}}{4/5}\right) - \ln\left(\frac{1+\sqrt{1-\left(\frac{12}{13}\right)^2}}{12/13}\right)$$

$$= \ln\left(\frac{5+\sqrt{9}}{4}\right) - \ln\left(\frac{13+\sqrt{25}}{12}\right) = \ln(2) - \ln\left(\frac{3}{2}\right) = \boxed{\ln\frac{4}{3}}$$

(79) $m \cdot \frac{dv}{dt} = mg - kv^2$

Trans #8 cont.

a) $v = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{gk}{m}}\, t\right)$

$\frac{dv}{dt} = \sqrt{\frac{mg}{k}} \operatorname{sech}^2\left(\sqrt{\frac{gk}{m}} \cdot t\right) \cdot \sqrt{\frac{gk}{m}} = \sqrt{g^2} \operatorname{sech}^2\left(\sqrt{\frac{gk}{m}} \cdot t\right) = g \operatorname{sech}^2\left(\sqrt{\frac{gk}{m}} \cdot t\right)$

$m\frac{dv}{dt} = mg \operatorname{sech}^2\left(\sqrt{\frac{gk}{m}} \cdot t\right) = mg\left(1 - \tanh^2\left(\sqrt{\frac{gk}{m}} \cdot t\right)\right)$

$= mg - mg\tanh^2\left(\sqrt{\frac{gk}{m}} \cdot t\right) = mg - k\underbrace{\left(\sqrt{\frac{mg}{k}}\right)^2 \tanh^2\left(\sqrt{\frac{gk}{m}} \cdot t\right)}_{v^2}$

$= mg - kv^2$

If $t = 0$, $v = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{gk}{m}} \cdot 0\right) = 0.$

b) $\lim_{t\to\infty} \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{gk}{m}}\, t\right) = \sqrt{\frac{mg}{k}} \lim_{t\to\infty} \tanh\left(\sqrt{\frac{gk}{m}}\, t\right) = \sqrt{\frac{mg}{k}}\,(1) = \sqrt{\frac{mg}{k}}$

$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1}{1} = 1$

c) $\sqrt{\frac{mg}{k}} = \sqrt{\frac{160}{0.005}} \doteq 179.89$ ft/sec

(80) $\lim_{x\to\infty} \frac{e^x}{e^{x/2}} = \lim_{x\to\infty} e^{x/2} = \infty$ $\therefore$ e^x grows faster than $e^{x/2}$

$\lim_{x\to\infty} \frac{e^x}{(\ln x)^x} = \lim_{x\to\infty} \left(\frac{e}{\ln x}\right)^x = 0$ $\therefore$ $(\ln x)^x$ grows faster than e^x

$\lim_{x\to\infty} \frac{(\ln x)^x}{x^x} = \lim_{x\to\infty} \left(\frac{\ln x}{x}\right)^x = 0$ $\therefore$ x^x grows faster than $(\ln x)^x$

slowest to fastest: $e^{x/2}$, e^x, $(\ln x)^x$, x^x

(81) a) $\lim_{x\to\infty} \frac{x}{x} = 1$; False

b) $\lim_{x\to\infty} \frac{x}{x+5} = 1$; False

c) $\lim_{x\to\infty} \frac{x}{x+5} = 1$; True

d) $\lim_{x\to\infty} \frac{x}{2x} = \frac{1}{2}$; True

e) $\lim_{x\to\infty} \frac{e^x + x}{e^x} = \lim_{x\to\infty} \frac{1 + \frac{x}{e^x}}{1} = 1 \to$ True

f) $\lim_{x\to\infty} \frac{x\ln x}{x^2} = \lim_{x\to\infty} \frac{\ln x}{x} = 0 \to$ True

g) $\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\ln x} = 0 \to$ True

h) $\lim_{x\to\infty} \frac{\ln x}{\ln(x^2+1)} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{2x}{x^2+1}} = \lim_{x\to\infty} \frac{x^2+1}{2x^2} = \frac{1}{2} \to$ False

(32) $\lim_{x\to\infty} \frac{\sqrt{10x+1}}{\sqrt{x}} = \sqrt{\lim_{x\to\infty} \frac{10x+1}{x}} = \sqrt{10}$ $\therefore$ $\sqrt{10x+1}$ & $\sqrt{x}$ gr[...]
at the same rat[...]

$\lim_{x\to\infty} \frac{\sqrt{x+1}}{\sqrt{x}} = \lim_{x\to\infty} \sqrt{\frac{x+1}{x}} \underset{\uparrow}{=} \sqrt{\lim_{x\to\infty} \frac{x+1}{x}} = \sqrt{1} = 1$ $\therefore$ $\frac{\sqrt{x+[...]}}{\sqrt{x}}$
at [...]

$\frac{x+1}{x}$ is cont
as $x\to\infty$

$\therefore$ $\sqrt{10x+1}$ & $\sqrt{x+1}$ grow at the sam e rate.
by the Transitive Property.